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THE INVARIANT STRAIGHT LINES OF AN AFFINE TRANSFORMATION IN $\mathbb{R}^n$ WITHOUT FIXED POINTS

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Abstract. In this note we prove that every affine transformation in $\mathbb{R}^n$ of the form $f(x) = Ax + a$ without fixed points has an invariant straight line if and only if when $g(a) \in \operatorname{Im}(g \circ g)$, where $g(x) = (A - I)x$.

In [1] it was proved that an affine transformation f in $\mathbb{R}^n$ without fixed points and which fulfils the inequality $|f(x)f(y)| \leq |xy|$ for any x and y has an invariant straight line. In this note we will solve this problem for arbitrary affine transformation in $\mathbb{R}^n$ without fixed points.

If an affine transformation f has an invariant straight line l, then the restriction of the transformation f to l is a translation by some nonzero vector v (f has no fixed points). For a given vector $v \in \mathbb{R}^n$ the set

$$H = \{x \in \mathbb{R}^n : Ax + a = x + v\}$$

is either empty or an affine subspace of the $\mathbb{R}^n$. If $v = 0$ then the set H is empty because the affine transformation has no fixed points.

In this note we are going to find a nonzero vectors v such that the set H would be nonempty and invariant under the transformation f.

Theorem 1. *An affine transformation f has an invariant affine subspace H of the $\mathbb{R}^n$ determined by some vector $v \neq 0$ if and only if when the vector a has a decomposition $a = v + u$, where $v \in \ker g$, $u \in \operatorname{Im} g$ and $g(x) = (A - I)x$.*

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Proof. Necessity. If the equation

$$Ax + a = x + v \tag{1}$$

has a solution, with respect to x, for some vector v, then $v - a \in \operatorname{Im} g$. The general solution of the system (1) has the form $x = s_1a_1 + \ldots + s_pa_p + x_0$, where the vectors $a_1, \ldots, a_p$ form a basis of the $\ker g$ and x_0 is a particular solution of the system (1). It is easy to check that $f(H) = H + v$ and thus $f(H) = H$ if and only if when $v \in \ker g$. Taking into account that $v - a \in \operatorname{Im} g$ we obtain the assertion.

Sufficiency. If $a = v + u$, where v is a nonzero vector which belongs to $\ker g$ and u belongs to $\operatorname{Im} g$, then the linear equation (1) can be written in the equivalent form

$$(A - I)x = -u. \tag{2}$$

Because $u \in \operatorname{Im} g$, so $(A - I)x_1 = -u$ for some $x_1 \in \mathbb{R}^n$, thus we can rewrite the system (2) in the form

$$(A - I)(x - x_1) = 0. \tag{3}$$

Since the affine transformation f has no fixed points, we have $\det(A - I) = 0$. Thus the system (3) has a nonzero solution. The affine subspace H determined by the vector v is invariant under the transformation f, because $v \in \ker g$. □

Theorem 1 is equivalent to the following

Theorem 2. *An affine transformation f has an invariant affine subspace H determined by some vector v if and only if when $g(a) \in \operatorname{Im}(g \circ g)$.*

Proof. Necessity. From Theorem 1 we have the decomposition $a = v + u$, where $v \in \ker g$ and $u \in \operatorname{Im} g$. Thus $g(a) = g(u)$, i.e. $g(a) \in \operatorname{Im}(g \circ g)$.

Sufficiency. If $g(a) \in \operatorname{Im}(g \circ g)$ then there exists some vector $w \in \mathbb{R}^n$ such that $g(a) = g(g(w))$. Thus $g(a - g(w)) = 0$. This denotes that $a - g(w) \in \ker g$. We conclude that $a = v + g(w)$ for some vector $v \in \ker g$. The vector v is nonzero because the affine transformation f has no fixed points. □

Corollary 1. *If a vector a has a decomposition $a = v + u$, where v is nonzero vector belonging to $\ker g$ and u belongs to $\operatorname{Im} g$, then the straight line l determined by the equation $x = x_0 + vt$, where x_0 is a solution of the system (1), is invariant under the affine transformation f.*

PROOF. If $x = x_0 + vt$ then $Ax + a = Ax_0 + tAv + a$. Because $v \in \ker g$, so $Av = v$ and $Ax_0 + a = x_0 + v$. Thus we obtain the equality $Ax + a = x_0 + (t+1)v$, i.e. $f(l) = l$. □

REMARK 1. The decomposition of the vector a in the form $a = v + u$, where $v \in \ker g$ and $u \in \operatorname{Im} g$, is unique.

Indeed, let us assume that $a = v_1 + u_1$ and $a = v_2 + u_2$, then from Corollary 1 it follows that the straight lines which are specified by the equations: $x = x_0 + v_1 t$ and $x = x_0 + v_2 t$ are invariant under the affine transformation f and thus the point x_0 is the fixed point of f, and this is not possible.

COROLLARY 2. *If a linear transformation g of an affine transformation f without fixed points fulfils the equality*

$$\ker g = \ker(g \circ g), \tag{4}$$

then f has an invariant straight line.

PROOF. If $\ker g = \ker(g \circ g)$ then $\mathbb{R}^n = \ker g \oplus \operatorname{Im} g$, i.e. $a = v + u$, where $v \in \ker g$ and $u \in \operatorname{Im} g$. The vector v is nonzero because the affine transformation f has no fixed points. From Corollary 1 we obtain our assertion. □

REMARK 2. In particular, the equality (4) is satisfied by affine transformations given by a symmetric matrix or by affine transformations such that the number 1 is a simple characteristic number of its matrix.

Reference

[1] Kasparek E., *Remark on invariant straight lines of some affine transformations in $\mathbb{R}^n$ without fixed points*, Ann. Math. Sil. **19** (2005), 19–21.

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