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WELL-POSEDNESS OF THE FIXED POINT PROBLEM FOR CERTAIN ASYMPTOTICALLY REGULAR MAPPINGS

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Abstract. We study the well-posedness of the fixed point problem for asymptotically regular self-mappings of a complete metric space (X, d) which satisfy the contractive condition (2.1) described below. This contractive condition is a variant of the contractive condition considered in [6]. The results of this paper provide some improvements and extensions to the results of Ćirić [6], Sharma and Yuel [19], and Guay and Singh [7]. This work is inspired and motivated by the paper [6].

1. Introduction

The notion of well-posednes of a fixed point problem has generated much interest to a several mathematicians, for examples, F.S. De Blassi and J. Myjak (see [2]), S. Reich and A. J. Zaslavski (see [16]), B.K. Lahiri and P. Das (see [9]) and V. Popa (see [14] and [15]).

Definition 1.1. *Let (X, d) be a metric space and $T\colon (X, d) \to (X, d)$ a mapping. The fixed point problem of T is said to be well posed if:*

(a) *T has a unique fixed point z in X;*
(b) *for any sequence $\{x_n\}$ of points in X such that $\lim_{n\to\infty} d(Tx_n, x_n) = 0$, we have $\lim_{n\to\infty} d(x_n, z) = 0$.*

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The notion of contractive mapping has been introduced by Banach in [1]. In the last thirty years there have appeared different types of generalizations of this concept. The connection between them has been studied in different papers (see [8], [10], [17] and [20]).

We recall that the idea of implicit relation is due to V. Popa (see [11] and [12]). Since the publication of the papers [11] and [12], almost all the contractive conditions involved in fixed or common fixed point results are defined by implicit relations.

Browder and Petryshyn (see [3]) defined the following notion.

Definition 1.2. *A selfmapping T on a metric space (X, d) is said to be asymptotically regular at a point x in X, if*

$$\lim_{n\to\infty} d(T^n x, T^n Tx) = 0, \tag{1.1}$$

where $T^n x$ denotes the n-th iterate of T at x.

Almost all of the contractive conditions ensuring the existence of fixed points and generalizing the Banach principle imply the asymptotic regularity of the mappings under consideration. So the investigation of the asymptotically regular maps plays an important role in fixed point theory.

Ćirić (see [6]) pointed out that Sharma and Yuel [19] and Guay and Singh [7] were among the first who used the concept of asymptotic regularity to prove fixed point theorems.

In [6], Ćirić generalized the results of Sharma and Yuel [19] and Guay and Singh [7] and studied a wide class of asymptotically regular mappings which possess fixed points in complete metric spaces and has proved the following theorem.

Theorem 1.1. *Let R^+ be the set of nonnegative reals and let $F_i\colon R^+ \to R^+$ be functions such that $F_i(0) = 0$ and F_i is continuous at 0 for $i = 1, 2$.*

Let (X, d) be a complete metric space and T a selfmapping on X satisfying the following condition:

$$\begin{aligned} d(Tx, Ty) \le{}& a_1 F_1(\min\{d(x, Tx), d(y, Ty)\}) + a_2 F_2(d(x, Tx)d(y, Ty)) \\ &+ a_3 d(x, y) + a_4[d(x, Tx) + d(y, Ty)] + a_5[d(x, Ty) + d(y, Tx)] \end{aligned} \tag{1.2}$$

for all x, y in X, where $a_i = a_i(x, y)$ $(i = 1, 2, 3, 4, 5)$ are nonnegative functions for which there exist three constants $K > 0$ and $\lambda_1, \lambda_2 \in (0, 1)$, such that

the following inequalities:

$$a_1(x,y), a_1(x,y) \le K, \tag{1.3}$$

$$a_4(x,y) + a_5(x,y) \le \lambda_1, \tag{1.4}$$

$$a_3(x,y) + 2a_5(x,y) \le \lambda_2, \tag{1.5}$$

are satisfied for all x, y in X.

If T is asymptotically regular at some x_0 in X, then T has a unique fixed point in X and at this point T is continuous.

In this paper, we introduce a general contractive condition similar to the condition (1.2). More precisely, we replace the two functions F_1 and F_2 by a single function F of two variables (see condition (2.1) below). The aim of this paper is to study the well posedness of the fixed point problem of asymptotically regular mappings satisfying the condition (2.1). We prove (see Theorem 2.1 below) that the fixed point problem for these mappings is well-posed. This work is inspired by the paper [6]. Our main result extends and unifies the results of Ćirić [6], Sharma and Yuel [19], and Guay and Singh [7].

The main result of this paper is Theorem 2.1. It is established in Section 2. In Section 3, we have gathered some consequences and corollaries. In Section 4, we have stated two versions of Theorem 2.1 in non complete spaces and in (non complete but) orbitally complete spaces respectively.

2. Main Result

The main result of this paper is the following theorem.

Theorem 2.1. *Let R^+ be the set of nonnegative reals and let $F\colon R^+ \times R^+ \to R^+$ be a function such that $F(t,0) = F(0,t) = 0$ and F is continuous at $(t,0)$ and $(0,t)$ for all $t \ge 0$.*

Let (X,d) be a complete metric space and T a selfmapping on X satisfying the following condition:

$$\begin{aligned} d(Tx,Ty) \le{} & a_0 F(d(x,Tx), d(y,Ty)) + a_1 d(x,y) \\ & + a_2[d(x,Tx) + d(y,Ty)] + a_3[d(x,Ty) + d(y,Tx)] \end{aligned} \tag{2.1}$$

for all x, y in X, where $a_i = a_i(x,y)$ $(i = 0,1,2,3)$ are nonnegative functions for which there exist three constants $K > 0$ and $\lambda_1, \lambda_2 \in (0,1)$, such that

the following inequalities:

$$a_0(x,y) \le K, \tag{2.2}$$

$$a_2(x,y) + a_3(x,y) \le \lambda_1, \tag{2.3}$$

$$a_1(x,y) + 2a_3(x,y) \le \lambda_2, \tag{2.4}$$

are satisfied for all x, y in X.

If T is asymptotically regular at some x_0 in X, then the fixed point problem of T is well-posed. Moreover, T is continuous at its unique fixed point.

PROOF. Let x_0 be a point of X at which T is asymptotically regular. Then as in the proof of Theorem 1 of [6], one can show that $\{x_n\}$ is a Cauchy sequence, where $x_n = T^n x_0$ and that $\{x_n\}$ converges to a fixed point $z \in X$.

To prove the uniqueness of z, let us suppose that u and v are two fixed points of T. From (2.1), with $a_i = a_i(u,v)$,

$$\begin{aligned} d(u,v) &= d(Tu,Tv) \\ &= a_0 F(0,0) + a_1 d(u,v) + a_2 \cdot 0 + 2a_3 d(u,v) \\ &= (a_1 + 2a_3) d(u,v). \end{aligned}$$

Hence, because of (2.4),

$$(1 - \lambda_2) d(u,v) \le 0, \tag{2.5}$$

which implies $v = u$.

Now, let $\{x_n\}$ be any arbitrary sequence satisfying $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. For every nonnegative integer n, we denote

$$d_n = d(x_n, Tx_n). \tag{2.6}$$

Using the triangle inequality, from (2.1) we have

$$\begin{aligned} d(x_n,x_m) &\le d_n + d(Tx_n,Tx_m) + d_m \\ &\le d_n + d_m + a_0 F(d_n,d_m) + a_1 d(x_n,x_m) + a_2(d_n + d_m) \\ &\quad + a_3[d(x_n,Tx_m) + d(x_m,Tx_n)], \end{aligned}$$

where $a_i = a_i(x_n,x_m)$ for $i = 0,1,2,3$.

Using again the triangle inequality, we get

$$d(x_n,x_m) \le (a_1 + 2a_3) d(x_n,x_m) + (1 + a_2 + a_3)(d_n + d_m) + a_0 F(d_n,d_m).$$

Hence, because of (2.3), (2.4) and (2.5), we obtain

$$(1-\lambda_2)d(x_n,x_m) \le (1+\lambda_1)(d_n+d_m)+KF(d_n,d_m). \tag{2.7}$$

Since $\lim_{n\to\infty} d(x_n,Tx_n)=0$ and F is continuous at $(0,0)$, then by taking the limit as m tends to infinity we obtain

$$(1-\lambda_2)\lim_{n\ge m\to\infty} d(x_n,x_m)=0, \tag{2.8}$$

which implies that $\{x_n\}$ is a Cauchy sequence.

Since X is complete, $\{x_n\}$ is convergent to a limit (say) u in X.

Now we show that u is equal to the unique fixed point z of T. We start by proving that $Tu=u$. To get a contradiction, let us suppose that $d(u,Tu)>0$. Then, from (2.1) we have

$$\begin{aligned} d(u,Tu) &\le d(u,Tx_n)+d(Tx_n,Tu)\\ &\le d(u,x_n)+d(x_n,Tx_n)+a_0F(d_n,d(u,Tu))\\ &\quad + a_1 d(x_n,u)+a_2[d_n+d(u,Tu)]\\ &\quad + a_3[d(x_n,Tu)+d(u,Tx_n)], \end{aligned}$$

where $a_i=a_i(x_n,u)$ for $i=1,2,3$.

Using the triangle inequality we get

$$\begin{aligned} d(u,Tu) \le\ & a_0F(d_n,d(u,Tu))+(a_2+a_3)d(u,Tu)\\ & + (1+a_2+a_3)d(x_n,Tx_n)+(1+a_1+2a_3)d(u,x_n). \end{aligned}$$

Therefore, from (2.2), (2.3) and (2.4),

$$\begin{aligned} d(u,Tu) \le\ & KF(d_n,d(u,Tu))+\lambda_1 d(u,Tu)\\ & + (1+\lambda_1)d(x_n,Tx_n)+(1+\lambda_2)d(u,x_n). \end{aligned}$$

Taking the limit and using the continuity of F at $(0,d(u,Tu))$, we get

$$d(u,Tu)\le \lambda_1 d(u,Tu) < d(u,Tu),$$

a contradiction. Therefore, $d(u,Tu)=0$ That is $Tu=u$. By uniqueness of z, we must have $z=u$. We conclude that the fixed point problem of T is well-posed.

To prove that T is continuous at z, suppose that $u_n \to z = Tz$. Then from (2.1),

$$\begin{aligned} d(Tu_n, z) &= d(Tu_n, Tz) \\ &\le a_0 F[(d(u_n, Tu_n), 0)] + a_1 d(u_n, z) \\ &\quad + a_2 d(u_n, Tu_n) + a_3[d(u_n, z) + d(Tu_n, z)] \\ &\le KF[(d(u_n, Tu_n), 0)] + (a_1 + a_2 + a_3) d(u_n, z) \\ &\quad + (a_2 + a_3) d(Tu_n, z), \end{aligned}$$

where $a_i = a_i(u_n, u)$ for $i = 1, 2, 3$.

Hence, using (2.3) and (2.4),

$$(1 - \lambda_1) d(Tu_n, u) \le KF[(d(u_n, Tu_n), 0)] + (\lambda_1 + \lambda_2) d(u_n, z). \tag{2.9}$$

By letting n go to infinity and using the fact that $F(t, 0) = 0$, we obtain

$$(1 - \lambda_1) \limsup_n d(Tu_n, z) \le 0,$$

which implies that $\lim_{n\to\infty} Tu_n = z$. This completes the proof. □

3. Consequences and Applications

We have the following corollaries.

COROLLARY 3.1. *Let R^+ be the set of nonnegative reals and let $F_i \colon R^+ \to R^+$ be functions such that $F_i(0) = 0$ and F_i is continuous at 0 for $(i = 1, 2)$.*

Let (X, d) be a complete metric space and T a selfmapping on X satisfying the following condition:

$$\begin{aligned} d(Tx, Ty) \le\ & b_1 F_1(\min\{d(x, Tx), d(y, Ty)\}) + b_2 F_2(d(x, Tx) d(y, Ty)) \\ & + b_3 d(x, y) + b_4[d(x, Tx) + d(y, Ty)] + b_5[d(x, Ty) + d(y, Tx)] \end{aligned} \tag{3.1}$$

for all x, y in X, where $b_i = b_i(x, y)$ $(i = 1, 2, 3, 4, 5)$ are nonnegative functions for which there exist three constants $K > 0$ and $\lambda_1, \lambda_2 \in (0, 1)$, such that

the following inequalities:

$$b_1(x,y), b_2(x,y) \le K, \tag{3.2}$$

$$b_4(x,y) + b_5(x,y) \le \lambda_1, \tag{3.3}$$

$$b_3(x,y) + 2b_5(x,y) \le \lambda_2, \tag{3.4}$$

are satisfied for all x, y in X.

If T is asymptotically regular at some x_0 in X, then the fixed point problem of T is well-posed. Moreover, T is continuous at its unique fixed point.

The proof follows from Theorem 2.1, by considering the functions:

$$F(s,t) := F_1(\min\{s,t\}) + F_2(st),$$
$$a_0(x,y) := \max\{b_1(x,y), b_2(x,y)\}, \quad \text{and} \quad a_1 := b_3,\ a_2 := b_4,\ a_3 := b_5.$$

Corollary 3.2. *Let $\alpha \ge 0$ and $\beta \in [0,1)$. Let (X,d) be a complete metric space and T a selfmapping on X satisfying the following condition:*

$$d(Tx,Ty) \le \alpha \frac{\min\{d(x,Tx), d(y,Ty) + d(x,Tx)d(y,Ty)\}}{1+d(x,y)} + \beta d(x,y) \tag{3.5}$$

for all x, y in X. If T is asymptotically regular at some x_0 in X, then the fixed point problem of T is well-posed. Moreover, T is continuous at its unique fixed point.

The proof follows from Theorem 2.1 by considering the functions

$$F(s,t) := \alpha[(\min\{s,t\}) + st],$$
$$a_0(x,y) := \frac{1}{1+d(x,y)}, \quad \text{and} \quad a_1 := \beta,\ a_2 := 0,\ a_3 := 0.$$

Beside these considerations, we can take $K = 1$, $\lambda_1 = \lambda_2 := \beta$.

We observe that the contractive condition (3.5) is more general than the one considered by Sharma and Yuel in [19].

Corollary 3.3. *Let (X,d) be a complete metric space and T a selfmapping on X satisfying the following condition:*

$$d(Tx,Ty) \le pd(x,y) + q[d(x,Tx) + d(y,Ty)] + r[d(x,Ty) + d(y,Tx)], \tag{3.6}$$

where $p, q,$ and r are fixed (nonegative) real numbers such that $q + r < 1$ and $p + 2r < 1$.

If T is asymptotically regular at some x_0 in X, then the fixed point problem of T is well-posed. Moreover, T is continuous at its unique fixed point.

The proof follows from Theorem 2.1, by considering the functions:

$$F(s,t) := 0, \quad a_0(x,y) := 0, \quad \text{and} \quad a_1 := p,\, a_2 := q,\, a_3 := r.$$

Beside these considerations, we can take $K = 0$, $\lambda_1 = q + r$ and $\lambda_2 := p + 2r$.
Condition (3.6) is the contractive condition, introduced and considered by Guay and Singh in [7].

4. General results in non complete metric spaces

In the case where the metric space (X, d) is not complete, we have the following general result.

THEOREM 4.1. *Let R^+ be the set of nonnegative reals and let $F\colon R^+ \times R^+ \to R^+$ be a function such that $F(t,0) = F(0,t) = 0$ and F is continuous at $(t,0)$ and $(0,t)$ for all $t \geq 0$. Let (X,d) be a metric space and T a selfmapping on X satisfying condition (2.1) for all x, y in X, where $a_i = a_i(x,y)$ $(i = 0,1,2,3)$ are nonnegative functions for which there exist three constants $K > 0$ and $\lambda_1, \lambda_2 \in (0,1)$, such that (2.2)–(2.4) are satisfied for all x, y in X.*

If T is asymptotically regular at some x_0 in X and the sequence of iterates $\{T^n x_0\}$ has a subsequence converging to a point z in X, then z is the the unique fixed point of T and the fixed point problem of T is well-posed. Moreover, T is continuous at z.

PROOF. As in the proof of Theorem 2.1 (see also the proof of Theorem 1 in [6]), the sequence of iterates $\{T^n x_0\}$ is a Cauchy sequence. Since it contains a subsequence which converges to the point z, we conclude that $\lim_{n\to\infty} T^n x_0 = z$. By using the contractive condition (2.1), one can prove that z is the unique fixed point of T. The rest of the result is obtained by the method of proof as in Theorem 2.1. This completes the proof. □

In 1974 Ćirić ([4]) has first introduced orbitally complete metric spaces.

DEFINITION 4.1. *Let T be a self mapping of a metric space (X,d). If for all x in X every Cauchy sequence of the orbit $O_X(T) = \{x, Tx, T^2x, \ldots\}$ is convergent in X, then the metric space (X,d) is said T-orbitally complete.*

REMARK 1. Every complete metric space is T-orbitally complete for any $T\colon X \to X$. An orbitally complete space may not be complete metric space (Example 1 [21]).

For (non complete but) orbitally complete metric spaces, we have the following result.

THEOREM 4.2. *Let R^+ be the set of nonnegative reals and let $F\colon R^+ \times R^+ \to R^+$ be a function such that $F(t,0) = F(0,t) = 0$ and F is continuous at $(t,0)$ and $(0,t)$ for all $t \geq 0$. Let (X,d) be a metric space and T a selfmapping on X satisfying condition (2.1) for all x, y in X, where $a_i = a_i(x,y)$ $(i = 0,1,2,3)$ are nonnegative functions for which there exist three constants $K > 0$ and $\lambda_1, \lambda_2 \in (0,1)$, such that (2.2)–(2.4) are satisfied for all x, y in X.*

If T is asymptotically regular at some x_0 in X and (X,d) is T-orbitally complete then the fixed point problem of T is well-posed. Moreover, T is continuous at its unique fixed point.

PROOF. As in the proof of Theorem 2.1 (see also the proof of Theorem 1 in [6]), the sequence of iterates $\{T^n x_0\}$ is a Cauchy sequence. Since (X,d) is T-orbitally complete then this sequence converges to a point z in X. By using the contractive condition (2.1), one can prove that z is the unique fixed point of T. The rest of the result is obtained by the method of proof as in Theorem 2.1. This completes the proof. $\square$

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