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JENSEN CONVEX FUNCTIONS BOUNDED ABOVE ON NONZERO CHRISTENSEN MEASURABLE SETS

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Abstract. We prove that every Jensen convex function mapping a real linear Polish space into $\mathbb{R}$ bounded above on a nonzero Christensen measurable set is convex.

Functions satisfying

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \tag{1}$$

for x, y from the domain being a convex set are called Jensen convex and they play very important role in many branches of mathematics (more information on such functions can be find in [5]). A lot of authors were interested in finding conditions which implies the continuity of f satisfying (1). Among others, W. Sierpiński, A. Ostrowski and M.R. Mehdi showed that every Jensen convex function which is Lebesgue measurable, or bounded above on a set of positive Lebesgue measure, or bounded above on a set of second category with the Baire property, has to be continuous (see [5, Theorems 9.3.1, 9.3.2, p.232 and Theorem 9.4.2, p.241]. P. Fischer and Z. Słodkowski generalized the result of Sierpiński; they proved that each Christensen measurable Jensen convex function mapping a real linear Polish space into $\mathbb{R}$ is continuous and convex (see [4, Theorem 2]). However the following problem seems to be open: does

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each Jensen convex function bounded above on a nonzero Christensen measurable set have to be continuous? This problem was formulated by K. Baron and R. Ger at the 21st International Symposium on Functional Equations (1983, Konolfingen, Switzerland) (see [6, 44, Problem (P239),pp. 285–286]).

We prove that each Jensen convex function $f\colon X \to \mathbb{R}$ mapping a real linear Polish space X into $\mathbb{R}$ bounded above on a nonzero Christensen measurable set is convex.

First, let us recall some basic definitions (cf. [2]–[4]) concerning Christensen measurability.

Let X be a real linear Polish space and let $\mathfrak{M}$ be the σ–algebra of all universally measurable subsets of X; i.e. $\mathfrak{M}$ is the intersection of all completions of the Borel σ–algebra of X with respect to probability Borel measures. In the following by a measure we mean a countable additive Borel measure extended to $\mathfrak{M}$.

Definition 1. A set $B \in \mathfrak{M}$ is *a Haar zero set* iff there exists a probability measure u on X such that $u(B + x) = 0$ for each $x \in X$. A set $P \subset X$ is *a Christensen zero set* iff P is a subset of a Haar zero set. A set $D \subset X$ is *a Christensen measurable set* iff there are $B \in \mathfrak{M}$ and a Christensen zero set P such that $D = B \cup P$. Finally, a function $f\colon X \to \mathbb{R}$ is said to be *Christensen measurable* iff $f^{-1}(U)$ is a Christensen measurable set for each open set $U \subset \mathbb{R}$.

Lemma 1 ([1, Lemma 14]). *Let $D \subset X$ be a nonzero Christensen measurable set and $x \in X \setminus \{0\}$. Then there exist a Borel set $D_x \subset D$ and $y_x \in X$ such that the set $k_x^{-1}(y_x + D_x) \subset \mathbb{R}$ has positive Lebesgue measure, where $k_x\colon \mathbb{R} \to X$ is given by $k_x(a) = ax$.*

Now we prove the announced result.

Theorem 1. *Assume $f\colon X \to \mathbb{R}$ is Jensen convex. If*

$$\sup f(C) < \infty \tag{2}$$

for a nonzero Christensen measurable $C \subset X$, then f is convex.

Proof. Fix $x \in X \setminus \{0\}$ and $z \in X$, define $\varphi\colon \mathbb{R} \to \mathbb{R}$ by

$$\varphi(\alpha) = f(\alpha x + z) \ \text{ for } \ \alpha \in \mathbb{R} \tag{3}$$

and note that it is Jensen convex. According to Lemma 1 there are a Borel set $B \subset \mathbb{R}$ of positive Lebesgue measure and a $y \in X$ such that

$$\alpha x - y \in C \ \text{ for } \ \alpha \in B.$$

Consequently, for $\alpha \in B$ we have

$$\varphi\left(\frac{\alpha}{2}\right) = f\left(\frac{(\alpha x - y) + (y + 2z)}{2}\right) \leq \frac{f(\alpha x - y) + f(y + 2z)}{2} \leq \frac{\sup f(C) + f(y + 2z)}{2}$$

This shows that $\sup \varphi\left(\frac{1}{2}B\right) < \infty$ and, according to theorem of Ostrowski [5, Theorem 9.3.1, p.232], φ is continuous. Hence, by [5, Theorem 5.3.5, p.133], φ is convex and to finish the proof it is enough to apply the following simple remark:

If X is a real linear space, then $f\colon X \to \mathbb{R}$ is convex if and only if for every $x \in X \setminus \{0\}$, $z \in X$ the function (3) is convex. □

Corollary 1. *Assume X is a real linear Polish space and $f\colon X \to \mathbb{R}$ is additive. If* (2) *holds for a nonzero Christensen measurable set $C \subset X$, then f is linear.*

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