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ON A JENSEN–HOSSZÚ EQUATION I

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Abstract. We solve functional equation of the form

$$f(x+y-xy)+f(xy)=2f\left(\frac{x+y}{2}\right)$$

in the class of functions transforming the space of all reals into itself. We also prove that this equation is stable in the Hyers-Ulam's sense.

1. Introduction

It is well-known that in the class of functions transforming the set of all reals into itself the Jensen functional equation

$$2f\left(\frac{x+y}{2}\right)=f(x)+f(y)$$

and the Hosszú functional equation

$$f(x+y-xy)+f(xy)=f(x)+f(y)$$

are equivalent and the general solution has the form $f(x)=a(x)+c,\ x\in\mathbb{R}$, where a is an additive function and c is an arbitrary constant [3]. Recall that

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$a\colon \mathbb{R} \to \mathbb{R}$ is an additive function if it satisfies the following Cauchy functional equation

$$a(x+y) = a(x) + a(y), \quad x, y \in \mathbb{R}.$$

We will consider the functional equation in which the left-hand side has the same form as in the Hosszú equation and right-hand side coincides with the left-hand side of the Jensen equation, i.e., the following functional equation

$$f(x+y-xy) + f(xy) = 2f\Big(\frac{x+y}{2}\Big), \quad x, y \in \mathbb{R}. \tag{1}$$

We will prove that equation (1) is also equivalent to the Hosszú (and in the same reason to the Jensen) equation and, moreover, equation (1) (similarly to each of them) is stable in the sense of Hyers and Ulam.

2. Results

Let $\delta \geq 0$ be fixed, $g\colon \mathbb{R} \to \mathbb{R}$ be a function and consider the following functional inequality

$$\Big|g(x+y-xy) + g(xy) - 2g\Big(\frac{x+y}{2}\Big)\Big| \leq \delta, \quad x, y \in \mathbb{R}. \tag{2}$$

First assume that $g(0) = 0$. Denote by g^+ and g^- the even and the odd part of g, i.e.,

$$g^+(x) := \frac{g(x) + g(-x)}{2}, \quad g^-(x) := \frac{g(x) - g(-x)}{2}, \quad x \in \mathbb{R}.$$

Putting $y = -x$ in (2) we obtain

$$|g(x^2) + g(-x^2)| \leq \delta, \quad x \in \mathbb{R},$$

which means that g^+ is a bounded function:

$$|g^+(x)| \leq \frac{1}{2}\delta, \quad x \in \mathbb{R}. \tag{3}$$

Because of the equality $g^-(x) = 2g(x) - g^+(x)$, $x \in \mathbb{R}$, and by virtue of (2) and (3) we get

(4) $$\Big|g^-(x+y-xy)+g^-(xy)-2g^-\Big(\frac{x+y}{2}\Big)\Big|\le 4\delta,\quad x,y\in\mathbb{R}.$$

For all $u\in\mathbb{R}$ and $v\le 0$ the quadratic equation

$$x^2-(u+v)x+v=0$$

has exactly two solutions x and y satisfying the following conditions:

$$x+y=u+v\quad\text{and}\quad xy=v.$$

Therefore

(5) $$\Big|g^-(u)+g^-(v)-2g^-\Big(\frac{u+v}{2}\Big)\Big|\le 4\delta,\qquad u\in\mathbb{R}, v\le 0.$$

Take arbitrary $u\in\mathbb{R}$ and $v>0$. According to the previous estimation and using also the oddness of g^- we obtain

$$\Big|g^-(u)+g^-(v)-2g^-\Big(\frac{u+v}{2}\Big)\Big|=\Big|2g^-\Big(\frac{-u-v}{2}\Big)-g^-(-u)-g^-(-v)\Big|\le 4\delta.$$

This means that inequality (5) is satisfied for all $u,v\in\mathbb{R}$. Putting $v=0$ and $u=2x$ we obtain

$$\Big|\frac{g^-(2x)}{2}-g^-(x)\Big|\le 2\delta,\quad x\in\mathbb{R}.$$

By a standard way one can prove that the sequence $(\frac{g^-(2^nx)}{2^n})_{n\in\mathbb{N}}$ is convergent to an additive function $a\colon\mathbb{R}\to\mathbb{R}$ fulfilling the following estimation

$$|g^-(x)-a(x)|\le 4\delta,\quad x\in\mathbb{R}$$

(a result of this type can be found in [1]–[4], for example).

In fact, we have proved the following theorem.

Theorem 1. *Let $\delta\ge 0$ be a real constant and let $g\colon\mathbb{R}\to\mathbb{R}$ be a function satisfying the following inequality*

$$\Big|g(x+y-xy)+g(xy)-2g\Big(\frac{x+y}{2}\Big)\Big|\le\delta$$

for all $x,y\in\mathbb{R}$. Then there exists an additive function $a\colon\mathbb{R}\to\mathbb{R}$ such that

$$|g(x)-a(x)-g(0)|\le\frac{9}{2}\delta$$

for every $x\in\mathbb{R}$.

Remark 1. The additive function a from Theorem 1 is determined uniquely. It is a consequence of the fact that if the difference between two additive functions is bounded on the space of all reals then they have to be equal.

Remark 2. Note that without any essential changes the same results may be obtained for functions transforming the space of all reals into a Banach space.

As a simple consequence of our Theorem 1 and Remark 2 we obtain

Theorem 2. *Let X be a Banach space. A function $f\colon \mathbb{R} \to X$ is a solution of equation* (1) *if and only if there exist an additive function $a\colon \mathbb{R} \to X$ and a constant $c \in X$ such that $f(x) = a(x) + c,\ x \in \mathbb{R}$.*

Remark 3. In the class of functions transforming the space of all reals into a Banach space the Jensen-Hosszú functional equation (1) is stable in the sense of Hyers and Ulam.

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