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EXISTENCE OF POSITIVE PERIODIC SOLUTIONS OF SOME DIFFERENTIAL EQUATIONS OF ORDER n $(n \geq 2)$

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Abstract. We study the existence of positive periodic solutions of the equations

$$x^{(n)}(t) - p(t)x(t) + \mu f(t, x(t), x'(t), \ldots, x^{(n-1)}(t)) = 0,$$
$$x^{(n)}(t) + p(t)x(t) = \mu f(t, x(t), x'(t), \ldots, x^{(n-1)}(t)),$$

where $n \geq 2, \mu > 0$, $p\colon (-\infty, \infty) \to (0, \infty)$ is continuous and 1–periodic, f is a continuous function and 1–periodic in the first variable and may take values of different signs. The Krasnosielski fixed point theorem on cone is used.

1. Introduction

Nonnegative solutions of varius boundary value problems for ordinary differential equations have been considered by several authors (see for instance in [1]–[6], [9]–[11]). This paper deals with existence of positive periodic solutions of the nonlinear differential equations of the form:

$$x^{(n)}(t) - p(t)x(t) + \mu f(t, x(t), x'(t), \ldots, x^{(n-1)}(t)) = 0, \tag{1.1}$$

$$x^{(n)}(t) + p(t)x(t) = \mu f(t, x(t), x'(t), \ldots, x^{(n-1)}(t)), \tag{1.2}$$

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where $p\colon (-\infty,\infty)\to(0,\infty)$ is continuous, 1–periodic, $\mu>0$, f is a continuous, 1–periodic function in t and may take values of different signs. Existence in this paper will be established using Krasnosielski fixed point theorem in a cone, which we state here for the convenience of the reader.

Theorem 1.1 (K. Deimling [5], D. Guo, V. Laksmikannthan [6]). *Let $E=(E,\|\cdot\|)$ be a Banach space and let $K\subset E$ be a cone in E. Assume Ω_1 and Ω_2 are bounded and open subsets of E with $0\in\Omega_1$ and $\overline{\Omega}_1\subset\Omega_2$ and let $A\colon K\cap(\overline{\Omega}_2\setminus\Omega_1)\to K$ be continuous and completely continuous. In addition suppose either*

$$\|Au\|\le\|u\|\quad \text{for } u\in K\cap\partial\Omega_1\quad \text{and}\quad \|Au\|\ge\|u\|\quad \text{for } u\in K\cap\partial\Omega_2$$

or

$$\|Au\|\ge\|u\|\quad \text{for } u\in K\cap\partial\Omega_1\quad \text{and}\quad \|Au\|\le\|u\|\quad \text{for } u\in K\cap\partial\Omega_2$$

hold. Then A has a fixed point in $K\cap(\overline{\Omega}_2\setminus\Omega_1)$.

2. Green's function and its sign

In this section we consider the Green functions of the problems:

$$x^{(n)}(t)-p(t)x(t)=0,\quad x^{(i)}(0)=x^{(i)}(1),\quad i=0,1,\dots,n-1; \tag{2.1}$$

$$x^{(n)}(t)+p(t)x(t)=0,\quad x^{(i)}(0)=x^{(i)}(1),\quad i=0,1,\dots,n-1; \tag{2.2}$$

for $n\ge 2$.

First we shall give some notation. We define $P_1^m(\mathbb{R})$ ($m\in\mathbb{N}$) to be the subspace of $B(\mathbb{R})$ (bounded, continuous real functions on $\mathbb{R}$) consisting of all 1–periodic mapping x such that $x^{(m)}$ is an 1–periodic and continuous function on $\mathbb{R}$. For $x\in P^{n-1}(\mathbb{R})$ we define

$$\|x\|_{n-1}=\sup_{t\in[0,1]}[|x(t)|+|x'(t)|+\ldots+|x^{(n-1)}(t)|].$$

Now we shall give conditions under which 1–periodic solution of equation (2.1) or (2.2) is a trivial one.

Theorem 2.1. *We assume that $p\colon(-\infty,\infty)\to(0,\infty)$ is continuous and 1–periodic.*

(a) *If* $n = 2k + 1$ $(k \in \mathbb{N})$*, then problem* (2.1) *or* (2.2) *has only the trivial solution.*
(b) *If* $n = 4k + 2$ $(k \in \mathbb{N} \cup \{0\})$*, then problem* (2.1) *has only the trivial solution.*
(c) *If* $n = 4k$ $(k \in \mathbb{N})$*, then problem* (2.2) *has only the trivial solution.*
(d) *If*

$$\alpha = \sup_{t \in [0,1]} p(t) < \pi(2\pi)^{n-1}, \tag{2.3}$$

then problem (2.1) *or* (2.2) *has only the trivial solution.*

THEOREM 2.2. *We assume that* $p\colon (-\infty, \infty) \to (0, \infty)$ *is continuous and* 1*–periodic. If*

$$\alpha = \sup_{t \in [0,1]} p(t) < 2(2\pi)^{n-2} \tag{2.4}$$

or

$$\beta = \int_0^1 p(t)dt < 1, \tag{2.4$'$}$$

then there exist two functions $G_1(t,s), G_2(t,s)$ *such that:*

1° G_1 *is the Green function of the problem* (2.1) *and* $G_1(t,s) < 0$ *for all* $(t,s) \in [0,1] \times [0,1]$ *and*
2° $G_2(t,s)$ *is the Green function of the problem* (2.2) *and* $G_2(t,s) > 0$ *for all* $(t,s) \in [0,1] \times [0,1]$.

In [7] the authors obtained the following results

THEOREM 2.3. *We assume that*

$$p\colon (-\infty, \infty) \to (0, \infty) \text{ is 1-periodic, } p \in L^1[0,1], \tag{e}$$

$$\lambda_{n-1} = \begin{cases} \frac{1}{2^n} \frac{1 \cdot 3 \cdots (n-1)}{2 \cdot 4 \cdots n}, & \text{if } n \text{ is even and } n \geq 2, \\ \frac{1}{2^n} \frac{1 \cdot 3 \cdots (n-2)}{2 \cdot 4 \cdots (n-1)}, & \text{if } n \text{ is odd and } n \geq 3, \end{cases} \tag{f}$$

$$\int_0^1 p(t)dt > 0, \quad \lambda_{n-1} \int_0^1 p(t)dt < 1. \tag{g}$$

Then problem (2.2) *has only the trivial solution.*

Theorem 2.4 ([7]). *We assume that*

(h) $$p\colon (-\infty,\infty)\to(-\infty,\infty) \text{ is } 1\text{–periodic, } p\in L^1[0,1],$$

(k) $$\int_0^1 p(t)dt>0,\quad \int_0^1 |p(t)|dt\le 16,\quad p(t)\not\equiv 0.$$

Then the problem

(2.2)′ $$x''(t)+p(t)x(t)=0,\quad x^{(i)}(0)=x^{(i)}(1),\quad i=0,1$$

has only the trivial solution.

From Corollary 2.3 in [10] it follows

Theorem 2.5. *If* $p\colon(-\infty,\infty)\to(0,\infty)$ *is continuous,* 1*–periodic, and* $\sup_{t\in[0,1]} p(t)<\pi^2$*, then the Green function* $G(t,s)$ *of the problem* (2.2)′ *has the positive sign.*

Before giving the proofs of Theorems 2.1–2.2 we formulate three lemmas.

Lemma 2.6. *If* $x\in C^1[a,b], t_0\in[a,b]$ *and* $x(t_0)=0$*, then*

(2.5) $$2\int_a^b x^2(t)dt\le(b-a)^2\int_a^b (x')^2(t)dt \quad \text{(see [8], p. 193).}$$

Lemma 2.7. *If* $x\in C^1[a,b]$ *and* $x(a)=x(b)=0$*, then*

(2.6) $$\pi^2\int_a^b x^2(t)dt\le(b-a)^2\int_a^b (x')^2(t)dt \quad \text{(see [8], p. 192).}$$

Lemma 2.8 (Wirtinger). *If* $x\in C^1[a,b]$*,* $x(a)=x(b)$ *and* $\int_a^b x(t)dt=0$*, then*

(2.7) $$(2\pi)^2\int_a^b x^2(t)dt\le(b-a)^2\int_a^b (x')^2(t)dt.$$

PROOF OF THEOREM 2.1. Let x be a solution of the problem (2.1) or (2.2). Then we have

$$\int_0^1 x^{(n)}(t)x(t)dt - \int_0^1 p(t)x^2(t)dt = 0 \tag{2.8}$$

or

$$\int_0^1 x^{(n)}(t)x(t)dt + \int_0^1 p(t)x^2(t)dt = 0.$$

Let $n = 2k+1$. Then integrating by parts k–times $x^{(2k+1)}(t)x(t)$ we get

$$x(t)x^{(2k)}(t)\Big|_0^1 + \ldots + (-1)^k \frac{(x^{(k)})^2(t)}{2}\Big|_0^1 - \int_0^1 p(t)x^2(t)dt = 0$$

or

$$x(t)x^{(2k)}(t)\Big|_0^1 + \ldots + (-1)^k \frac{(x^{(k)})^2(t)}{2}\Big|_0^1 + \int_0^1 p(t)x^2(t)dt = 0.$$

Hence we have

$$\int_0^1 p(t)x^2(t)dt = 0.$$

Consequently $x \equiv 0$. Notice also for $n = 4k+2$ or $n = 4k$ that

$$\int_0^1 x^{(4k+2)}(t)x(t)dt - \int_0^1 p(t)x^2(t)dt = (-1)^{2k+1}\int_0^1 (x^{(2k+1)})^2(t)dt - \int_0^1 p(t)x^2(t)dt = 0$$

or

$$\int_0^1 x^{(4k)}(t)x(t)dt + \int_0^1 p(t)x^2(t)dt = (-1)^{2k}\int_0^1 (x^{(2k)})^2(t)dt + \int_0^1 p(t)x^2(t)dt = 0.$$

This yields $x \equiv 0$.

Now we will examine case (d). If x is a solution of the problem (2.1) or (2.2) and $x(t) \geq 0$ ($x(t) \leq 0$) for all $t \in [0,1]$, then

$$0 = \int_0^1 x^{(n)}(t)dt = \int_0^1 p(t)x(t)dt$$

or

$$0 = \int_0^1 x^{(n)}(t)dt = -\int_0^1 p(t)x(t)dt.$$

The last equalities yield $x \equiv 0$.

Let x be a sign–changing solution of the problem (2.1) or (2.2) and let $x(t_0) = 0$. Then $x(t_0 + 1) = x(t_0) = 0$. By Lemmas 2.7–2.8 we get

$$\pi^2 \int_{t_0}^{t_0+1} x^2(t)dt = \pi^2 \int_0^1 x^2(t)dt \leq \int_{t_0}^{t_0+1} (x')^2(t)dt = \int_0^1 (x')^2(t)dt, \tag{2.9}$$

$$(2\pi)^2 \int_0^1 (x')^2(t)dt \leq \int_0^1 (x'')^2(t)dt, \tag{2.10}$$

$$\vdots$$

$$(2\pi)^2 \int_0^1 (x^{(n-1)})^2(t)dt \leq \int_0^1 (x^{(n)})^2 dt = \int_0^1 p^2(t)x^2(t)dt. \tag{2.11}$$

Relations (2.9)–(2.11) imply

$$\int_0^1 x^2(t)dt \leq \frac{1}{\pi^2} \frac{1}{(2\pi)^{2(n-1)}} \alpha^2 \int_0^1 x^2(t)dt,$$

which contradicts (2.3). The proof of Theorem 2.1 is finished. □

Proof of Theorem 2.2. Case 1°. As G_1 is a continuous function defined on $[0,1] \times [0,1]$, we only have to prove that it does not vanish in every point. Let us suppose, to derive a contradiction, that there exists $(t_0, s_0) \in [0,1] \times [0,1]$ such that $G_1(t_0, s_0) = 0$. First, let us assume that $(t_0, s_0) \in (0,1) \times [0,1]$. It is known that for a given $s_0 \in (0,1), G_1(t, s_0)$ as a function of t is a solution of (2.1) in the intervals $[0, s_0)$ and $(s_0, 1]$ such that

$$\frac{\partial^i G_1(0,s_0)}{\partial t^i} = \frac{\partial^i G_1(1,s_0)}{\partial t^i}, \quad i = 0,1,\ldots,n-1. \tag{2.12}$$

We define

$$x(t) = \begin{cases} G_1(t,s_0), & \text{for } t \in [s_0,1], \\ G_1(t-1,s_0), & \text{for } t \in [1,s_0+1]. \end{cases} \tag{2.13}$$

The function x is of the class C^{n-1} and in consequence is a solution of equation (2.1) in the whole interval $[s_0, s_0+1]$,

$$x^{(i)}(s_0) = x^{(i)}(s_0+1) \quad \text{for } i = 0,1,\ldots,n-2, \tag{2.14}$$

and

$$x^{(n-1)}(s_0) - x^{(n-1)}(s_0+1) = 1. \tag{2.15}$$

There exists a point $\bar{t} \in [s_0, s_0+1]$ such that $x^{(n-1)}(\bar{t}) = 0$. From the equalities

$$x(t) = \int_{t_0}^{t} x'(s)ds, \quad x^{(n-1)}(t) = \int_{\bar{t}}^{t} x^{(n)}(s)ds, \quad t \in [s_0, s_0+1], \tag{2.16}$$

and Lemma 2.6 it follows

$$2\int_{s_0}^{s_0+1} x^2(t)dt \le \int_{s_0}^{s_0+1} (x')^2(t)dt \tag{2.17}$$

and

$$2\int_{s_0}^{s_0+1} (x^{(n-1)})^2(t)dt \le \int_{s_0}^{s_0+1} (x^{(n)})^2(t)dt. \tag{2.17$'$}$$

On the other hand by Lemma 2.8 we get

$$(2\pi)^2\int_{s_0}^{s_0+1} (x')^2(t)dt \le \int_{s_0}^{s_0+1} (x'')^2(t)dt, \tag{2.18}$$

$$\vdots$$

$$(2\pi)^2\int_{s_0}^{s_0+1} (x^{(n-2)})^2(t)dt \le \int_{s_0}^{s_0+1} (x^{(n-1)})^2(t)dt. \tag{2.19}$$

Conditions (2.17)–(2.19) yield

$$\int_{s_0}^{s_0+1} x^2(t)dt \le \frac{\alpha^2}{2^2(2\pi)^{2(n-2)}} \int_{s_0}^{s_0+1} x^2(t)dt. \tag{2.20}$$

Thus $x \equiv 0$ for $t \in [s_0, s_0+1]$, in contradiction with elementary properties of Green's function. Analogously, if $t_0 \in [0, s_0)$, we get a contradiction.

Finally, if $s_0 = 0$ or $s_0 = 1$, then $G_1(t, s_0)$ is a solution of (2.1) in $[0,1]$ such that

$$\frac{\partial^i G_1(0, s_0)}{\partial t^i} = \frac{\partial^i G_1(1, s_0)}{\partial t^i}, \quad i = 0, 1, \ldots, n-2,$$

and the same arguments as before lead to a contradiction. Similarly we conclude for $t_0 = 0$ or $t_0 = 1$.

Now we will consider case $\beta < 1$.

From conditions (2.14) we deduce that there exist points $t_1, \ldots, t_{n-1}$ such that $t_1, \ldots, t_{n-1} \in [s_0, s_0+1]$ and

$$x(t_0) = x'(t_1) = \ldots = x^{(n-1)}(t_{n-1}) = 0,$$

where x is defined by (2.13). Hence

$$\begin{aligned}
\sup_{t\in[s_0,s_0+1]} |x(t)| &= \sup_{t\in[s_0,s_0+1]} \Big| \int_{t_0}^{t} x'(s)ds \Big| \\
&\le \sup_{t\in[s_0,s_0+1]} |x'(t)| \le \ldots \le \sup_{t\in[s_0,s_0+1]} \big|x^{(n-1)}(t)\big| \\
&= \sup_{t\in[s_0,s_0+1]} \Big| \int_{t_{n-1}}^{t} x^{(n)}(s)ds \Big| \\
&= \sup_{t\in[s_0,s_0+1]} \Big| \int_{t_{n-1}}^{t} p(s)x(s)ds \Big| \\
&\le \sup_{t\in[s_0,s_0+1]} |x(t)| \int_{s_0}^{s_0+1} p(s)ds \\
&\le \sup_{t\in[s_0,s_0+1]} |x(t)| \int_0^1 p(s)ds = \beta \sup_{t\in[s_0,s_0+1]} |x(t)|,
\end{aligned} \tag{2.21}$$

which contradicts (2.4)$'$.

Thus G_1 has constant sign. Let us prove that this sign is negative. The unique 1–periodic solution of the equation

$$x^{(n)}(t) - p(t)x(t) = 1 \tag{2.22}$$

is just

$$x(t) = \int_0^1 G_1(t,s)ds. \tag{2.23}$$

On the other hand integrating (2.22) from 0 to 1 we find

$$-\int_0^1 p(t)x(t)dt = 1.$$

As by hypothesis $p(t) > 0$ (for all $t \in [0,1]$), $x(t) < 0$ for some t and as a consequence $G_1(t,s) < 0$ for all $(t,s) \in [0,1] \times [0,1]$. Proof of case 2° is similar to that of proof of case 1°. □

Remark 2.9. Let $L_n \colon F^n_{a,b} \to L^1[a,b]$ be operator defined by $L_n \equiv D^n + MI$, where $D = \frac{d}{dt}, I$ is the identity operator, M is a real constant different from zero and

$$F^n_{a,b} = \{u \in W^{n,1}[a,b] \colon u^{(i)}(a) = u^{(i)}(b), i = 0,\dots,n-2, u^{(n-2)}(a) \geq u^{(n-1)}(b)\}.$$

We say that L_n is inverse positive in $F^n_{a,b}$ if $L_n u \geq 0$ implies $u \geq 0$ for all $u \in F^n_{a,b}$ and L_n is inverse negative in $F^n_{a,b}$ if $L_n\, u \geq 0$ implies $u \leq 0$ for all $u \in F^n_{a,b}$.

In [4] the author obtained the following results. Let $c = \pi/(b-a)$.

(A) The operator L_2 is inverse positive in $F^2_{a,b}$ if and only if $M \in (0, c^2]$.
(B) The operator L_3 is inverse positive in $F^3_{a,b}$ if and only if $M \in (0, (2cM_3)^3]$, where $M_3 \approx 0,8832205$.
(C) The operator L_3 is inverse negative in $F^3_{a,b}$ if and only if $M \in [-(2cM_3)^3, 0)$.
(D) The operator L_4 is inverse negative in $F^4_{a,b}$ if and only if $M \in [-(2cM_4)^4, 0)$, where $M_4 \approx 0,7528094$.

EXAMPLE 2.10. If $p(t) \equiv k > 0$, then

$$\tilde{G_1}(t,s) = -\frac{1}{2k(e^k-1)}\begin{cases} e^{k(1-s+t)} + e^{k(s-t)}, & 0 \le t \le s \le 1, \\ e^{k(t-s)} + e^{k(1+s-t)}, & 0 \le s \le t \le 1, \end{cases}$$

is the Green function of the problem

$$x''(t) - k^2 x(t) = 0, \quad x(0) = x(1), \quad x'(0) = x'(1),$$

and $\tilde{G_1}(t,s) < 0$ for all $(t,s) \in [0,1] \times [0,1]$.

EXAMPLE 2.11. If $p(t) \equiv k > 0$ and $k \neq 2l\pi$ for all $l \in \mathbb{N}$, then

$$\tilde{G_2}(t,s) = \frac{1}{2k \sin k/2} \cos k\big[1/2 - |s-t|\big]$$

is the Green function of the problem

$$x''(t) + k^2 x(t) = 0, \quad x(0) = x(1), \quad x'(0) = x'(1).$$

If $k \in (0,\pi)$, then $\tilde{G_2}(t,s) > 0$ for all $(t,s) \in [0,1] \times [0,1]$.

EXAMPLE 2.12. We consider the problem

$$x^{(4)}(t) - k^4 x(t) = 0, \quad x^{(i)}(0) = x^{(i)}(1), \quad i = 0,1,2,3, \tag{2.24}$$

where $k > 0$ and $k \neq 2l\pi$ for $l \in \mathbb{N}$. The problem (2.24) has only the trivial solution. To see this let

$$x(t) = c_1 e^{kt} + c_2 e^{-kt} + c_3 \cos kt + c_4 \sin kt, \tag{2.25}$$

where c_1, c_2, c_3, c_4 are constants. From (2.24)–(2.25) we get a system of equations

$$\begin{cases} c_1(1-e^k) + c_2(1-e^{-k}) + c_3(1-\cos k) - c_4 \sin k = 0, \\ c_1(1-e^k) + c_2(e^{-k}-1) + c_3 \sin k + c_4(1-\cos k) = 0, \\ c_1(1-e^k) + c_2(1-e^{-k}) + c_3(\cos k - 1) + c_4 \sin k = 0, \\ c_1(1-e^k) + c_2(e^{-k}-1) - c_3 \sin k + c_4(\cos k - 1) = 0. \end{cases} \tag{2.26}$$

Let W denote the determinant of the matrix of system (2.26). Then

$$W = -16(1-e^k)(1-e^{-k})(1-\cos k) \neq 0. \tag{2.27}$$

It is not hard to verify that the Green function G_1^* of the problem (2.24) is given by the expression

$$G_1^*(t,s) = -\frac{1}{4k^3}\begin{cases} \frac{e^{k(t-s+1)}+e^{k(s-t)}}{e^k-1} + \frac{\cos k(s-t-\frac{1}{2})}{\sin k/2}, & 0 \le t \le s \le 1, \\ \frac{e^{k(t-s)}+e^{k(s-t+1)}}{e^k-1} + \frac{\cos k(s-t+\frac{1}{2})}{\sin k/2}, & 0 \le s \le t \le 1. \end{cases} \tag{2.28}$$

Now we shall introduce some notation. We denote

$$\overline{M}_i = \sup_{t,s\in[0,1]} |G_i(t,s)|, \quad \overline{m}_i = \inf_{t,s\in[0,1]} |G_i(t,s)|,$$

$$\overline{M}_{ij} = \sup_{t,s\in[0,1]} \left|\frac{\partial^j G_i(t,s)}{\partial t^j}\right|, \quad \overline{m}_{ij} = \inf_{t,s\in[0,1]} \left|\frac{\partial^j G_i(t,s)}{\partial t^j}\right|,$$

for $i = 1, 2$ and $j = 1, \dots, n-1$.

The properties of the Green functions G_i ($i = 1, 2$) needed later are described by the following lemmas.

LEMMA 2.13. *We assume that* $p\colon (-\infty,\infty) \to (0,\infty)$ *is continuous and* 1*–periodic and* p *has property* (2.3) *or* (g). *Let* $f\colon \mathbb{R}^{1+n} \to \mathbb{R}$ *be continuous. Then*

(i) $x \in C^n[a,b]$ *is a solution of the problem* (1.1) *if and only if* x *satisfies the integral equation*

$$x(t) = -\mu \int_0^1 G_1(t,s) f(s, x(s), x'(s), \dots, x^{(n-1)}(s)) ds; \tag{2.29}$$

(ii) $x \in C^n[a,b]$ *is a solution of the problem* (1.2) *if and only if* x *satisfies the equation*

$$x(t) = \mu \int_0^1 G_2(t,s) f(s, x(s), x'(s), \dots, x^{(n-1)}(s)) ds; \tag{2.30}$$

where G_1 *is the Green function of the problem* (2.1) *and* G_2 *is the Green function of the problem* (2.2).

Lemma 2.14. *Let all assumptions of Theorem 2.2 be satisfied. Then*

$$d_{0i}|G_i(t,s)| - \Big|\frac{\partial G_i(t,s)}{\partial t}\Big| - \ldots - \Big|\frac{\partial^{n-1} G_i(t,s)}{\partial t^{n-1}}\Big| \geq |G_i(s,s)| + \Big|\frac{\partial G_i(s,s)}{\partial t}\Big| + \ldots + \Big|\frac{\partial^{n-1} G_i(s-0,s)}{\partial t^{n-1}}\Big| \tag{2.31}$$

for $s,t \in [0,1]$ *and*

$$d_{0i}|G_i(t,s)| - \Big|\frac{\partial G_i(t,s)}{\partial t}\Big| - \ldots - \Big|\frac{\partial^{n-1} G_i(t,s)}{\partial t^{n-1}}\Big| \geq |G_i(s,s)| + \Big|\frac{\partial G_i(s,s)}{\partial t}\Big| + \ldots + \Big|\frac{\partial^{n-1} G_i(s+0,s)}{\partial t^{n-1}}\Big|$$

for $s,t \in [0,1]$, $i = 1,2$, *where* $\Big|\frac{\partial^{n-1} G_i(s-0,s)}{\partial t^{n-1}}\Big|$ $\Big(\Big|\frac{\partial^{n-1} G_i(s+0,s)}{\partial t^{n-1}}\Big|\Big)$ *denotes the left–hand (the right–hand) side derivative of order* $n-1$ *of* G_i *at the point* (s,s) *and*

$$d_{0i} \geq \frac{\overline{M}_i + 2\overline{M}_{i1} + \ldots + 2\overline{M}_{in-1}}{\overline{m}_i},$$

$$|G_i(s,s)| + \Big|\frac{\partial G_i(s,s)}{\partial t}\Big| + \ldots + \Big|\frac{\partial^{n-1} G_i(s-0,s)}{\partial t^{n-1}}\Big| \geq M_{0i}\Big(|G_i(t,s)| + \Big|\frac{\partial G_i(t,s)}{\partial t}\Big| + \ldots + \Big|\frac{\partial^{n-1} G_i(t,s)}{\partial t^{n-1}}\Big|\Big) \tag{2.32}$$

for $s,t \in [0,1], i = 1,2,$ *and*

$$M_{0i} \in \Big(0, \frac{\overline{m}_i + \overline{m}_{i_1} + \ldots + \overline{m}_{i_{n-1}}}{\overline{M}_i + \overline{M}_{i_1} + \ldots + \overline{M}_{i_{n-1}}}\Big),$$

$$|G_i(s,s)| + \Big|\frac{\partial G_i(s,s)}{\partial t}\Big| + \ldots + \Big|\frac{\partial^{n-1} G_i(s+0,s)}{\partial t^{n-1}}\Big| \geq M_{0i}\Big(|G_i(t,s)| + \Big|\frac{\partial G_i(t,s)}{\partial t}\Big| + \ldots + \Big|\frac{\partial^{n-1} G_i(t,s)}{\partial t^{n-1}}\Big|\Big).$$

Throughout the paper

$$\mathbb{R}_0^+ = [0,\infty), \quad \mathbb{R}_0^- = (-\infty,0], \quad \mathbb{R} = (-\infty,\infty),$$
$$D_0 = \mathbb{R}_0^+ \times \mathbb{R}^{n-1}, \quad D = \mathbb{R}^{n+1}, \tilde{D} = \mathbb{R} \times \mathbb{R}_0^- \times \mathbb{R}^{n-1},$$

$p\colon (-\infty,\infty) \to (0,\infty)$ is continuous and 1–periodic $L > 0, \mu > 0$,

$$\phi_i(t) = \mu L \int_0^1 |G_i(t,s)|ds \quad \text{for } t \in [0,1],$$
$$\overline{\phi}_i\colon (-\infty,\infty) \to (-\infty,\infty), \quad \overline{\phi}_i \in P_1^n(\mathbb{R}),$$

$\overline{\phi}_i(t) = \phi_i(t)$ for $t \in [0,1]$ and

$$m_i = \sup_{t\in[0,1]} \int_0^1 |G_i(t,s)|ds + \sup_{t\in[0,1]} \int_0^1 \Big|\frac{\partial G_i(t,s)}{\partial t}\Big|ds + \ldots + \sup_{t\in[0,1]} \int_0^1 \Big|\frac{\partial^{n-1} G_i(t,s)}{\partial t^{n-1}}\Big|ds \quad \text{for } i = 1,2. \tag{2.33}$$

3. Positive periodic solutions

In this section we present results on the existence of positive, 1–periodic solutions of equations (1.1) and (1.2).

THEOREM 3.1. *Assume condition* (2.4) *or* (2.4)$'$. *Let a continuous function* $f\colon \mathcal{D} \to (-\infty,\infty)$ *and a constant* $L > 0$ *be such that*

$$\begin{aligned} &f(t+1, v_0, v_1, \ldots, v_{n-1}) = f(t, v_0, v_1, \ldots, v_{n-1}), \\ &f(t, v_0, v_1, \ldots, v_{n-1}) + L \ge 0 \quad \text{for all } (t, v_0, v_1, \ldots, v_{n-1}) \in \mathcal{D}. \end{aligned} \tag{3.1}$$

Suppose that there exists a continuous nondecreasing function $\psi\colon [0,\infty) \to [0,\infty)$ *such that* $\psi(u) > 0$ *for* $u > 0$ *and*

$$f(t, v_0, v_1, \ldots, v_{n-1}) + L \le \psi(v_0 + |v_1| + \ldots + |v_{n-1}|) \quad \text{on } \mathcal{D}, \tag{3.2}$$

and that there exist $C_1 > 0$ *and* $r > 0$ *such that* $r \geq \mu L C_1 d_{01}$,

$$(3.3)\quad \int_0^1 |G_1(t,s)|ds \leq M_{01}C_1, \quad t \in [0,1], \quad \textit{and} \quad \frac{r}{\psi(r + \|\overline{\phi}_1\|_{n-1})} \geq \mu m_1,$$

where d_{01}, M_{01}, m_1 *have properties* (2.31)–(2.33). *Assume, additionally, that*

$$(3.4)\qquad f(t, v_0, v_1, \ldots, v_{n-1}) + L \geq \tau(t)g(v_0)$$

where $\tau\colon (-\infty, \infty) \to [0, \infty)$ *is continuous,* 1*-periodic, and* $g\colon [0,\infty) \to [0,\infty)$ *is continuous, nondecreasing, and* $g(u) > 0$ *for* $u > 0$. *Suppose that there exists* $R > 0$ *such that* $R > r$ *and*

$$(3.5)\quad d_{01}R \leq \int_0^1 \tau(s)\Big[d_{01}\Big|G_1\Big(\frac{1}{2}, s\Big)\Big| - \Big|\frac{\partial G_1(\frac{1}{2}, s)}{\partial t}\Big| - \ldots - \Big|\frac{\partial^{n-1} G_1(\frac{1}{2}, s)}{\partial t^{n-1}}\Big|\Big] g\Big(\frac{\varepsilon M_{01} R}{d_{01}}\Big) ds,$$

where $\varepsilon > 0$ *is any constant such that*

$$1 - \frac{\mu L C_1 d_{01}}{R} \geq \varepsilon.$$

Then (1.1) *has a positive solution* $x \in P_1^n(\mathbb{R})$.

Proof. The proof of Theorem 3.1 is similar to that of Theorem 2.1 in [1]. To show (1.1) has a positive 1–periodic solution we will look at

$$(3.6)\quad x(t) = -\mu \int_0^1 G_1(t,s) f_+^*(s, x(s) - \overline{\phi}_1(s), x'(s) - \overline{\phi}_1'(s), \ldots, x^{(n-1)}(s) - \overline{\phi}^{(n-1)}(s))ds,$$

where

$$f_+^*(t, v_0, \ldots, v_{n-1}) = \begin{cases} f(t, v_0, v_1, \ldots, v_{n-1}) + L, & \text{if } f(t, v_0, \ldots, v_{n-1}) \in \mathcal{D}_0, \\ f(t, 0, v_1, \ldots, v_{n-1}) + L, & \text{if } f(t, v_0, \ldots, v_{n-1}) \in \tilde{\mathcal{D}}. \end{cases}$$

We will show that there exists a solution x_1 to (3.6) with $x_1(t) \geq \overline{\phi}_1(t)$ for $t \in [0,1]$. If this is true, then $u(t) = x_1(t) - \phi_1(t)$ is a positive solution of (3.6), since for $t \in [0,1]$ we have

$$\begin{aligned} u(t) &= -\mu \int_0^1 G_1(t,s)[f_+^*(s, x(s) - \overline{\phi}_1(s), \\ &\qquad x'(s) - \overline{\phi_1}'(s), \ldots, x^{(n-1)}(s) - \overline{\phi_1}^{(n-1)}(s))ds + \mu L \int_0^1 G_1(t,s)ds \\ &= -\mu \int_0^1 G_1(t,s) f(s, u(s), u'(s), \ldots, u^{(n-1)}(s))ds. \end{aligned}$$

We concentrate our study on (3.6). Let $E = (P_1^{n-1}(\mathbb{R}), \|\cdot\|_{n-1})$ and

$$K_1 = \{u \in P_1^{n-1}(\mathbb{R})\colon \min_{t\in[0,1]}[d_{01}u(t) - |u'(t)| - \ldots - |u^{(n-1)}(t)] \geq M_{01}\|u\|_{n-1}\}.$$

Obviously K_1 is a cone of E. Let

$$\Omega_1 = \{u \in P_1^{n-1}(\mathbb{R})\colon \|u\|_{n-1} < r\} \tag{3.7}$$

and

$$\Omega_2 = \{u \in P_1^{n-1}(\mathbb{R})\colon \|u\|_{n-1} < R\}. \tag{3.8}$$

Now let $A_1\colon K_1 \cap (\overline{\Omega}_2|\Omega_1) \to P_1^{n-1}(\mathbb{R})$ be defined by $A_1\varphi = x_\varphi$, where $\varphi \in K_1 \cap (\overline{\Omega}_2|\Omega_1)$ and x_φ is the unique 1–periodic solution of the equation

$$x^{(n)}(t) - p(t)x(t) + \mu h(t, \varphi(t) - \overline{\phi}_1(t)) = 0, \tag{3.9}$$

where

$$h(t, \varphi(t) - \overline{\phi}_1(t)) = f_+^*(t, \varphi(t) - \overline{\phi}_1(t), \ldots, \varphi^{(n-1)}(t) - \overline{\phi}_1^{(n-1)}(t)).$$

First we show $A_1\colon K_1 \cap (\overline{\Omega}_2|\Omega_1) \to K_1$. If $\varphi \in K \cap (\overline{\Omega}_2|\Omega_1)$ and $t \in [0,1]$, then by Lemma 2.13 we have

$$(A_1\varphi)(t) = -\mu \int_0^1 G_1(t,s)h(s, \varphi(s) - \overline{\phi}_1(s))ds. \tag{3.10}$$

To shorten notation, we let $h(s,\varphi)$ stand for $h(s,\varphi(s)-\overline{\phi}_1(s))$. Relations (2.31)–(2.23) imply

$$\begin{aligned}
&d_{01}(A_1\varphi)(t)-|(A_1\varphi)'(t)|-\ldots-|(A_1\varphi)^{(n-1)}(t)| \\
&\quad=\mu d_{01}\int_0^1 -G_1(t,s)h(s,\varphi)ds-\mu\Big|\Big(\int_0^1 -G_1(t,s)h(s,\varphi)ds\Big)'\Big| \\
&\qquad-\ldots-\mu\Big|\Big(\int_0^1 -G_1(t,s)h(s,\varphi)ds\Big)^{(n-1)}\Big| \\
&\quad\geq\mu\int_0^t\Big[d_{01}|G_1(t,s)|-\Big|\frac{\partial G_1(t,s)}{\partial t}\Big|-\ldots-\Big|\frac{\partial^{n-1}G_1(t,s)}{\partial t^{n-1}}\Big|\Big]h(s,\varphi)ds \\
&\qquad+\mu\int_t^1\Big[d_{01}|G_1(t,s)|-\Big|\frac{\partial G_1(t,s)}{\partial t}\Big|-\ldots-\Big|\frac{\partial^{n-1}G_1(t,s)}{\partial t^{n-1}}\Big|\Big]h(s,\varphi)ds \\
&\quad\geq\mu\int_0^t\Big[|G_1(s,s)|+\Big|\frac{\partial G_1(s,s)}{\partial t}\Big|+\ldots+\Big|\frac{\partial^{n-1}G_1(s+0,s)}{\partial t^{n-1}}\Big|\Big]h(s,\varphi)ds \\
&\qquad+\mu\int_t^1\Big[|G_1(s,s)|+\Big|\frac{\partial G_1(s,s)}{\partial t}\Big|+\ldots+\Big|\frac{\partial^{n-1}G_1(s-0,s)}{\partial t^{n-1}}\Big|\Big]h(s,\varphi)ds \\
&\quad\geq\mu M_{01}\int_0^1\Big[|G_1(\bar t,s)|+\Big|\frac{\partial G_1(\bar t,s)}{\partial t}\Big|+\ldots+\Big|\frac{\partial^{n-1}G_1(\bar t,s)}{\partial t^{n-1}}\Big|\Big]h(s,\varphi)ds \\
&\qquad+\mu M_{01}\int_1^t\Big[|G_1(\bar t,s)|+\Big|\frac{\partial G_1(\bar t,s)}{\partial t}\Big|+\ldots+\Big|\frac{\partial^{n-1}G_1(\bar t,s)}{\partial t^{n-1}}\Big|\Big]h(s,\varphi)ds \\
&\quad\geq\mu M_{01}\int_0^1\Big[|G_1(\bar t,s)|+\Big|\frac{\partial G_1(\bar t,s)}{\partial t}\Big|+\ldots+\Big|\frac{\partial^{n-1}G_1(\bar t,s)}{\partial t^{n-1}}\Big|\Big]h(s,\varphi)ds \\
&\quad\geq M_{01}[(A_1\varphi)(\bar t)+|(A_1\varphi)'(\bar t)|+\ldots+|(A_1\varphi)^{(n-1)}(\bar t)|,
\end{aligned}$$

where $\bar t\in[0,1]$. Hence

$$\begin{aligned}
d_{01}(A_1\varphi)(t)&\geq d_{01}(A_1\varphi)(\bar t)-|(A_1\varphi)'(\bar t)|-\ldots-|(A_1\varphi)^{(n-1)}(\bar t)| \\
&\geq M_{01}\|A_1\varphi\|_{n-1}.
\end{aligned}\tag{3.11}$$

Consequently $A_1\varphi\in K_1$. So $A_1\colon K_1\cap(\overline{\Omega}_2|\Omega_1)\to K_1$.

We now show

$$\|A_1\varphi\|_{n-1} \le \|\varphi\|_{n-1} \quad \text{for } \varphi \in K_1 \cap \partial\Omega_1. \tag{3.12}$$

To see this let $\varphi \in K_1 \cap \partial\Omega_1$. Then $\|\varphi\|_{n-1} = r$ and $\varphi(t) \ge \frac{M_{01}r}{d_{01}}$ for $t \in \mathbb{R}$. From (3.2)–(3.3) we have

$$(A_1\varphi)(t) + |(A_1\varphi)'(t)| + \ldots + |(A_1\varphi)^{(n-1)}(t)| \le \mu\psi(r + \|\overline{\phi}_1\|_{n-1})m_1 \le r \le \|\varphi\|_{n-1}.$$

So (3.12) holds. Next we show

$$\|A_1\varphi\|_{n-1} \ge \|\varphi\|_{n-1} \quad \text{for } \varphi \in K_1 \cap \partial\Omega_2. \tag{3.13}$$

To see it let $\varphi \in K_1 \cap \partial\Omega_2$. Then $\|\varphi\|_{n-1} = R$ and $d_{01}\varphi(t) \ge RM_{01}$ for $t \in \mathbb{R}$. Let ε be as in (3.5). From (3.3) we have

$$\begin{aligned}\varphi(t) - \overline{\phi}_1(t) &= \varphi(t) - \mu L \int_0^1 (-G_1(t,s))ds \\ &\ge \varphi(t) - \frac{\mu L C_1 M_{01} R}{R} \ge \varphi(t)\Big(1 - \frac{\mu L C_1 d_{01}}{R}\Big) \\ &\ge \varepsilon\varphi(t) \ge \frac{\varepsilon R M_{01}}{d_{01}} > \frac{\varepsilon r M_{01}}{d_{01}} > 0\end{aligned}$$

(note $\varphi(t) - \overline{\phi}_1(t) > 0$ for $\varphi \in K_1 \cap (\overline{\Omega}_2 \setminus \Omega_1)$ and $t \in \mathbb{R}$). This together with (3.4)–(3.5) yields

$$\begin{aligned}d_{01}\|(A_1\varphi)\|_{n-1} &\ge d_{01}(A_1\varphi)\Big(\frac{1}{2}\Big) - \Big|(A_1\varphi)'\Big(\frac{1}{2}\Big)\Big| - \ldots - \Big|(A_1\varphi)^{(n-1)}\Big(\frac{1}{2}\Big)\Big| \\ &\ge \mu \int_0^1 \Big[d_{01}\Big|G_1\Big(\frac{1}{2},s\Big)\Big| - \Big|\frac{\partial G_1(\frac{1}{2},s)}{\partial t}\Big| \\ &\quad - \ldots - \Big|\frac{\partial^{n-1} G_1(\frac{1}{2},s)}{\partial t^{n-1}}\Big|\Big]\tau(s)g(\varphi(s) - \overline{\phi}_1(s))ds \\ &\ge \mu \int_0^1 \tau(s)\Big[d_{01}\Big|G_1\Big(\frac{1}{2},s\Big)\Big| - \Big|\frac{\partial G_1(\frac{1}{2},s)}{\partial t}\Big| \\ &\quad - \ldots - \Big|\frac{\partial^{n-1} G_1(\frac{1}{2},s)}{\partial t^{n-1}}\Big|\Big]g\Big(\frac{\varepsilon M_{01} R}{d_{01}}\Big)ds \ge d_{01}R.\end{aligned}$$

Hence we have (3.13). It is not difficult to observe that A_1 is continuous. By the Arzela–Ascoli Theorem we conclude that $A_1\colon K_1 \cap (\overline{\Omega}_2|\Omega_1) \to K_1$ is compact. Theorem 1.1 implies A_1 has a fixed point $x \in K_1 \cap (\overline{\Omega}_2|\Omega_1)$, i.e. $r \le \|x\|_{n-1} \le R$ and $x(t) \ge M_{01}r/d_{01}$, which completes the proof. □

Theorem 3.2. *Assume conditions* (3.1), (3.2), (3.4) *and* (2.4) *or* (2.4)′. *Suppose that there exist* $C_2 > 0$ *and* $r > 0$ *such that* $r \ge \mu L C_2 d_{02}$,

$$(3.14)\quad \int_0^1 G_2(t,s)ds \le C_2 M_{02}, \quad t \in [0,1], \quad \text{and} \quad r \ge \psi(r + \|\overline{\phi}_2\|_{n-1})\mu m_2,$$

where d_{02}, M_{02}, *and* m_2 *have properties* (2.31)–(2.33), *and that there exists* $R > 0$ *such that* $R > r$ *and*

$$(3.15)\quad d_{02}R \le \mu \int_0^1 \tau(s)\Big[d_{02}G_2\Big(\frac{1}{2},s\Big) - \Big|\frac{\partial G_2(\frac{1}{2},s)}{\partial t}\Big| - \ldots - \Big|\frac{\partial^{n-1}G_2(\frac{1}{2},s)}{\partial t^{n-1}}\Big|\Big] g\Big(\frac{\varepsilon M_{02}R}{d_{02}}\Big)ds,$$

where $\varepsilon > 0$ *is any constant such that*

$$1 - \frac{\mu L C_2 d_{02}}{R} \ge \varepsilon.$$

Then (1.2) *has a positive solution* $x \in P_1^n(\mathbb{R})$.

Proof. Let E, Ω_1, and Ω_2 be as in Theorem 3.1. Let

$$K_2 = \{u \in P_1^{n-1}(\mathbb{R})\colon \min_{t\in[0,1]}[d_{02}u(t) - |u'(t)| - \ldots - |u^{(n-1)}(t)|] \ge M_{02}\|u\|_{n-1}\}.$$

Then K_2 is a cone of E. Now, let $\varphi \in K_2 \cap (\overline{\Omega}_2|\Omega_1)$ and let x_φ be the unique 1–periodic solution of the problem

$$x^{(n)}(t) + p(t)x(t) = \mu f_+^*(t, \varphi(t) - \overline{\phi}_2(t), \varphi'(t) - \overline{\phi}_2'(t), \ldots, \varphi^{(n-1)}(t) - \overline{\phi}_2^{(n-1)}(t)),$$

where f_+^* is defined by (3.6). Finally let $A_2\colon K_2 \cap (\overline{\Omega}_2|\Omega_1) \to P_1^{n-1}(\mathbb{R})$ be defined by $A_2\varphi = x_\varphi$. It is not difficult to prove that $A_2\colon K_2 \cap (\overline{\Omega}_2|\Omega_1) \to K_2$, A_2 is continuous and compact. Similar arguments as in Theorem 3.1 guarantee that

$$\|A_2\varphi\|_{n-1} \le \|\varphi\|_{n-1} \quad \text{for } \varphi \in K_2 \cap \partial\Omega_1$$

and

$$\|A_2\varphi\|_{n-1} \ge \|\varphi\|_{n-1} \quad \text{for } \varphi \in K_2 \cap \partial\Omega_2.$$

Theorem 1.1 implies that A_2 has a fixed point $x \in K_2 \cap (\overline{\Omega}_2|\Omega_1)$, i.e. $x(t) \ge M_{02}r/d_{02}$ for $t \in \mathbb{R}$, which completes the proof. □

EXAMPLE 3.3. We consider the problem

$$x^{(4)}(t) - x(t) + \mu|\sin\pi t|[(x(t) + |x'(t)| + |x''(t)| + |x^{(3)}(t)|)^2 - 1] = 0, \tag{3.16}$$

$x^{(i)}(0) = x^{(i)}(1), i = 0, 1, 2, 3.$

It is not difficult to verify that the problem (3.16) has a solution $x \in P_1^4(\mathbb{R})$ (for sufficiently small μ) such that $x(t) > 0$ for $t \in \mathbb{R}$. To see this we apply Theorem 3.1 with $p(t) \equiv 1, L = 1,$ $\tau(t) = |\sin\pi t|$, $d_{01} = 26$, $M_{01} = 0,07$, $\mu = 0,004$, $g(u) = u^2 = \psi(u)$, $\overline{\phi}_1 = \frac{1}{2}\mu$, $C_1 = 8$, $r = 1$, $\overline{\alpha}_4 = 1$ with sufficiently large R $(R > 1)$.

COROLLARY 3.4. *Assume condition* (2.4) *or* (2.4)$'$. *Let*

$$f\colon \mathcal{D} \to [0,\infty) \quad \textit{be continuous} \tag{3.17}$$

and such that

$$f(t+1, v_0, v_1, \dots, v_{n-1}) = f(t, v_0, v_1, \dots, v_{n-1}) \tag{3.18}$$

for all $(t, v_0, v_1, \dots, v_{n-1}) \in \mathcal{D}$. *Suppose that there exists a continuous non-decreasing function* $\psi\colon [0,\infty) \to [0,\infty)$ *such that* $\psi(u) > 0$ *for* $u > 0$ *and*

$$f(t, v_0, v_1, \dots, v_{n-1}) \le \psi(v_0 + |v_1| + \dots + |v_{n-1}|) \quad \textit{on } \mathcal{D}, \tag{3.19}$$

and that there exists r *such that*

$$r \ge \psi(r)\mu m_1. \tag{3.20}$$

Assume, additionally, that there exist functions τ *and* g *such that*

$$f(t, v_0, v_1, \dots, v_{n-1}) \ge \tau(t)g(v_0) \quad \textit{for all } (t, v_0, v_1, \dots, v_{n-1}) \in \mathcal{D}, \tag{3.21}$$

where $g\colon [0,\infty) \to [0,\infty)$ *is continuous, nondecreasing, and* $g(u) > 0$ *for* $u > 0$, *and* $\tau\colon (-\infty,\infty) \to [0,\infty)$ *is continuous and* 1*–periodic, and that there exists* $R > 0$ *such that* $R > r$ *and*

$$(3.22)\quad d_{01}R \le \mu \int\limits_0^1 \tau(s)\Big[d_{01}\Big|G_1\Big(\frac{1}{2},s\Big)\Big| - \Big|\frac{\partial G_1(\frac{1}{2},s)}{\partial t}\Big| - \ldots - \Big|\frac{\partial^{n-1} G_1(\frac{1}{2},s)}{\partial t^{n-1}}\Big|\Big] g\Big(\frac{M_{01}R}{d_{01}}\Big)ds.$$

Then (2.1) *has a positive solution* $x \in P_1^n(\mathbb{R})$.

Corollary 3.5. *Assume conditions* (3.17)–(3.19), (3.21) *and* (2.4) *or* (2.4)′. *Suppose that there exists* $r > 0$ *such that*

$$(3.23)\qquad r \ge \psi(r)\mu m_2$$

and that there exists $R > 0$ *such that* $R > r$ *and*

$$(3.24)\quad d_{02} \le \mu \int\limits_0^1 \tau(s)\Big[d_{02}\Big|G_2\Big(\frac{1}{2},s\Big)\Big| - \Big|\frac{\partial G_2(\frac{1}{2},s)}{\partial t}\Big| - \ldots - \Big|\frac{\partial^{n-1} G_2(\frac{1}{2},s)}{\partial t^{n-1}}\Big|\Big] g\Big(\frac{M_{02}R}{d_{02}}\Big)ds.$$

Then (2.2) *has a positive solution* $x \in P_1^u(\mathbb{R})$.

Proof of Corollary 3.4. The proof is similar to that of Theorem 3.1. Let E, Ω_1, Ω_2, and K_1 be as in Theorem 3.1. Now let $\varphi \in K_1 \cap (\overline{\Omega}_2|\Omega_1)$ and let x_φ be the unique 1–periodic solution of the equation

$$x^{(n)}(t) - p(t)x(t) + \mu f(t,\varphi(t),\varphi'(t),\ldots,\varphi^{(n-1)}(t)) = 0$$

and let $A_3\colon K_1 \cap (\overline{\Omega}_2|\Omega_1) \to P_1^{n-1}(\mathbb{R})$ be defined by $A_3\varphi = x_\varphi$. It is easy to check that $A_3\colon K_1 \cap (\overline{\Omega}_2|\Omega_1) \to K_1, A_3$ is continuous and compact, $\|A_3\varphi\|_{n-1} \le \|\varphi\|_{n-1}$ for $\varphi \in K_1 \cap \partial\Omega_1$ and $\|A_3\varphi\|_{n-1} \ge \|\varphi\|_{n-1}$ for $\varphi \in K_1 \cap \partial\Omega_2$. Applying Theorem 1.1 we can show that equation (2.1) has a positive solution $x \in P_1^n(\mathbb{R})$. □

References

[1] Agarwal R.P., Grace S.R., O'Regan D., *Existence of positive solutions of semipositone Fredholm integral equations*, Funkcial. Ekvac. **45** (2002), 223–235.

[2] Agarwal R.P., O'Regan D., Wang P.J.Y., *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer Academic Publishers, Dordrecht, 1999.

[3] Agarwal R.P., O'Regan D., *Infinite Interval Problems for Differential, Difference and Integral Equations*, Kluwer Academic Publishers, Dordrecht, 2001.

[4] Cabada A., *The method of lower and upper solutions for second, third, fourth and higher order boundary value problems*, J. Math. Anal. Appl. **185** (1994), 302–320.

[5] Deimling K., *Nonlinear Functional Analysis*, Springer, New York, 1985.

[6] Guo D., Laksmikannthan V., *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, 1988.

[7] Lasota A., Opial Z., *Sur les solutions périodiques des équations différentielles ordinaires*, Ann. Polon. Math. **16** (1964), 69–94.

[8] Rektorys K., *Variational Methods in Mathematics, Science and Engineering*, D. Reidel Publishing Co., Dordrecht, 1980.

[9] Šeda V., Nieto J.J., Gera M., *Periodic boundary value problems for nonlinear higher order ordinary differential equations*, Appl. Math. Comput. **48** (1992), 71–82.

[10] Torres P.J., *Existence of one–signed periodic solution of some second–order differential equations via a Krasnosielskii fixed point theorem*, J. Differential Equations **190** (2003), 643–662.

[11] Zima M., *Positive Operators in Banach Spaces and Their Applications*, Wydawnictwo Uniwersytetu Rzeszowskiego, Rzeszów, 2005.

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