Annales Mathematicae Silesianae **23** (2009), 83–101
Prace Naukowe Uniwersytetu Śląskiego nr 2769, Katowice

THE GROUP OF BALANCED AUTOMORPHISMS OF A SPHERICALLY HOMOGENEOUS ROOTED TREE

Adam Woryna

Abstract. Let X^* be a tree of words over the changing alphabet $(X_0, X_1, \ldots)$ with $X_i = \{0, 1, \ldots, m_i - 1\}$, $m_i > 1$. We consider the group $Aut(X^*)$ of automorphisms of a tree X^*. A cyclic automorphism of X^* is called constant if its root permutations at any two words from the same level of X^* coincide. In this paper we introduce the notion of a *balanced* automorphism which is obtained from a constant automorphism by changing root permutations at all words ending with an odd letter for their inverses. We show that the set of all balanced automorphisms forms a subgroup of $Aut(X^*)$ if and only if $2 \nmid m_i$ implies $m_{i+1} = 2$ for $i = 0, 1, \ldots$. We study, depending on a branch index of a tree, the algebraic properties of this subgroup.

1. Introduction

Nowadays, the groups of automorphisms of a spherically, homogeneous rooted tree are the subject of intensive investigations. In the majority of cases these works are concentrated on groups of automorphisms of a regular tree (see [2, 4, 5] for example) with their self-similar, branch and other exotic subgroups with recursive properties ([1, 3, 4]). These constructions are usually based on several important types of automorphisms like rooted automorphisms, directed automorphisms or automorphisms defined by finite state automata (see [3] for example).

Received: 10.03.2009. Revised: 26.09.2009.
(2010) Mathematics Subject Classification: 20E22, 20E08.
Key words and phrases: tree of words, rooted tree, automorphism of a rooted tree, group of automorphisms of a rooted tree.

In this paper we introduce a new type of automorphisms of an arbitrary spherically, homogeneous rooted tree X^*, which we call as balanced automorphisms. The original approach suggested in the paper does not use the language of wreath products of groups or a wreath recursion, which is common in describing groups of automorphisms of a spherically homogeneous rooted tree. Nevertheless, ideas presented in the paper allows to study the structure based on balanced automorphisms effectively. Among others, we provide general condition on a branch index $\overline{m}$ of the tree X^* which yields the set $\mathcal{B}_{\overline{m}}$ of all balanced automorphisms of X^* forms a subgroup of $Aut(X^*)$. We describe, depending on $\overline{m}$, the algebraic properties of the group $\mathcal{B}_{\overline{m}}$. In particular, this construction provides a new concrete realization of an uncountable family of uncountable metabelian groups.

The structure of the paper is the following. Section 2 recalls only necessary definitions concerning automorphisms of a spherically, homogeneous rooted tree, which will be used in our considerations. We define the notions of a tree of words over the changing alphabet, a branch index and an automorphism of such a tree, sections, root permutations and a portrait of an automorphism. We recall some formulas useful in computations over automorphisms of a tree.

In Section 3 we provide the definition of a balanced automorphism as well as we define the auxiliary group $\mathcal{Z}^{\pm}$ useful in the computation over balanced automorphisms. In Theorem 3.1 we characterize all branch indexes $\overline{m}$ which yield the set $\mathcal{B}_{\overline{m}}$ is a group and we present $\mathcal{B}_{\overline{m}}$ as a quotient of $\mathcal{Z}^{\pm}$. We describe the case in which $\mathcal{B}_{\overline{m}}$ is an infinite cartesian product of finite dihedral groups (Corollary 3.2). In particular, for suitable $\overline{m}$ the group $\mathcal{B}_{\overline{m}}$ has the universal embedding property for finite dihedral groups.

The last section contains our main Theorem 4.1 describing, depending on $\overline{m}$, the algebraic properties of $\mathcal{B}_{\overline{m}}$. For instance we characterize the lower and the upper central series of $\mathcal{B}_{\overline{m}}$. We show that $\mathcal{B}_{\overline{m}}$ is metabelian for each $\overline{m}$. Moreover, $\mathcal{B}_{\overline{m}}$ is either of finite exponent or contains a free abelian group of uncountable rank. We prove that $\mathcal{B}_{\overline{m}}$ is a product of its abelian subgroups but, in general, it is not a product of its abelian subgroups one of which is normal.

In the text we denote by $(n)_m$ the rest from dividing of n by m. By $\equiv_m$ we denote the congruence relation modulo m.

2. Tree of words and its automorphisms

An infinite, spherically homogeneous one-rooted tree of finite valency may be defined as a tree of words over the so-called changing alphabet, namely over the infinite sequence

$$X = (X_i)_{i\in\mathbb{N}_0}$$

of finite, nonempty sets X_i (*sets of letters*) indexed by the set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ of nonnegative integers. A *word* over X is an empty (denoted by ε) or a finite sequence of letters $x_0x_1 \ldots x_n$, where $x_i \in X_i$ for $i = 0, 1, \ldots, n$, $n \in \mathbb{N}_0$. Then the set X^* of all words over X has the structure of a spherically homogeneous rooted tree which we denote also by X^*. Namely, the root of X^* is the empty word ε and the children of any $w \in X^*$ constitute words of the form wx, where $x \in X_n$ and $n = |w|$ denotes the length of w. The set X^n of all words of a given length $n \in \mathbb{N}_0$ forms the *n-th level* of a tree X^*. In particular, the number of children of any vertex from the n-th level is equal to $m_n = |X_n|$. The sequence $\overline{m} = (m_i)_{i\in\mathbb{N}_0}$ is called *the branch index* of the tree X^*.

We write $Aut(X^*)$ for the group of automorphisms of the tree X^*. This is the set of bijections

$$g\colon X^* \to X^*, \quad w \mapsto w^g,$$

that fix the root vertex ε and preserve the vertex-adjacency. For $g \in Aut(X^*)$ we have $|w^g| = |w|$ and, if v is a prefix of w, then v^g is a prefix of w^g.

From an arbitrary changing alphabet $X = (X_i)_{i\in\mathbb{N}_0}$ we may build for any $n \in \mathbb{N}_0$ the changing alphabet $X_{(n)}$, where

$$X_{(n)} = (X_{n+i})_{i\in\mathbb{N}_0}.$$

By $X^*_{(n)}$ we denote a tree of words over $X_{(n)}$ and by $X^m_{(n)}$ - the m-th level of $X^*_{(n)}$, $m \in \mathbb{N}_0$. Let $g \in Aut(X^*)$ and $n \in \mathbb{N}_0$. For any $w \in X^n$ the mapping

$$g|_w\colon X^*_{(n)} \to X^*_{(n)}, \quad u \mapsto u^{g|_w},$$

defined by the equality

$$(wu)^g = w^g u^{g|_w}$$

is called a *section* of g at w or simply a w*-section* of g. It is worth to see that $g|_w \in Aut(X^*_{(n)})$ and $g|_w|_v = g|_{wv}$ for any $v \in X^*_{(n)}$. Let $\pi_{g,w}$ be the restriction of $g|_w$ to the set of one letter words. We may treat $\pi_{g,w}$ as an element of the symmetric group $S(X_n)$ of the set X_n

$$\pi_{g,w}\colon X_n \to X_n, \quad \pi_{g,w}(x) = x^{g|_w}.$$

The permutation $\pi_{g,w}$ is called a *root permutation* of g at w. Let us put the set X^* in a lexicographical order

$$X^* = \{w_0 = \varepsilon, w_1, w_2, \ldots\}.$$

The sequence $(\pi_{g,w_i})_{i\in\mathbb{N}_0}$ of root permutations of $g \in Aut(X^*)$ is called a *portrait* of g. The automorphism g is characterized by its portrait. Namely, if $w = x_0x_1\ldots x_n$ then

$$w^g = \pi_{g,\varepsilon}(x_0)\pi_{g,x_0}(x_1)\ldots\pi_{g,x_0\ldots x_{n-1}}(x_n).$$

In reverse, any sequence $(\pi_{w_i})_{i\in\mathbb{N}_0} \in S(X_{|w_0|}) \times S(X_{|w_1|}) \times \ldots$ constitutes a portrait of a unique automorphism $g \in Aut(X^*)$.

If $g, g' \in Aut(X^*)$ and $w \in X^*$ then one may verify (see for example [3]) the following equalities

$$gg'|_w = g|_w g'|_{w^g}, \quad g^{-1}|_w = \left(g|_{w^{g^{-1}}}\right)^{-1}. \tag{1}$$

In particular for the root permutation of the product gg' and the root permutation of the inverse g^{-1} at w we obtain formulas

$$\pi_{gg',w} = \pi_{g,w}\pi_{g',w^g}, \quad \pi_{g^{-1},w} = \left(\pi_{g,w^{g^{-1}}}\right)^{-1}. \tag{2}$$

3. The group $\mathcal{B}_{\overline{m}}$ of balanced automorphisms

From now we will consider the changing alphabet $X = (X_i)_{i\in\mathbb{N}_0}$ in which

$$X_i = \{0, 1, \ldots, m_i - 1\}, \tag{3}$$

where $m_i > 1$ for every $i \in \mathbb{N}_0$. Further we will consider the so called cyclic automorphisms of a tree X^*. An automorphism $g \in Aut(X^*)$ is called *cyclic* if for every $i \in \mathbb{N}_0$ all its root permutations at words from the i-th level of X^* are powers of the cycle

$$\sigma_i = (0, 1, \ldots, m_i - 1).$$

By formulas (2) we see that the product of cyclic automorphisms as well as an inverse to a cyclic automorphism is cyclic. Thus the set $CAut(X^*)$ of cyclic automorphisms is a subgroup of the group $Aut(X^*)$. Obviously, this subgroup is proper and uncountable. In the subgroup $CAut(X^*)$ we consider the set of *constant* automorphisms; namely $g \in CAut(X^*)$ is constant if $\pi_{g,w} = \pi_{g,w'}$ for any w, w' from the same level of X^*. Thus for every $i \in \mathbb{N}_0$ the restriction of the portrait of any constant automorphism g to the i-th level of X^* is uniquely defined by a number $\alpha_i \in \{0, 1, \ldots, m_i - 1\}$ for which we have $\pi_{g,w} = \sigma_i^{\alpha_i}$ for every $w \in X^i$. One easily checks that the set of constant automorphisms is a proper, uncountable subgroup of $CAut(X^*)$ isomorphic, via $g \mapsto (\alpha_0, \alpha_1, \alpha_2, \ldots)$, to the infinite cartesian product $\prod_{i \in \mathbb{N}_0} \mathbb{Z}_{m_i}$ of cyclic groups.

The main idea of this paper is based on the following generalization of the above concept of a constant automorphism.

Definition 3.1. An automorphism $g \in CAut(X^*)$ is called *balanced* if the root permutations of g at any words w, w' from the same level of X^* depend merely on the parity of last letters of w and w' in the following way: if the last letters of w and w' are of the same parity then $\pi_{g,w'} = \pi_{g,w}$ and if these letters are of different parity then $\pi_{g,w'} = \pi_{g,w}^{-1}$.

In other ways every balanced automorphism may be obtained from the corresponding constant automorphism by replacing all its root permutations at words ending with an odd letter with their inverses.

The set of all balanced automorphisms of a tree X^* is defined uniquely by the branch index $\overline{m} = (m_i)_{i \in \mathbb{N}_0}$ of X^*. We denote this set by $\mathcal{B}_{\overline{m}}$.

Let $\mathbb{Z}^{\mathbb{N}_0}$ be the set of all mappings $\alpha \colon \mathbb{N}_0 \to \mathbb{Z}$. According to the above definition $g \in \mathcal{B}_{\overline{m}}$ if and only if there exists $\alpha \in \mathbb{Z}^{\mathbb{N}_0}$ such that the root permutation of g at any $w \in X^i$ ($i \in \mathbb{N}_0$) is equal to

$$\pi_{g,w} = \begin{cases} \sigma_0^{\alpha(0)}, & \text{if } i = 0, \\ \sigma_i^{\alpha(i)\cdot(-1)^x}, & \text{if } i > 0, \end{cases}$$

where $x \in X_{i-1}$ is the last letter of w. We denote by g_α the balanced automorphism defined by $\alpha \in \mathbb{Z}^{\mathbb{N}_0}$.

For any $\alpha, \beta \in \mathbb{Z}^{\mathbb{N}_0}$ we define a product $\alpha\beta \in \mathbb{Z}^{\mathbb{N}_0}$ as follows

$$(\alpha\beta)(i) = \alpha(i) + \beta(i) \cdot (-1)^{\alpha(i-1)}, \quad i \in \mathbb{N}_0. \tag{4}$$

In the above formula we assume $\alpha(-1) = 0$.

Proposition 3.1. *The set $\mathbb{Z}^{\mathbb{N}_0}$ with the product* (4) *forms a group, which we denote by $\mathcal{Z}^{\pm}$. The zero-mapping $\theta(i) \equiv 0$ is a neutral element in $\mathcal{Z}^{\pm}$ and for the inverse to α we have*

$$\alpha^{-1}(i) = -\alpha(i) \cdot (-1)^{\alpha(i-1)}, \quad i \in \mathbb{N}_0.$$

Proof. We directly verify that $\alpha\theta = \theta\alpha = \alpha$ and $\alpha^{-1}\alpha = \theta$ for any $\alpha \in \mathbb{Z}^{\mathbb{N}_0}$. Moreover, for any $\alpha, \beta, \gamma \in \mathbb{Z}^{\mathbb{N}_0}$ and any $i \in \mathbb{N}_0$ we have

$$\begin{aligned}((\alpha\beta)\gamma)(i) &= (\alpha(\beta\gamma))(i)\\ &= \alpha(i) + (-1)^{\alpha(i-1)} \cdot \beta(i) + (-1)^{\alpha(i-1)+\beta(i-1)} \cdot \gamma(i).\end{aligned}$$

Thus $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ and $\mathbb{Z}^{\mathbb{N}_0}$ forms a group with the product (4). □

Theorem 3.1. *The set $\mathcal{B}_{\overline{m}}$ of all balanced automorphisms forms a subgroup in $Aut(X^*)$ if and only if the branch index $\overline{m}$ satisfies*

$$2 \nmid m_{i-1} \;\Rightarrow\; m_i = 2 \quad \textit{for } i > 0. \tag{5}$$

In case of (5) *the following equalities hold for any $\alpha, \beta \in \mathcal{Z}^{\pm}$*

$$g_\alpha g_\beta = g_{\alpha\beta}, \quad (g_\alpha)^{-1} = g_{\alpha^{-1}}. \tag{6}$$

Proof. Suppose that $\mathcal{B}_{\overline{m}}$ is a group and $2 \nmid m_{i-1}$ for some fixed $i > 0$. Let $\alpha, \beta \in \mathcal{Z}^{\pm}$ be such that $\alpha(i-1) = 1$, $\alpha(i) = 0$, $\beta(i) = 1$. Let $g = g_\alpha$ and $g' = g_\beta$. Let $w \in X^i$ and let $x \in X_{i-1}$ be the last letter of w. Then the root permutations of g and g' at w are equal to

$$\begin{aligned}\pi_{g,w} &= \sigma_i^{\alpha(i)\cdot(-1)^x} = Id_{X_i},\\ \pi_{g',w} &= \sigma_i^{\beta(i)\cdot(-1)^x} = \sigma_i^{(-1)^x}.\end{aligned}$$

Thus from the equality $\pi_{gg',w} = \pi_{g,w}\pi_{g',w^g}$ we obtain

$$\pi_{gg',w} = \sigma_i^{(-1)^z},$$

where z is the last letter of w^g. But $w = vx$ for some $v \in X^{i-1}$ and thus $w^g = v^g x^{g|_v} = v^g \pi_{g,v}(x)$. Hence $z = \pi_{g,v}(x)$. For $\pi_{g,v}$ we have two possibilities

$$\pi_{g,v} = \sigma_{i-1}^{\alpha(i-1)} = \sigma_{i-1} \quad \text{or} \quad \pi_{g,v} = \sigma_{i-1}^{-\alpha(i-1)} = \sigma_{i-1}^{-1}.$$

In consequence $z = (x+1)_{m_{i-1}}$ or $z = (x-1)_{m_{i-1}}$. Thus

$$\pi_{gg',w} = \sigma_i^{(-1)^{(x+1)_{m_{i-1}}}} \quad \text{or} \quad \pi_{gg',w} = \sigma_i^{(-1)^{(x-1)_{m_{i-1}}}}.$$

Since $\mathcal{B}_{\overline{m}}$ is a group there is $\gamma \in \mathcal{Z}^{\pm}$ defining gg'. Hence $\pi_{gg',w} = \sigma_i^{\gamma(i)\cdot(-1)^x}$. Since w was chosen arbitrarily and the order of $\sigma_i \in S(X_i)$ is m_i we obtain

$$\gamma(i) \cdot (-1)^x \equiv_{m_i} (-1)^{(x+1)m_{i-1}} \quad \text{for all } x \in X_{i-1}$$

or

$$\gamma(i) \cdot (-1)^x \equiv_{m_i} (-1)^{(x-1)m_{i-1}} \quad \text{for all } x \in X_{i-1}.$$

In the first case by substitution $x = 0$ and $x = m_{i-1} - 1$ we have respectively $\gamma(i) \equiv_{m_i} -1$ and $\gamma(i) \equiv_{m_i} 1$. In the second case by substitution $x = 0$ and $x = m_{i-1} - 1$ we have respectively $\gamma(i) \equiv_{m_i} 1$ and $\gamma(i) \equiv_{m_i} -1$. Thus in each case we obtain $0 \equiv_{m_i} 2$. In consequence $m_i = 2$.

Now, we show that the condition (5) implies the equalities (6). In consequence we obtain that $\mathcal{B}_{\overline{m}}$ is a group. So, let $\alpha, \beta \in \mathcal{Z}^{\pm}$ and let $g = g_\alpha$, $g' = g_\beta$, $g'' = g_\gamma$, where $\gamma = \alpha\beta$. Let w be any word of the length $i > 0$ and let x be the last letter of w. Then the root permutations of g, g' and g'' at w are equal to

$$\pi_{g,w} = \sigma_i^{\alpha(i)\cdot(-1)^x}, \quad \pi_{g',w} = \sigma_i^{\beta(i)\cdot(-1)^x}, \quad \pi_{g'',w} = \sigma_i^{\gamma(i)\cdot(-1)^x},$$

where $\gamma(i) = (\alpha\beta)(i) = \alpha(i) + \beta(i) \cdot (-1)^{\alpha(i-1)}$. As before, we compute the root permutation of gg' at w

$$\pi_{gg',w} = \pi_{g,w}\pi_{g',w^g} = \sigma_i^{\alpha(i)\cdot(-1)^x} \sigma_i^{\beta(i)\cdot(-1)^z} = \sigma_i^{\alpha(i)\cdot(-1)^x + \beta(i)\cdot(-1)^z},$$

where z is the last letter of the word w^g. As before, we obtain $z = \sigma_{i-1}^{\pm\alpha(i-1)}(x)$. Hence, if m_{i-1} is even we have $z \equiv_2 x + \alpha(i-1)$. In consequence in this case

$$\pi_{gg',w} = \sigma_i^{\alpha(i)\cdot(-1)^x + \beta(i)\cdot(-1)^z} = \sigma_i^{\gamma(i)\cdot(-1)^x} = \pi_{g'',w}.$$

If m_{i-1} is odd then $m_i = 2$ and $\sigma_i = (0, 1)$. From the congruence

$$\alpha(i) \cdot (-1)^x + \beta(i) \cdot (-1)^z \equiv_2 \gamma(i) \cdot (-1)^x,$$

we have in this case $\pi_{gg',w} = \sigma_i^{\gamma(i)\cdot(-1)^x} = \pi_{g'',w}$. The root permutations of g'' and gg' at the empty word coincide and are equal to $\sigma_0^{\alpha(0)+\beta(0)}$. Thus the portraits of g'' and gg' coincide and $g'' = gg'$. Now, since $g_\theta = Id_{X^*}$ we have $g_\alpha g_{\alpha^{-1}} = g_{\alpha\alpha^{-1}} = g_\theta = Id_{X^*}$. $\square$

Since m_i is the order of the cycle $\sigma_i \in S(X_i)$ we obtain for any $\alpha, \beta \in \mathcal{Z}^{\pm}$

$$g_\alpha = g_\beta \quad \text{iff} \quad \alpha(i) \equiv_{m_i} \beta(i) \quad \text{for } i \in \mathbb{N}_0.$$

In particular,

$$g_\alpha = Id_{X^*} \quad \text{iff} \quad \alpha(i) \equiv_{m_i} 0 \quad \text{for } i \in \mathbb{N}_0.$$

The following statement is a direct consequence of Theorem 3.1 and the above observation.

Corollary 3.1. *If the branch index $\overline{m} = (m_i)_{i\in\mathbb{N}_0}$ satisfies* (5) *then the function*

$$\psi\colon \mathcal{Z}^{\pm} \to \mathcal{B}_{\overline{m}}, \quad \psi(\alpha) = g_\alpha$$

is a group epimorphism with the kernel

$$\ker(\psi) = \{\alpha \in \mathcal{Z}^{\pm} \colon \alpha(i) \equiv_{m_i} 0 \textit{ for } i \in \mathbb{N}_0\}.$$

If G_i $(i \in \mathbb{N}_0)$ is a group then by $\prod_i G_i$ we will denote the infinite cartesian product of G_i's

$$\prod_i G_i = G_0 \times G_1 \times G_2 \times \ldots.$$

Corollary 3.2. *If $\overline{m} = (2, m_1, 2, m_3, 2, m_5, \ldots)$ then $\mathcal{B}_{\overline{m}}$ is isomorphic to the product $\prod_i D_{2n_i}$ of finite dihedral groups D_{2n_i}, where $n_i = m_{2i+1}$ for $i \in \mathbb{N}_0$. In particular, for the branch index of the form $\overline{m} = (2, 3, 2, 4, 2, 5, \ldots)$ the group $\mathcal{B}_{\overline{m}}$ has the universal embedding property for finite dihedral groups.*

Proof. The finite dihedral group D_{2n} is a group of all symmetries of a regular polygon with n sides. It is known that D_{2n} is isomorphic to the semi-direct product $\mathbb{Z}_n \rtimes \mathbb{Z}_2$ with the action of $\mathbb{Z}_2$ by inverting elements. For any $\alpha \in \mathcal{Z}^{\pm}$ let us denote $\Lambda_\alpha = (\Lambda_\alpha(i))_{i\in\mathbb{N}_0}$, where

$$\Lambda_\alpha(i) = ((\alpha(2i+1))_{n_i}, (\alpha(2i))_2), \quad i \in \mathbb{N}_0.$$

We have $\Lambda_\alpha(i) \in D_{2n_i}$ and $\Lambda_\alpha \in \prod_i D_{2n_i}$. Moreover, $g_\alpha = g_\beta$ iff $\Lambda_\alpha = \Lambda_\beta$. Thus

$$\phi\colon \mathcal{B}_{\overline{m}} \to \prod_i D_{2n_i}, \quad \phi(g_\alpha) = \Lambda_\alpha$$

is a well defined, one-to-one mapping. Obviously, ϕ is onto. Moreover, for any $\alpha, \beta \in \mathcal{Z}^{\pm}$ and any $i \in \mathbb{N}_0$ we compute

$$\begin{aligned}\Lambda_\alpha(i)\Lambda_\beta(i) &= ((\alpha(2i+1))_{n_i}, (\alpha(2i))_2) \cdot ((\beta(2i+1))_{n_i}, (\beta(2i))_2)\\ &= ((\alpha(2i+1) + (-1)^{\alpha(2i)} \cdot \beta(2i+1))_{n_i}, (\alpha(2i) + \beta(2i))_2)\\ &= ((\alpha\beta(2i+1))_{n_i}, (\alpha\beta(2i))_2) = \Lambda_{\alpha\beta}(i).\end{aligned}$$

In consequence $\Lambda_\alpha\Lambda_\beta = \Lambda_{\alpha\beta}$ and ϕ is an isomorphism by Theorem 3.1. $\square$

4. Algebraic properties of $\mathcal{B}_{\overline{m}}$

The lower central series of any group G we define in a standard way $\Gamma_0(G) = G$, $\Gamma_{s+1}(G) = [\Gamma_s(G), G]$ for $s \in \mathbb{N}_0$. We also denote

$$\Gamma(G) = \bigcap_{s\in\mathbb{N}_0} \Gamma_s(G).$$

It is known that G is residually nilpotent if and only if $\Gamma(G) = \{1_G\}$. The upper central series of G is defined as follows. $Z^0(G) = \{1_G\}$ and $Z^{s+1}(G)$ is the unique subgroup of G such that

$$Z^{s+1}(G)/Z^s(G) = Z(G/Z^i(G)),$$

where $Z(G)$ denotes the center of G. For any group G the following equalities hold: $Z^1(G) = Z(G)$ and

$$Z^{s+1}(G) = \{x \in G\colon [x,y] \in Z^s(G) \text{ for } y \in G\}, \quad s \in \mathbb{N}_0. \tag{7}$$

The subgroup $Z^s(G)$ is called the s-th center of G. One can continue the upper central series (7) to infinite ordinal numbers via transfinite recursion; for a limit ordinal λ, define $Z^\lambda(G) = \bigcup_{\delta<\lambda} Z^\delta(G)$. The limit of this process (the union of the higher centers) is called the hypercenter of the group. If the transfinite upper central series stabilizes at the whole group, then the group is called hypercentral. Hypercentral groups have many properties of nilpotent groups, such as the normalizer condition (the normalizer of a proper subgroup properly contains the subgroup), elements of coprime orders commute, and periodic hypercentral groups are the direct product of their Sylow p-subgroups.

Proposition 4.1. *For $s > 0$ we have*
(i) $\Gamma_s(\mathcal{Z}^\pm) = \{\alpha \in \mathcal{Z}^\pm \colon \alpha(0) = 0$ *and* $\alpha \in (2^s\mathbb{Z})^{\mathbb{N}_0}\}$,
(ii) $Z^s(\mathcal{Z}^\pm) = Z(\mathcal{Z}^\pm) = \{\alpha \in \mathcal{Z}^\pm \colon \alpha(0) \in 2\mathbb{Z}$ *and* $\alpha(i) = 0$ *for* $i > 0\}$.
In particular $\Gamma(\mathcal{Z}^\pm) = \{\theta\}$ *and thus* $\mathcal{Z}^\pm$ *is residually nilpotent.*

Proof. (i) Let Δ_s be the right side of (i). Directly by definition of $\mathcal{Z}^\pm$ we see that Δ_s is a subgroup of $\mathcal{Z}^\pm$. For any $\alpha, \beta \in \mathcal{Z}^\pm$ we have

$$\begin{aligned} [\alpha,\beta](i) = {} & -\alpha(i)\cdot(-1)^{\alpha(i-1)}\cdot(1-(-1)^{\beta(i-1)}) \\ & + \beta(i)\cdot(-1)^{\beta(i-1)}\cdot(1-(-1)^{\alpha(i-1)}), \quad i \in \mathbb{N}_0. \end{aligned} \tag{8}$$

In particular $[\alpha,\beta](0) = 0$ and $[\alpha,\beta] \in (2\mathbb{Z})^{\mathbb{N}_0}$. Thus $[\alpha,\beta] \in \Delta_1$. Since Δ_1 is a subgroup of $\mathcal{Z}^\pm$ we obtain $\Gamma_1(\mathcal{Z}^\pm) \subseteq \Delta_1$. Conversely, for any $\gamma \in \Delta_1$ we easily verify that $[\alpha,\beta] = \gamma$, where $\alpha, \beta \in \mathcal{Z}^\pm$ are defined as follows. $\beta(i) = 1$ for $i \in \mathbb{N}_0$ and α is defined recursively by $\alpha(0) = 0$ and $\alpha(i) = (1/2)\cdot(1-(1+\gamma(i))\cdot(-1)^{\alpha(i-1)})$. In consequence $\Delta_1 = \Gamma_1(\mathcal{Z}^\pm)$. Let us assume that $\Gamma_s(\mathcal{Z}^\pm) = \Delta_s$ for some $s \geq 1$. Then for any $\alpha \in \Delta_s$ and any $\beta \in \mathcal{Z}^\pm$ we have

$$[\alpha,\beta](i) = -\alpha(i)\cdot(1-(-1)^{\beta(i-1)}) \in 2^{s+1}\mathbb{Z}.$$

Thus $[\alpha,\beta] \in \Delta_{s+1}$ and in consequence $\Gamma_{s+1}(\mathcal{Z}^\pm) \subseteq \Delta_{s+1}$. Conversely, for any $\gamma \in \Delta_{s+1}$ we have $[\alpha,\beta] = \gamma$, where $\alpha \in \Delta_s$ and $\beta \in \mathcal{Z}^\pm$ are defined as follows: $\alpha(i) = -\gamma(i)/2$ and $\beta(i) = 1$ for $i \in \mathbb{N}_0$. Hence $\Delta_{s+1} \subseteq \Gamma_{s+1}(\mathcal{Z}^\pm)$ and finally $\Delta_{s+1} = \Gamma_{s+1}(\mathcal{Z}^\pm)$. The inductive argument finishes the proof.

(ii) Let $\alpha \in Z^1(\mathcal{Z}^\pm)$ and let $i > 0$. Let $\beta \in \mathcal{Z}^\pm$ be such that $\beta(i) = 0$ and $\beta(i-1) = 1$. From

$$\alpha\beta(i) = \beta\alpha(i) \tag{9}$$

we obtain $\alpha(i) = 0$. For $\beta \in \mathcal{Z}^\pm$ such that $\beta(i) = 1$ and $\beta(i-1) = 0$ we have $(-1)^{\alpha(i-1)} = 1$, by (9). Hence $\alpha(i-1) \in 2\mathbb{Z}$. In consequence $\alpha(0) \in 2\mathbb{Z}$ and $\alpha(i) = 0$ for $i > 0$. Conversely, let $\alpha \in \mathcal{Z}^\pm$ be such that $\alpha(0) \in 2\mathbb{Z}$ and $\alpha(i) = 0$ for $i > 0$. We easily verify that (9) holds for any $\beta \in \mathcal{Z}^\pm$ and any $i \in \mathbb{N}_0$. Thus $\alpha \in Z^1(\mathcal{Z}^\pm)$. Let us assume that the thesis holds for some $s \geq 1$. Let $\alpha \in Z^{s+1}(\mathcal{Z}^\pm)$. Thus for any $\beta \in \mathcal{Z}^\pm$ we have $[\alpha,\beta] \in Z^s(\mathcal{Z}^\pm)$. In particular, by induction assumption $[\alpha,\beta](i) = 0$ for any $i > 0$. Let $i > 0$ and let $\beta \in \mathcal{Z}^\pm$ be such that $\beta(i) = 0$ and $\beta(i-1) = 1$. By (8) we obtain

$$[\alpha,\beta](i) = -2\cdot\alpha(i)\cdot(-1)^{\alpha(i-1)}.$$

Hence $\alpha(i) = 0$. In consequence, for $\beta \in \mathcal{Z}^{\pm}$ with $\beta(i) = 1$ and $\beta(i-1) = 0$ we have

$$[\alpha, \beta](i) = 1 - (-1)^{\alpha(i-1)}.$$

Hence $\alpha(i-1) \in 2\mathbb{Z}$. In consequence $\alpha(0) \in 2\mathbb{Z}$ and $\alpha(i) = 0$ for $i > 0$. Conversely, let $\alpha \in \mathcal{Z}^{\pm}$ be such that $\alpha(0) \in 2\mathbb{Z}$ and $\alpha(i) = 0$ for $i > 0$. Then, as we have shown above $\alpha \in Z^1(\mathcal{Z}^{\pm})$. In consequence $\alpha \in Z^{s+1}(\mathcal{Z}^{\pm})$. The inductive argument finishes the proof. □

Let us fix the branch index $\overline{m} = (m_i)_{i \in \mathbb{N}_0}$, which satisfies condition (5), so that the set

$$\mathcal{B} = \mathcal{B}_{\overline{m}}$$

is a group. For each $s \in \mathbb{N}$ we consider the sequence $n(s) = (n_i(s))_{i \in \mathbb{N}_0}$, where

$$n_0(s) = 1, \quad n_i(s) = m_i / \gcd(2^s, m_i) \quad \text{for } i > 0.$$

As well as we consider the set

$$J_s = \{i \in \mathbb{N} \colon m_i \nmid 2^s\}.$$

We also consider the sequence $n = (n_i)_{i \in \mathbb{N}_0}$, where

$$n_0 = 1, \quad n_i = m_i / \max\{2^l \colon 2^l \mid m_i\} \quad \text{for } i > 0.$$

In other words n_i for $i > 0$ constitutes the "odd factor" of m_i.

Proposition 4.2. *For $\alpha \in \mathcal{Z}^{\pm}$ and $s > 0$ we have*

(i) *$g_\alpha \in \Gamma_s(\mathcal{B})$ if and only if $\alpha(i) \equiv_{m_i/n_i(s)} 0$ for $i \in \mathbb{N}_0$.*
(ii) *$g_\alpha \in \Gamma(\mathcal{B})$ if and only if $\alpha(i) \equiv_{m_i/n_i} 0$ for $i \in \mathbb{N}_0$.*
(iii) *$g_\alpha \in Z^s(\mathcal{B})$ if and only if $\alpha(i-1) \equiv_2 0$ and $2^s \cdot \alpha(i) \equiv_{m_i} 0$ for $i \in J_s$.*

Proof. (i) By Corollary 3.1 for any $\alpha \in \mathcal{Z}^{\pm}$ and any $s \in \mathbb{N}$ we have $g_\alpha \in \Gamma_s(\mathcal{B})$ if and only if $\alpha(i) \equiv_{m_i} \gamma(i)$ for some $\gamma \in \Gamma_s(\mathcal{Z}^{\pm})$ and any $i \in \mathbb{N}_0$. Now, the assertion follows from Proposition 4.1.

(ii) By (i) we have $g_\alpha \in \Gamma(\mathcal{B})$ if and only if for each $s \in \mathbb{N}$ the congruence $\alpha(i) \equiv_{m_i/n_i(s)} 0$ holds for any $i \in \mathbb{N}_0$. Simple calculations give the assertion.

(iii) We use the induction on s. So, let $g_\alpha \in Z^1(\mathcal{B}) = Z(\mathcal{B})$ and let $i \in J_1$. Then for any $\beta \in \mathcal{Z}^{\pm}$ we have

$$\alpha\beta(i) \equiv_{m_i} \beta\alpha(i). \tag{10}$$

Let $\beta \in \mathcal{Z}^{\pm}$ be such that $\beta(i) = 1$ and $\beta(i-1) = 0$. From (10) we have $(-1)^{\alpha(i-1)} \equiv_{m_i} 1$. Since $m_i \nmid 2$ we have in consequence $\alpha(i-1) \equiv_2 0$. Let $\beta \in \mathcal{Z}^{\pm}$ be such that $\beta(i) = 0$ and $\beta(i-1) = 1$. From (10) we obtain $2 \cdot \alpha(i) \equiv_{m_i} 0$. Conversely, let $\alpha \in \mathcal{Z}^{\pm}$ be such that $\alpha(i-1) \equiv_2 0$ and $2 \cdot \alpha(i) \equiv_{m_i} 0$ for all $i \in J_1$. Then $(-1)^{\alpha(i-1)} = 1$ and $2 \cdot \alpha(i) \equiv_{m_i} 0$ for all $i \in \mathbb{N}_0$. In consequence (10) holds for any $\beta \in \mathcal{Z}^{\pm}$ and any $i \in \mathbb{N}_0$. Thus $g_\alpha \in Z^1(\mathcal{B})$ and the thesis of the point (iii) is true for $s = 1$. Let us assume that the thesis holds fore some $s \geq 1$. Let $g_\alpha \in Z^{s+1}(\mathcal{B})$. Then $[g_\alpha, g_\beta] = g_{[\alpha,\beta]} \in Z^s(\mathcal{B})$ for any $\beta \in \mathcal{Z}^{\pm}$. By inductive assumption we have for any $\beta \in \mathcal{Z}^{\pm}$ and any $i \in J_s$

$$2^s \cdot [\alpha, \beta](i) \equiv_{m_i} 0.$$

Let $i \in J_{s+1}$ and let $\beta \in \mathcal{Z}^{\pm}$ be such that $\beta(i-1) = 0$ and $\beta(i) = 1$. Then

$$2^s \cdot [\alpha, \beta](i) = 2^s \cdot (1 - (-1)^{\alpha(i-1)}).$$

Since $J_{s+1} \subseteq J_s$ we have $2^s \cdot (1 - (-1)^{\alpha(i-1)}) \equiv_{m_i} 0$. Since $m_i \nmid 2^{s+1}$ we have in consequence $\alpha(i-1) \equiv_2 0$. Now, let $\beta \in \mathcal{Z}^{\pm}$ be such that $\beta(i-1) = 1$. Then we compute

$$2^s \cdot [\alpha, \beta](i) = -2^{s+1} \cdot \alpha(i).$$

In consequence $2^{s+1} \cdot \alpha(i) \equiv_{m_i} 0$. Conversely, let $\alpha \in \mathcal{Z}^{\pm}$ be such that $\alpha(i-1) \equiv_2 0$ and $2^{s+1} \cdot \alpha(i) \equiv_{m_i} 0$ for any $i \in J_{s+1}$. Let $\beta \in \mathcal{Z}^{\pm}$. Then

$$[\alpha, \beta](i-1) \equiv_2 0 \quad \text{and} \quad 2^s \cdot [\alpha, \beta](i) \equiv_{m_i} 0 \quad \text{for } i \in J_s. \tag{11}$$

The first congruence is obvious. For the second one note that if $i \in J_{s+1}$ then from $\alpha(i-1) \equiv_2 0$ we obtain

$$2^s \cdot [\alpha, \beta](i) = \begin{cases} 0, & \text{if } \beta(i-1) \equiv_2 0, \\ -2^{s+1} \cdot \alpha(i), & \text{if } \beta(i-1) \equiv_2 1. \end{cases}$$

Thus $2^s \cdot [\alpha, \beta](i) \equiv_{m_i} 0$. If $i \notin J_{s+1}$ then $m_i \mid 2^{s+1}$ and from $[\alpha, \beta](i) \equiv_2 0$ we have

$$2^s \cdot [\alpha, \beta](i) \equiv_{2^{s+1}} 0.$$

Thus $2^s \cdot [\alpha, \beta](i) \equiv_{m_i} 0$. By inductive assumption we have from (11)

$$[g_\alpha, g_\beta] = g_{[\alpha,\beta]} \in Z^s(\mathcal{B})$$

for $\beta \in \mathcal{Z}^{\pm}$. In consequence $g_\alpha \in Z^{s+1}(\mathcal{B})$. $\square$

Proposition 4.3. *If N is a normal, abelian subgroup of $\mathcal{B}$ and $g_\alpha \in N$ then for each $i \in \mathbb{N}_0$ we have $\alpha(i-1) \equiv_2 0$ or $m_i \in \{2,4\}$. In particular, if $m_i > 4$ for each $i \in \mathbb{N}_0$ then the commutator subgroup $\mathcal{B}' = \Gamma_1(\mathcal{B})$ is the greatest normal abelian subgroup of $\mathcal{B}$ as well as $\mathcal{B}'$ is a maximal abelian subgroup of $\mathcal{B}$.*

Proof. Let us fix $i \in \mathbb{N}_0$ and let $\beta \in \mathcal{Z}^{\pm}$ be such that $\beta(i-1) = 0$ and $\beta(i) = 1$. Since N is normal, we have $g_{\beta\alpha\beta^{-1}} = g_\beta g_\alpha {g_\beta}^{-1} \in N$. Since N is abelian, we have

$$g_{\alpha^{-1}\beta\alpha\beta^{-1}} = {g_\alpha}^{-1} \cdot (g_{\beta\alpha\beta^{-1}}) = (g_{\beta\alpha\beta^{-1}}) \cdot {g_\alpha}^{-1} = g_{\beta\alpha\beta^{-1}\alpha^{-1}}.$$

In particular $(\alpha^{-1}\beta\alpha\beta^{-1})(i) \equiv_{m_i} (\beta\alpha\beta^{-1}\alpha^{-1})(i)$. We compute

$$(\alpha^{-1}\beta\alpha\beta^{-1})(i) = (-1)^{\alpha(i-1)} - 1$$

and

$$(\beta\alpha\beta^{-1}\alpha^{-1})(i) = 1 - (-1)^{\alpha(i-1)}.$$

Hence $(-1)^{\alpha(i-1)} - 1 \equiv_{m_i} 1 - (-1)^{\alpha(i-1)}$. In consequence $\alpha(i-1) \equiv_2 0$ or $m_i \in \{2,4\}$. $\square$

Remark 4.1. If $m_i = 4$ for any $i \in \mathbb{N}_0$ then the set $\{g_\alpha \colon \alpha \in \{0,2\}^{\mathbb{N}_0}\} \cup \{g_\alpha \colon \alpha \in \{1,3\}^{\mathbb{N}_0}\}$ is a normal, abelian subgroup of $\mathcal{B}$ which contains the commutator subgroup $\mathcal{B}' = \{g_\alpha \colon \alpha \in \{0,2\}^{\mathbb{N}_0}\}$ properly.

For $\epsilon \in \{0,1\}$ let us consider the following set

$$\mathcal{K}_\epsilon = \{g_\alpha \colon \alpha(2\mathbb{N}_0 + \epsilon) = \{0\}\}.$$

Proposition 4.4. *$\mathcal{K}_\epsilon$ is a subgroup of $\mathcal{B}$, which is isomorphic to $\prod_i \mathbb{Z}_{m_{\iota_i}}$, where $\iota_i = 2i + (\epsilon+1)_2$.*

Proof. Let $g_\alpha, g_\beta \in \mathcal{K}_\epsilon$. Then $\alpha(2i+\epsilon) = \beta(2i+\epsilon) = 0$ and

$$(\alpha\beta^{-1})(2i+\epsilon) = \alpha(2i+\epsilon) - \beta(2i+\epsilon) \cdot (-1)^{\alpha(2i+\epsilon-1)+\beta(2i+\epsilon-1)} = 0$$

for $i \in \mathbb{N}_0$. In consequence $g_\alpha (g_\beta)^{-1} = g_{\alpha\beta^{-1}} \in \mathcal{K}_\epsilon$. Hence $\mathcal{K}_\epsilon$ is a subgroup of $\mathcal{B}$. Moreover, $g_\alpha = g_\beta$ iff $\alpha(\iota_i) \equiv_{m_{\iota_i}} \beta(\iota_i)$ for $i \in \mathbb{N}_0$. Thus

$$\phi \colon \mathcal{K}_\epsilon \to \prod_i \mathbb{Z}_{m_{\iota_i}}, \quad \phi(g_\alpha) = ((\alpha(\iota_i))_{m_{\iota_i}})_{i \in \mathbb{N}_0}, \quad i \in \mathbb{N}_0,$$

defines a bijection. Moreover, since $\alpha(\iota_i-1)=\alpha(2i-\epsilon)=0$, we have for the i-th coordinate of $\phi(g_\alpha g_\beta)$

$$\begin{aligned}\phi(g_\alpha g_\beta)(i)&=\phi(g_{\alpha\beta})(i)=(\alpha\beta(\iota_i))_{m_{\iota_i}}\\&=(\alpha(\iota_i)+\beta(\iota_i))_{m_{\iota_i}}\\&=\phi(g_\alpha)(i)\phi(g_\beta)(i).\end{aligned}$$

Thus ϕ is an isomorphism. □

THEOREM 4.1. (i) *For $s>0$ the group $\Gamma_s(\mathcal{B})$ is isomorphic to $\prod_{i>0}\mathbb{Z}_{n_i(s)}$. The group $\Gamma(\mathcal{B})$ is isomorphic to $\prod_{i>0}\mathbb{Z}_{n_i}$.*

(ii) *$\mathcal{B}$ is nilpotent of a class s if and only if m_i is a power of two for $i>0$ and $\max\{m_i\colon i\in\mathbb{N}\}=2^s$; $\mathcal{B}$ is residually nilpotent if and only if m_i is a power of two for any $i>0$.*

(iii) *The center $Z(\mathcal{B})$ is trivial if and only if the following conditions hold:*
(a) *$m_0=2$,*
(b) *$m_i=2 \Rightarrow 2\nmid m_{i-1}$ for $i>0$,*
(c) *$4\nmid m_i$ for $i>0$.*

(iv) *The hypercenter of $\mathcal{B}$ is equal to $Z^\omega(\mathcal{B})=\bigcup_{s\in\mathbb{N}_0}Z^s(\mathcal{B})$. Moreover, $\mathcal{B}$ is hypercentral if and only if it is a nilpotent group.*

(v) *$\mathcal{B}$ is metabelian with a semigroup law $x^2y^2=y^2x^2$.*

(vi) *The abelianization $\mathcal{B}/\mathcal{B}'$ is isomorphic to $\mathbb{Z}_{m_0}\times\prod_{i\in I}\mathbb{Z}_2^{(i)}$, where $\mathbb{Z}_2^{(i)}$ is an isomorphic copy of $\mathbb{Z}_2$ and $I=\{i\in\mathbb{N}\colon 2\mid m_i\}$.*

(vii) *$\mathcal{B}=\mathcal{K}_0\mathcal{K}_1$ and $\mathcal{K}_0\cap\mathcal{K}_1=\{Id_{X^*}\}$.*

(viii) *If there is $s>0$ such that $m_s,m_{s+1}\notin\{2,4\}$ and $4\mid m_s$ then $\mathcal{B}$ is not a product of its abelian subgroups of which one is normal; in particular $\mathcal{B}$ is not a semidirect product of its abelian subgroups.*

(ix) *Let $M=\sup\overline{m}$. If $M<\infty$ then $\mathcal{B}$ is of finite exponent. If $M=\infty$ then $\mathcal{B}$ contains a free abelian group of an uncountable rank.*

PROOF. (i) From the point (i) of Proposition 4.2 for any $g_\alpha,g_\beta\in\Gamma_s(\mathcal{B})$ we obtain by easy calculations $g_\alpha=g_\beta$ iff

$$\frac{\alpha(i)}{m_i/n_i(s)}\equiv_{n_i(s)}\frac{\beta(i)}{m_i/n_i(s)}\quad\text{for } i>0.$$

Thus $\phi_s\colon\Gamma_s(\mathcal{B})\to\prod_{i>0}\mathbb{Z}_{n_i(s)}$, where

$$\phi_s(g_\alpha)=\Big(\Big(\frac{\alpha(i)}{m_i/n_i(s)}\Big)_{n_i(s)}\Big)_{i>0}$$

is a well defined, one to one mapping. Moreover, ϕ_s is onto. Indeed, since for $i>0$ the numbers $n_i(s)$ and $t_i(s)=2^s\cdot n_i(s)/m_i$ are coprime, there are integers a_i,b_i such that $a_i\cdot n_i(s)+b_i\cdot t_i(s)=1$. For any $k=(k_i)_{i>0}\in\prod_{i>0}\mathbb{Z}_{n_i(s)}$ we define $\alpha\in\mathcal{Z}^{\pm}$ as follows $\alpha(0)=0$ and $\alpha(i)=2^s\cdot b_i\cdot k_i$ for $i>0$. Then $g_\alpha\in\Gamma_s(\mathcal{B})$ and $\phi_s(g_\alpha)=k$. To show ϕ_s is a homomorphism we consider for any $i>0$ two cases: $2\nmid m_{i-1}$ and $2\mid m_{i-1}$. In the first case $m_i=2$ by condition (5) and thus $n_i(s)=1$. In consequence the i-th coordinate $\phi_s(g_\alpha)(i)$ of $\phi_s(g_\alpha)$ is equal to 0 for any $g_\alpha\in\Gamma_s(\mathcal{B})$. In consequence for any $g_\beta\in\Gamma_s(\mathcal{B})$ we have

$$\phi_s(g_\alpha g_\beta)(i)=\phi_s(g_\alpha)(i)\phi_s(g_\beta)(i)=0.$$

In the second case $g_\alpha\in\Gamma_s(\mathcal{B})$ implies $\alpha(i-1)\equiv_2 0$ by the point (i) of Proposition 4.2. Thus $(-1)^{\alpha(i-1)}=1$ and for any $g_\beta\in\Gamma_s(\mathcal{B})$ the i-th coordinate of $\phi_s(g_\alpha g_\beta)$ is equal to

$$\begin{aligned}\phi_s(g_{\alpha\beta})(i)&=\Big(\frac{\alpha\beta(i)}{m_i/n_i(s)}\Big)_{n_i(s)}\\&=\Big(\frac{\alpha(i)}{m_i/n_i(s)}+\frac{\beta(i)}{m_i/n_i(s)}\Big)_{n_i(s)}\\&=\phi_s(g_\alpha)(i)\phi_s(g_\beta)(i).\end{aligned}$$

In consequence $\phi_s(g_\alpha g_\beta)=\phi_s(g_\alpha)\phi_s(g_\beta)$ for any $g_\alpha,g_\beta\in\Gamma_s(\mathcal{B})$. Thus ϕ_s is an isomorphism. In the similar way for any $g_\alpha,g_\beta\in\Gamma(\mathcal{B})$ we obtain $g_\alpha=g_\beta$ iff

$$\frac{\alpha(i)}{m_i/n_i}\equiv_{n_i}\frac{\beta(i)}{m_i/n_i}\quad\text{for } i>0.$$

Thus $\phi\colon\Gamma(\mathcal{B})\to\prod_{i>0}\mathbb{Z}_{n_i}$, where

$$\phi(g_\alpha)=\Big(\Big(\frac{\alpha(i)}{m_i/n_i}\Big)_{n_i}\Big)_{i>0}$$

is a well defined, one to one mapping. Since for $i>0$ the numbers n_i and $t_i=m_i/n_i$ are coprime, there are integers c_i,d_i such that $c_i\cdot n_i+d_i\cdot t_i=1$. Then for any $k=(k_i)_{i>0}\in\prod_{i>0}\mathbb{Z}_{n_i}$ we take $\alpha\in\mathcal{Z}^{\pm}$ such that $\alpha(0)=0$ and $\alpha(i)=k_i\cdot d_i\cdot t_i$ for $i>0$. Then $g_\alpha\in\Gamma(\mathcal{B})$ and $\phi(g_\alpha)=k$. In consequence ϕ is bijective. As before we show that ϕ is a homomorphism.

(ii) By (i) the group $\mathcal{B}$ is nilpotent of a class $s>0$ iff $n_i(s)=1$ for each $i>0$ and there is i_0 such that $n_{i_0}(s-1)\neq 1$. Equivalently, $m_i\mid 2^s$ for $i>0$ and $m_{i_0}\nmid 2^{s-1}$ for some $i_0>0$. That is to say m_i is a power of two for any $i > 0$ and $\max\{m_i : i \in \mathbb{N}\} = 2^s$. Next, by (i) the group $\mathcal{B}$ is residually nilpotent iff $n_i = 1$ for $i > 0$. Equivalently, m_i is a power of two for each $i > 0$.

(iii) If $m_0 \neq 2$ then $Z(\mathcal{B})$ is not trivial. Indeed, for $\alpha \in \mathcal{Z}^{\pm}$ such that $\alpha(0) = 2$ and $\alpha(i) = 0$ for $i > 0$ we have $g_\alpha \neq Id_{X^*}$. Moreover, by the point (iii) of Proposition 4.2 we have $g_\alpha \in Z(\mathcal{B})$. Similarly, if $m_i = 2$ and $2 \mid m_{i-1}$ for some $i > 0$ then $g_\alpha \neq Id_{X^*}$ and $g_\alpha \in Z(\mathcal{B})$ for any $\alpha \in \mathcal{Z}^{\pm}$ such that $\alpha(i-1) = m_{i-1}/2$ and $\alpha(j) = 0$ for $j \neq i-1$. If $4 \mid m_i$ for some $i \in \mathbb{N}_0$ then for the element g_α with $\alpha \in \mathcal{Z}^{\pm}$ such that $\alpha(i) = m_i/2$ and $\alpha(j) = 0$ for $j \neq i$ we have: $g_\alpha \in Z(\mathcal{B})$ and $g_\alpha \neq Id_{X^*}$. Thus if $Z(\mathcal{B})$ is trivial then the conditions (a)-(c) hold. Conversely, let us assume that (a)-(c) hold. Let $g_\alpha \in Z(\mathcal{B})$ for some $\alpha \in \mathcal{Z}^{\pm}$. We must show that $\alpha(i) \equiv_{m_i} 0$ for all $i \in \mathbb{N}_0$. It follows by (a) and (b) that $1 \in J_1$. Thus $\alpha(0) \equiv_2 0$ by the point (iii) of Proposition 4.2. Since $m_0 = 2$ we have $\alpha(0) \equiv_{m_0} 0$. Let $i > 0$. We distinguish the following cases.

Case 1: $2 \nmid m_i$. Thus $i \in J_1$ and by the point (iii) of Proposition 4.2 we have $2 \cdot \alpha(i) \equiv_{m_i} 0$. Since the numbers 2 and m_i are coprime we obtain $\alpha(i) \equiv_{m_i} 0$.

Case 2: $m_i = 2$. Thus $i+1 \in J_1$ by (b). By the point (iii) of Proposition 4.2 we have $\alpha(i) \equiv_2 0$ and since $m_i = 2$ we have $\alpha(i) \equiv_{m_i} 0$.

Case 3: $m_i \neq 2$ and $2 \mid m_i$. Thus $i \in J_1$ and by the point (iii) of Proposition 4.2 we have $2 \cdot \alpha(i) \equiv_{m_i} 0$. Since $\gcd(2, m_i) = 2$, we obtain from the last congruence $\alpha(i) \equiv_{m_i/2} 0$. Moreover, $i+1 \in J_1$ by (b). Thus $\alpha(i) \equiv_2 0$ by the point (iii) of Proposition 4.2. By condition (c) the numbers $m_i/2$ and 2 are coprime. Thus $\alpha(i) \equiv_{m_i} 0$.

(iv) Let $g_\alpha \in \mathcal{B}$ be such that $[g_\alpha, g_\beta] = g_{[\alpha,\beta]} \in Z^\omega(\mathcal{B})$ for any $g_\beta \in \mathcal{B}$. Since $Z^\omega(\mathcal{B})$ is a normal subgroup of $\mathcal{B}$, it is sufficient to prove $g_\alpha \in Z^\omega(\mathcal{B})$. By the point (iii) of Proposition 4.2, for each $\beta \in \mathcal{Z}^{\pm}$ there is $s \in \mathbb{N}$ such that

$$2^s \cdot [\alpha, \beta](i) \equiv_{m_i} 0 \quad \text{for } i \in J_s.$$

Let $\beta \in \mathcal{Z}^{\pm}$ be such that $\beta(i) = 2$ for any $i \in \mathbb{N}_0$ and let $s_0 \in \mathbb{N}$ be the corresponding number. Then we have

$$2^{s_0} \cdot [\alpha, \beta](i) = 2^{s_0+1} \cdot (1 - (-1)^{\alpha(i-1)}) \equiv_{m_i} 0 \quad \text{for } i \in J_{s_0}.$$

Since $J_{s_0+2} \subseteq J_{s_0}$ we obtain $\alpha(i-1) \equiv_2 0$ for any $i \in J_{s_0+2}$. Let $\beta' \in \mathcal{Z}^{\pm}$ be such that $\beta'(i) = 1$ for any $i \in \mathbb{N}_0$ and let s_1 be the corresponding number for β'. Then we have

$$2^{s_1} \cdot [\alpha, \beta'](i) = 2^{s_1} \cdot ((-1)^{\alpha(i-1)} \cdot (1 - 2 \cdot \alpha(i)) - 1) \equiv_{m_i} 0 \quad \text{for } i \in J_{s_1}.$$

For $s = \max(s_0 + 2, s_1 + 1)$ we have $J_s \subseteq J_{s_1}$ and $J_s \subseteq J_{s_0+2}$. Hence from the last congruence we obtain $2^{s_1+1} \cdot \alpha(i) \equiv_{m_i} 0$ for $i \in J_s$. In consequence

$\alpha(i-1) \equiv_2 0$ and $2^s \cdot \alpha(i) \equiv_{m_i} 0$ for $i \in J_s$. Thus $g_\alpha \in Z^s(\mathcal{B})$ by the point (iii) of Proposition 4.2. In consequence $g_\alpha \in Z^\omega(\mathcal{B})$. To prove the second part of (iv) let us assume that $Z^\omega(\mathcal{B}) = \mathcal{B}$. Let $\alpha \in \mathcal{Z}^\pm$ be such that $\alpha(i) = 1$ for any $i \in \mathbb{N}_0$. There is $s \in \mathbb{N}$ such that $g_\alpha \in Z^s(\mathcal{B})$. By the point (iii) of Proposition 4.2 we have in particular $2^s \cdot \alpha(i) \equiv_{m_i} 0$ for any $i \in J_s$. Since $\alpha(i) = 1$ the set J_s must be empty. In consequence $Z^s(\mathcal{B}) = \mathcal{B}$ and $\mathcal{B}$ is nilpotent.

(v) For any $\alpha, \beta \in \mathcal{Z}^\pm$ we have $[\alpha, \beta] \in (2\mathbb{Z})^{\mathbb{N}_0}$. Hence for any $\alpha, \beta, \gamma, \delta \in \mathcal{Z}^\pm$ we have

$$[\alpha, \beta][\gamma, \delta](i) = [\gamma, \delta][\alpha, \beta](i) = [\alpha, \beta](i) + [\gamma, \delta](i).$$

In consequence $\mathcal{Z}^{\pm''} = [\mathcal{Z}^{\pm'}, \mathcal{Z}^{\pm'}] = \{\theta\}$. Moreover, $\alpha^2 \in (2\mathbb{Z})^{\mathbb{N}_0}$ for any $\alpha \in \mathcal{Z}^\pm$. Thus $\alpha^2\beta^2 = \beta^2\alpha^2$ for any $\alpha, \beta \in \mathcal{Z}^\pm$. Now, the assertion follows from Corollary 3.1.

(vi) By the point (i) of Proposition 4.2 we have $g_\alpha\mathcal{B}' = g_\beta\mathcal{B}'$ iff

$$\alpha\beta^{-1}(i) \equiv_{m_i/n_i(1)} 0 \quad \text{for } i \in \mathbb{N}_0. \tag{12}$$

Since $n_0(1) = 1$, $n_i(1) = m_i/2$ for $i \in I$ and $n_i(1) = m_i$ for $i \notin I \cup \{0\}$ we obtain that (12) is equivalent to the following set of congruences

$$\alpha(0) \equiv_{m_0} \beta(0), \quad \alpha(i) \equiv_2 \beta(i) \quad \text{for } i \in I.$$

Thus

$$\phi\colon \mathcal{B}/\mathcal{B}' \to \mathbb{Z}_{m_0} \times \prod_{i\in I} \mathbb{Z}_2^{(i)}, \quad \phi(g_\alpha\mathcal{B}')(i) = \begin{cases} (\alpha(0))_{m_0}, & \text{for } i = 0, \\ (\alpha(i))_2, & \text{for } i \in I \end{cases}$$

is a well defined bijection. Moreover, for any $i \in I \cup \{0\}$ one easily verifies

$$\phi(g_\alpha g_\beta\mathcal{B}')(i) = \phi(g_{\alpha\beta}\mathcal{B}')(i) = \phi(g_\alpha\mathcal{B}')(i)\phi(g_\beta\mathcal{B}')(i).$$

Thus ϕ is an isomorphism.

(vii) Obviously $\mathcal{K}_0 \cap \mathcal{K}_1 = \{g_\theta\} = \{Id_{X^*}\}$. Now, for any $\alpha \in \mathcal{Z}^\pm$ and $\epsilon \in \{0, 1\}$ we define the element $\alpha_\epsilon \in \mathcal{Z}^\pm$ as follows

$$\alpha_\epsilon(i) = \alpha(i) \cdot ((i)_2 - \epsilon)^{\alpha(i-1)}, \quad i \in \mathbb{N}_0.$$

In the right side of the above formula we assume $0^0 = 1$. Then $g_{\alpha_\epsilon} \in \mathcal{K}_\epsilon$ and $\alpha = \alpha_0\alpha_1$. Thus $g_\alpha = g_{\alpha_0}g_{\alpha_1}$ and $\mathcal{B} = \mathcal{K}_0\mathcal{K}_1$.

(viii) Suppose not and let $\mathcal{B} = NK$, where N and K are abelian subgroups and N is normal. For each $\gamma \in \mathcal{Z}^\pm$ there are $g_\alpha \in N$, $g_\beta \in K$ such that

$g_\gamma = g_\alpha g_\beta = g_{\alpha\beta}$. In particular $\gamma(s-1) \equiv_{m_{s-1}} \alpha\beta(s-1)$ and $\gamma(s) \equiv_{m_s} \alpha\beta(s)$. Thus

$$\gamma(s-1) \equiv_{m_{s-1}} \alpha(s-1) + \beta(s-1) \cdot (-1)^{\alpha(s-2)} \tag{13}$$

and

$$\gamma(s) \equiv_{m_s} \alpha(s) + \beta(s) \cdot (-1)^{\alpha(s-1)}. \tag{14}$$

Both m_{s-1} and m_s are even (otherwise $m_s = 2$ or $m_{s+1} = 2$ by Theorem 3.1). Moreover, by Proposition 4.3 we have $\alpha(s-1) \equiv_2 \alpha(s) \equiv_2 0$. Thus from (13) and (14) we obtain $\gamma(s-1) \equiv_2 \beta(s-1)$ and $\gamma(s) \equiv_2 \beta(s)$. Now, let $\gamma \in \mathcal{Z}^{\pm}$ be such that $\gamma(s-1) = 0$ and $\gamma(s) = 1$. Then there is $g_\beta \in K$ such that

$$0 \equiv_2 \beta(s-1), \tag{15}$$

$$1 \equiv_2 \beta(s). \tag{16}$$

Similarly, let $\gamma' \in \mathcal{Z}^{\pm}$ be such that $\gamma'(s-1) = 1$. Then there is $g_{\beta'} \in K$ such that

$$1 \equiv_2 \beta'(s-1). \tag{17}$$

Since K is abelian we have $g_{\beta\beta'} = g_\beta g_{\beta'} = g_{\beta'} g_\beta = g_{\beta'\beta}$. In particular, $\beta\beta'(s) \equiv_{m_s} \beta'\beta(s)$ or equivalently

$$\beta(s) + (-1)^{\beta(s-1)} \cdot \beta'(s) \equiv_{m_s} \beta'(s) + (-1)^{\beta'(s-1)} \cdot \beta(s).$$

By using (15) and (17) the last congruence we may rewrite as

$$\beta(s) + \beta'(s) \equiv_{m_s} \beta'(s) - \beta(s).$$

Thus $2\beta(s) \equiv_{m_s} 0$. Since $4 \mid m_s$ we obtain $\beta(s) \equiv_2 0$ contrary to (16).

(ix) For any integer n and any $\alpha \in \mathcal{Z}^{\pm}$ the n-th power of α is equal to

$$(\alpha^n)(i) = \alpha(i) \cdot \left(n + \left[\frac{n}{2}\right] \cdot \left((-1)^{\alpha(i-1)} - 1\right)\right), \quad i \in \mathbb{N}_0.$$

The proof of the above formula is straightforward or by induction on n. Now, if $M < \infty$ then for the $M!$-th power of α we have

$$(\alpha^{M!})(i) = M! \cdot \alpha(i) \cdot \frac{(1 + (-1)^{\alpha(i-1)})}{2} \equiv_{m_i} 0 \quad \text{for } i \in \mathbb{N}_0.$$

Thus $(g_\alpha)^{M!} = Id_{X^*}$. On the other hand, if $M = \infty$ then for any $s > 0$ the sequence $n(s) = (n_i(s))_{i\in\mathbb{N}_0}$ is unbounded. It is known that in this case the product $\prod_{i>0} \mathbb{Z}_{n_i(s)}$ contains a free abelian group of an uncountable rank. Thus by (i) the subgroup $\Gamma_s(\mathcal{B})$ and in consequence the whole group $\mathcal{B}$ contains such a free abelian group. □

Corollary 4.1. *There are uncountable many pairwise non-isomorphic groups in the set* $\{\mathcal{B}_{\overline{m}} \colon \overline{m} \in (2\mathbb{N})^{\mathbb{N}_0}\}$.

Proof. Let $P = \{p_1, p_2, \ldots\}$ and $P' = \{p'_1, p'_2, \ldots\}$ be two infinite subsets of the set of all primes such that $P \neq P'$. Let us consider the following branch indexes

$$\overline{m}_P = (2, 2p_1, 2p_2, \ldots), \quad \overline{m}_{P'} = (2, 2p'_1, 2p'_2, \ldots).$$

As a direct consequence of the point (i) of Theorem 4.1 we obtain that the groups $\Gamma_1(\mathcal{B}_{\overline{m}_P})$ and $\Gamma_1(\mathcal{B}_{\overline{m}_{P'}})$ are not isomorphic. In consequence $\mathcal{B}_{\overline{m}_P}$ and $\mathcal{B}_{\overline{m}_{P'}}$ are not isomorphic. □

References

[1] Bartholdi L., Grigorchuk R., Nekrashevych V., *From fractal groups to fractal sets*, in: Fractals in Graz 2001, Trends Math., Vol. 19, Birkhäuser, Basel, 2003, pp. 25–118.

[2] Grigorchuk R., Nekrashevych V., Sushchanskii V., *Automata, dynamical systems and groups*, Proc. Steklov Inst. Math. **231** (2000), 128–203.

[3] Grigorchuk R., *Just infinite branch groups*, in: New Horizons in pro-p Groups, Progr. Math., Vol. 184, Birkhäuser Boston, 2000, pp. 121–179.

[4] Nekrashevych V., *Self-similar Groups*, Math. Surveys Monogr., Vol.117, Amer. Math. Soc., Providence, RI., 2005.

[5] Sidki S., *Regular Trees and their Automorphisms*, Monografias de Matematica, Vol. 56, IMPA, Rio de Janeiro, 1998.

Institute of Mathematics
Silesian University of Technology
Kaszubska 23
44-100 Gliwice
Poland
e-mail: Adam.Woryna@polsl.pl