

ON PSEUDOADDITIVE MAPPINGS

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Abstract. We solve the following functional equation

$$f(x + y + z) + g(x + y) = q(z) + p(y) + h(x),$$

where f, g, q, p, h are unknown functions transforming a semigroup into a group.

Byung Do Kim in a paper [1] posed the following definition: Let G and H be groups. We say that $f, g, h, p, q : G \rightarrow H$ are the pseudoadditive mappings of the mixed quadratic and Pexider type in G if

$$f(x + y + z) + g(x + y) - h(x) - p(y) - q(z) = 0$$

for all $x, y, z \in G$. We will show that functions which solves these equation are the sums of some additive functions and some constants. This makes the above definition very unfortunately. In this note we assume that $(G, +)$ is a semigroup with a neutral element 0 and $(H, +)$ is an arbitrary group (both not necessarily commutative), and the functions f, g, q, p and h transform G into H . Recall that a function $A : G \rightarrow H$ is additive if and only if it fulfils the following Cauchy functional equation

$$A(x + y) = A(x) + A(y), \quad x, y \in G.$$

It is easily seen that we have the following remark.

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REMARK. Let $M : G \rightarrow H$ be a function fulfilling the following equation

$$M(x + y) = M(y) + M(x), \quad x, y \in G.$$

Then $N : G \rightarrow H$ given by the formula $N(x) := -M(x)$, $x \in G$, is an additive function.

Our main result reads as follows.

THEOREM. *The functions $f, g, q, p, h : G \rightarrow H$ forms a solution of the functional equation of the form*

$$(1) \quad f(x + y + z) + g(x + y) = q(z) + p(y) + h(x), \quad x, y, z \in G,$$

if and only if there exist additive functions $A, B : G \rightarrow H$ and constants $a, b, c, d \in H$ such that $f(x) = -B(x) + a$, $g(x) = -a + B(x) + a + b - A(x)$, $q(x) = -B(x) + a + b - c - d$, $p(x) = d + c - A(x) - c$, $h(x) = c - A(x)$.

PROOF. Assume that f, g, q, p and h forms a solution of equation (1) and we put

$$a := f(0), \quad b := g(0), \quad c := h(0), \quad \text{and } d := p(0).$$

Setting in (1) $x = y = 0$ we obtain

$$(2) \quad q(z) = f(z) + b - c - d, \quad z \in G.$$

According to (1) we have

$$f(x + y + z) + g(x + y) = f(z) + b - c - d + p(y) + h(x), \quad x, y, z \in G.$$

Hence

$$(3) \quad p(y) = d + c - b - a + f(y) + g(y) - c, \quad y \in G.$$

By virtue of (1) we get

$$f(x + y + z) + g(x + y) = f(z) - a + f(y) + g(y) - c + h(x), \quad x, y, z \in G.$$

Putting here $y = z = 0$ we obtain

$$(4) \quad h(x) = c - b - a + f(x) + g(x), \quad x \in G.$$

This together with (1) gives

$$f(x+y+z) + g(x+y) = f(z) - a + f(y) + g(y) - b - a + f(x) + g(x), \quad x \in G,$$

which implies (if $z = 0$) that

$$f(x+y) + g(x+y) = f(y) + g(y) - b - a + f(x) + g(x), \quad x, y \in G.$$

Defining function $\varphi : G \rightarrow H$ by the formula

$$\varphi(x) := -b - a + f(x) + g(x), \quad x \in G,$$

observe that

$$(5) \quad \varphi(x+y) = \varphi(y) + \varphi(x), \quad x, y \in G.$$

On account of the definition of φ we have

$$(6) \quad g(x) = -f(x) + a + b + \varphi(x), \quad x \in G.$$

Using (6), (5), (4), (3) and (2) we can rewrite equation (1) to the form

$$f(x+y+z) - f(x+y) + a = f(z), \quad x, y, z \in G.$$

This means that the function $\psi(x) := f(x) - a$, $x \in G$ fulfils the following equation

$$(7) \quad \psi(x+y) = \psi(y) + \psi(x), \quad x, y \in G.$$

It follows from our Remark, (5) and (7) that the functions $A(x) := -\varphi(x)$, $x \in G$, and $B(x) := -\psi(x)$, $x \in G$, are additive and, moreover, using the definitions of φ , ψ , (6), (5), (4), (3) and (2) we infer that

$$\begin{aligned} f(x) &= -B(x) + a, \\ g(x) &= -a + B(x) + a + b - A(x), \\ h(x) &= c - A(x), \\ p(x) &= d + c - A(x), \\ q(x) &= -B(x) + a + b - c - d \end{aligned}$$

for every $x \in G$. This ends the proof of the "necessity" of our assertion.

To the end of the proof it is enough to show that if f, g, q, p, h have the form as above, where A and B are additive functions and a, b, c, d and e are

arbitrary constants then they forms a solution of equation (1). It follows from the definitions of A, B , and from the forms of functions f, g, q, p and h that

$$\begin{aligned} f(x + y + z) + g(x + y) &= \\ &= -B(x + y + z) + a - a + B(x + y) + a + b - A(x + y) \\ &= -B(z) + a + b - A(y) - A(x) \\ &= -B(z) + a + b - c - d + d + c - A(y) - c + c - A(x) \\ &= q(z) + p(y) + h(x) \end{aligned}$$

for all $x, y, z \in G$. The proof of the theorem is finished. \square

References

- [1] Byung Do Kim, *On Hyers-Ulam-Rassias Stability of Functional Equations*, Acta Mathematica Sinica, English Series, Mar., **24** (2008), no. 3, 353–372.

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