WITT EQUIVALENCE OF RINGS OF REGULAR FUNCTIONS

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Abstract. In this paper we show that the rings of regular functions on two
real algebraic curves over the same real closed field are Witt equivalent (i.e.
their Witt rings are isomorphic) if and only if the curves have the same number
of semi-algebraically connected components. Moreover, in the second part of
the paper, we prove that every strong isomorphism of Witt rings of rings of
regular functions can be extended to an isomorphism of Witt rings of fields of
rational functions. This extension is not unique, though.

1. Introduction

The set of similarity classes of nonsingular bilinear forms over a fixed
commutative ring A, equipped with operations induced by the orthogonal sum
and the tensor product, has a natural structure of a ring. This ring is called
the *Witt ring WA* of A. The Witt ring encodes numerous information of its
ground ring. Unfortunately the complete theory of Witt rings is known only
over fields (cf. [9, 15]). The theory of Witt rings over integral domains has
been intensively developed since 1970s by many authors (see e.g. [11]). This
case is far more challenging than the previous one. So far the most progress
has been done for Dedekind domains. The ultimate question in algebraic
theory of quadratic forms is: when the Witt rings of two rings A and B are
isomorphic? If this is the case we say that the rings A, B are *Witt equivalent*.
This problem is difficult even over fields and has been investigated in more

than 40 scientific papers. So far it has been solved only in a very few cases. The three main are: fields having no more than 32 square classes (see [2]), global fields — this area has been most actively investigated in previous years (see e.g. [12, 16, 17]) and fields of rational functions on algebraic curves (see [6, 7, 8]). The pursue for criteria of Witt equivalence of rings has started only recently (see e.g. [13, 14]).

In this paper we cope with the problem of Witt equivalence of rings of regular functions on two smooth complete real curves. We prove (see Proposition 4.1) that two such rings are Witt equivalent if and only if the underlying curves consist of equal numbers of semi-algebraically connected components. If this is the case, the isomorphism of the Witt rings of rings of regular functions is obtained by restricting the tame Harrison map (see [7] for the definition) of their fields of fractions. Moreover, we show (see Theorem 4.4) that every *strong* isomorphism of Witt rings of two rings of regular functions can be extended (in a non-unique way) to an isomorphism of Witt rings of fields of rational functions. The proof of the latter result occupies the subsection 4.1. The paper is organized as follows. In Section 2 we introduce all the necessary terminology and gather the needed tools. In Section 3 we present a number of results concerning the structure of the Witt ring of the ring of regular functions on a real curve. Analogous results have already been presented in [4] for a Witt ring of a real curve, which is a coarser object. Unfortunately, to the best of our knowledge, they have never been explicitly formulated for the Witt ring of a ring of regular functions. Thus, having no convenient source of reference, wee feel obligated to explicate these results in our setup. However, as they can hardly be considered new we omit most of the proofs in this section. Finally, Section 4 constitutes the kernel of this paper and is completely devoted to our main results mentioned above.

2. Preliminaries

Let $\Bbbk$ be a real closed field. It will silently remain fixed throughout this whole paper. The letters K, L will always denote the formally real algebraic function fields of one variable over $\Bbbk$ (in particular $\operatorname{tr.deg}_{\Bbbk} K = \operatorname{tr.deg}_{\Bbbk} L = 1$). Let $\Omega(K)$ be a set of all points of K trivial on $\Bbbk$. The completion of K with respect to a point $\mathfrak{p} \in \Omega(K)$ is denoted by $K_{\mathfrak{p}}$ while its residue field by $K(\mathfrak{p})$. The associated valuation is denoted by $\operatorname{ord}_{\mathfrak{p}}^{K}$. Among all the points $\mathfrak{p} \in \Omega(K)$ we select those having the formally real (hence isomorphic to $\Bbbk$) residue fields. Following [3, 4] we call such points *real* and we write $\gamma(K)$ for the set of all the real points of K. It is a real algebraic curve over $\Bbbk$. The field K can be treated as the field of rational functions on this curve.

On the curve $\gamma(K)$ we consider Euclidean topology (see [1]) induced by the unique ordering of $\Bbbk$ (note that in [3, 4] this topology is called "strong topology"). The curve $\gamma(K)$ consists of a finite number of semi-algebraically connected components $\gamma_1^K, \ldots, \gamma_N^K$. With every real point $\mathfrak{p} \in \gamma(K)$ we associate two orderings $P_+(\mathfrak{p})$ and $P_-(\mathfrak{p})$ of the field K compatible with $\mathfrak{p}$:

$$P_+(\mathfrak{p}) = \left\{ f \in K : \exists_{\mathfrak{p}' \in \gamma(K)} \forall_{\mathfrak{q} \in (\mathfrak{p}, \mathfrak{p}')} f(\mathfrak{q}) > 0 \right\},$$
$$P_-(\mathfrak{p}) = \left\{ f \in K : \exists_{\mathfrak{p}' \in \gamma(K)} \forall_{\mathfrak{q} \in (\mathfrak{p}', \mathfrak{p})} f(\mathfrak{q}) > 0 \right\}.$$

(Note that the left/right neighborhoods $(\mathfrak{p}', \mathfrak{p})$ and $(\mathfrak{p}, \mathfrak{p}')$ of $\mathfrak{p}$ are relative to an orientation of the curve $\gamma(K)$, which we assume to be fixed.) This permits us to introduce the notion of a signature of a square class. Namely for any square class $f \in \dot{K}/\dot{K}^2$ (to simplify the notation we use the same symbol f to denote both an element of the field and its square class) we define:

$$\operatorname{sgn}_{\mathfrak{p}}^K f := \begin{cases} 1, & \text{if } f \in P_+(\mathfrak{p}) \cap P_-(\mathfrak{p}), \\ 0, & \text{if either } f \in P_+(\mathfrak{p}) \cap -P_-(\mathfrak{p}) \text{ or } f \in -P_+(\mathfrak{p}) \cap P_-(\mathfrak{p}), \\ -1, & \text{if } -f \in P_+(\mathfrak{p}) \cap P_-(\mathfrak{p}). \end{cases}$$

(In [3, 4] this function is denoted by $\tau_{\mathfrak{p}}$.) The following observation is immediate, anyway we formulate it explicitly for future references.

OBSERVATION 2.1. $\operatorname{sgn}_{\mathfrak{p}}^K f = 0$ *if and only if* $\operatorname{ord}_{\mathfrak{p}}^K f \equiv 1 \pmod 2$.

Now let $R_K := \{ f \in K : \operatorname{ord}_{\mathfrak{p}}^K f \geq 0 \text{ for every } \mathfrak{p} \in \gamma(K) \}$ be the subring of the field K consisting of all the functions regular on $\gamma(K)$. It is a Dedekind domain, hence its Witt ring WR_K injects into the Witt ring WK of its field of fractions (see [11, Corollary IV.3.3]). In fact we know more:

THEOREM 2.2 ([4, Theorem 11.2]). *If the curve $\gamma(K)$ consists of N semi-algebraically connected components $\gamma_1^K, \ldots, \gamma_N^K$, then the following sequence is exact:*

$$0 \to WR_K \xrightarrow{\; i_K \;} WK \xrightarrow{\; \partial_K \;} \bigoplus_{\mathfrak{p} \in \gamma(K)} WK(\mathfrak{p}) \xrightarrow{\; \lambda_K \;} \mathbb{Z}^N \to 0.$$

Here $i_K \colon WR_K \hookrightarrow WK$ is the canonical injection induced by the inclusion $R_K \subset K$. Next, for every $\mathfrak{p} \in \gamma(K)$ the map $\partial_{\mathfrak{p}} \colon WK \to WK(\mathfrak{p}) \cong \mathbb{Z}$ is the second residue homomorphism associated with $\mathfrak{p}$ and ∂_K denotes the

compound map. Finally $\bigoplus WK(\mathfrak{p}) \cong \mathbb{Z}^{(\gamma(K))}$ *and* $\lambda_K : \mathbb{Z}^{(\gamma(K))} \to \mathbb{Z}^N$ *is the epimorphism defined by*

$$\lambda_K\big((a_\mathfrak{p})_{\mathfrak{p}\in\gamma(K)}\big) := \big(\sum_{\mathfrak{p}\in\gamma_1^K} a_\mathfrak{p}, \ldots, \sum_{\mathfrak{p}\in\gamma_N^K} a_\mathfrak{p}\big).$$

In the rest of this paper we tend to identify WR_K with its image under the canonical injection $WR_K \hookrightarrow WK$ and so we conveniently treat WR_K as the subring of WK. Define the subgroup $\mathbb{E}_K < \dot{K}/\dot{K}^2$ by

$$\mathbb{E}_K := \big\{f \in \dot{K}/\dot{K}^2 : \mathrm{ord}_\mathfrak{p}^K f \equiv 0 \pmod 2 \text{ for all } \mathfrak{p} \in \gamma(K)\big\}.$$

Observation 2.1 allows us to rewrite this condition in the following way:

$$\mathbb{E}_K := \big\{f \in \dot{K}/\dot{K}^2 : \mathrm{sgn}_\mathfrak{p}^K f \neq 0 \text{ for all } \mathfrak{p} \in \gamma(K)\big\}.$$

Consequently, the signature of the square class f belonging to $\mathbb{E}_K$ is constant on every semi-algebraically connected component γ_i^K. We denote it by $\mathrm{sgn}_i^K f$ (see also Proposition 3.1 below). Now, by Theorem 2.2, a unary form $\langle f\rangle$ lies in WR_K if and only if it belongs to the kernel of ∂_K and this is the case if and only if f is the class of a unit at the completion $K_\mathfrak{p}$ for every $\mathfrak{p} \in \gamma(K)$. This, in turn, means that f belongs to $\mathbb{E}_K$. So we have:

COROLLARY 2.3. *Let* $f \in \dot{K}/\dot{K}^2$ *be a square class, then* $f \in \mathbb{E}_K$ *if and only if the unary form* $\langle f\rangle$ *belongs to* WR_K, *i.e.*

$$f \in \mathbb{E}_K \iff \langle f\rangle \in WR_K.$$

The above corollary suggests that it may be fruitful to investigate the subset of WR_K consisting of all the classes of unary forms. Thus, we define

$$\langle\mathbb{E}_K\rangle := \big\{\langle f\rangle \in WK : f \in \mathbb{E}_K\big\} \subset WR_K.$$

Notice that $\langle\mathbb{E}_K\rangle$ is closed under multiplication but not under addition hence it is not a subgroup of the "Witt group" WR_K.

Now, fix a single point $\mathfrak{p}_i$ in every component γ_i^K of $\gamma(K)$. Recall (cf. [3, §6]) that for every two distinct points $\mathfrak{p}, \mathfrak{q}$ belonging to the same component γ_i^K there exists an element $\chi_{(\mathfrak{p},\mathfrak{q})}$ of K such that $\chi_{(\mathfrak{p},\mathfrak{q})}$ is definite on $\gamma \setminus \{\mathfrak{p}, \mathfrak{q}\}$, positive definite on $\gamma \setminus \gamma_i$ and fulfills $\partial_\mathfrak{p}\chi_{(\mathfrak{p},\mathfrak{q})} = -1$, $\partial_\mathfrak{q}\chi_{(\mathfrak{p},\mathfrak{q})} = 1$. Following [3] we call $\chi_{(\mathfrak{p},\mathfrak{q})}$ an *interval function* for the pair $(\mathfrak{p},\mathfrak{q})$. An interval function is unique only upto multiplication by a totally positive element (see [3, §6]). Hence, in what follows, for every $\mathfrak{p}, \mathfrak{p}_i \in \gamma_i^K$, we assume that $\chi_{(\mathfrak{p},\mathfrak{p}_i)}$ is an arbitrarily chosen and fixed interval function associated with the pair $(\mathfrak{p}, \mathfrak{p}_i)$.

The group of square classes $\dot{K}/\dot{K}^2$ may be treated as an $\mathbb{F}_2$-vector space. The subgroup $\mathbb{E}_K$ is its subspace. We identify its completion:

LEMMA 2.4. *The $\mathbb{F}_2$-vector space $\dot{K}/\dot{K}^2$ decomposes into*

$$\dot{K}/\dot{K}^2 = \mathbb{E}_K \oplus \lim_{\mathbb{F}_2}\{\chi_{(\mathfrak{p},\mathfrak{p}_i)} \in \dot{K}/\dot{K}^2 : \mathfrak{p} \in \gamma_i^K, 1 \le i \le N\}.$$

PROOF. Take any square class $f \in \dot{K}/\dot{K}^2$ and let

$$\mathfrak{p}_{1,1}, \ldots, \mathfrak{p}_{1,n_1}, \mathfrak{p}_{2,1}, \ldots, \mathfrak{p}_{2,n_2}, \ldots, \mathfrak{p}_{N,n_N}, \qquad \mathfrak{p}_{i,j} \in \gamma_i^K,$$

be all the points of $\gamma(K)$ where f has an odd valuation and so changes sign. Consider now a square class $\hat{f} \in \lim_{\mathbb{F}_2}\{\chi_{(\mathfrak{p},\mathfrak{p}_i)} \in \dot{K}/\dot{K}^2 : \mathfrak{p} \in \gamma_i^K, 1 \le i \le N\}$ given by the condition

$$\hat{f} := \prod_{\substack{1 \le i \le N \\ 1 \le j \le n_i \\ \mathfrak{p}_{i,j} \neq \mathfrak{p}_i}} \chi_{(\mathfrak{p}_{i,j}, \mathfrak{p}_i)}.$$

Then $\hat{f}$ changes sign precisely at the same points as f does. Hence the product $f \cdot \hat{f}$ has the constant sign on every semi-algebraically connected component. Thus $f \cdot \hat{f} \in \mathbb{E}_K$. $\qquad\square$

Of course the notion of a real curve, semi-algebraically connected components, local signatures, ... etc. can be—in the same manner as above—defined over the other field, which we denoted L. Therefore $\Omega(L)$, $\gamma(L)$, γ_i^L, $L_\mathfrak{p}$, $L(\mathfrak{p})$, $\mathrm{ord}_\mathfrak{p}^L$, $\mathrm{sgn}_\mathfrak{p}^L$, WR_L, $\mathbb{E}_L$ are the L-counterparts of the objects defined above for K. If it is clear from the context which field we discuss, we tend to omit the letters K and L. All the terminology and notation used in this paper and not introduced so far is standard and follows the convention established by [1, 3, 4, 9, 15]. As it was mentioned before, in order to simplify the notation, we use the same symbol to denote both an element of the field and its square class. Likewise, we use the same symbol for a quadratic form and its Witt class. Throughout all this paper an orientation of $\gamma(K)$ (resp. $\gamma(L)$) is arbitrarily chosen and fixed. Intervals on both curves are silently defined with respect to this fixed orientation.

3. Structure of the Witt group WR

We discuss here the structure of the Witt group WR of the ring R of regular functions on γ. All the results presented in this section are fully analogous to the ones presented in [4]. In [4], however, they were formulated and proved for a coarser group. Thus, we feel obligated to state all of the results explicitly in our different set-up. On the other hand, since the proofs are completely analogous, we feel free to reduce some of them to only short sketches and to omit the rest, giving instead the references to the original theorems.

PROPOSITION 3.1 ([4, Proposition 10.3]). *Let φ be an element of the Witt ring WK. If $\varphi \in WR$, then φ has a constant signature on every semi-algebraically connected component of γ.*

For φ denote by $\operatorname{sgn}_i \varphi$ the signature of φ on γ_i $(1 \le i \le N)$.

PROPOSITION 3.2 ([4, Theorem 10.4 (i)]). *Every element φ of WR is uniquely determined by its discriminant $\operatorname{disc} \varphi$ and its signatures $\operatorname{sgn}_1, \dots, \operatorname{sgn}_N$ on the components $\gamma_1, \dots, \gamma_N$ of γ.*

Consider now a subset $S \subseteq WR$ defined

$$S := \big\{ \langle 1, -f \rangle : f \in \Sigma K^2 \big\}.$$

Clearly S is a subgroup of WR. Moreover, $2 \cdot S = \{0\}$. We claim that S is the nilradical of WR.

PROPOSITION 3.3 ([4, Theorem 10.4 (ii)]). $S = \operatorname{Nil} WR$.

SKETCH OF THE PROOF. The inclusion $S \subseteq \operatorname{Nil} WR$ follows from the fact that:

$$\operatorname{Nil} WR = WR \cap \operatorname{Nil} WK = \big\{ \varphi \in WR : \operatorname{sgn}_i \varphi = 0 \text{ for } 1 \le i \le N \big\}.$$

As for the other inclusion, take any $\varphi \in \operatorname{Nil} WR$. Let $\langle f_1, \dots, f_n \rangle$ be a diagonalization of φ over the field K. Let further f be the discriminant of φ. For every $1 \le i \le N$ we have $\operatorname{sgn}_i \varphi = 0$ and hence at almost every point $\mathfrak{p} \in \gamma$ exactly half of f_j's are negative, the other half is positive. Consequently, $\operatorname{sgn}_\mathfrak{p} f = 1$ at every $\mathfrak{p} \in \gamma$. Thus the Witt classes of φ and $\langle 1, -f \rangle$ over R are equal, but clearly $\langle 1, -f \rangle \in S$. $\square$

Fix now $z_1, \ldots, z_N \in \mathbb{E}$ such that $\operatorname{sgn}_i z_i = -1$ and $\operatorname{sgn}_j z_i = 1$ for $j \neq i$.

PROPOSITION 3.4 ([4, Theorem 10.4 (iii)]). *The fundamental ideal $IR \lhd WR$ decomposes into*

$$IR = \bigoplus_{i=1}^{N} \mathbb{Z}[\langle 1, -z_i \rangle] \oplus \operatorname{Nil} WR.$$

SKETCH OF THE PROOF. Take any $\varphi \in IR$. If $\operatorname{sgn}_i \varphi = 0$ for all $1 \leq i \leq N$ then $\varphi \in \operatorname{Nil} WR$. Thus, assume that not all of $\operatorname{sgn}_i \varphi$ are null. We have $\operatorname{sgn}_i \varphi \equiv \dim \varphi \pmod 2$ and so $\operatorname{sgn}_i \varphi$ is even for $1 \leq i \leq N$. Take now $\hat{\varphi}_1, \ldots, \hat{\varphi}_N$ defined by the formula

$$\hat{\varphi}_i := (\operatorname{sgn}_i \varphi / 2) \cdot \langle 1, -z_i \rangle.$$

Let further $\hat{\psi} := \hat{\varphi}_1 + \cdots + \hat{\varphi}_N$. For almost every $\mathfrak{p} \in \gamma_i$ $(1 \leq i \leq N)$ we have

$$\operatorname{sgn}_{\mathfrak{p}} \operatorname{disc} \varphi = (-1)^{\operatorname{sgn}_i \varphi} \operatorname{sgn}_{\mathfrak{p}} \operatorname{disc} \hat{\psi} = \operatorname{sgn}_{\mathfrak{p}} \operatorname{disc} \hat{\psi}.$$

Therefore $\operatorname{disc} \varphi = g \cdot \operatorname{disc} \hat{\psi}$, for some $g \in \Sigma K^2$. Thus, by the Witt theorem [4, Theorem 9.5] we get

$$\varphi = \hat{\psi} + \langle 1, -g \rangle \in \bigoplus_{i=1}^{N} \mathbb{Z}[\langle 1, -z_i \rangle] \oplus \operatorname{Nil} WR. \qquad \square$$

COROLLARY 3.5. *The Witt ring WR is generated (as a ring) by the set $\{\langle f \rangle : f \in \mathbb{E}\}$.*

4. Main results

We are now ready to present our main results. Consider again two formally real algebraic function fields K, L both having the same real closed field of constants $\Bbbk$.

PROPOSITION 4.1. *Let $\gamma(K)$, $\gamma(L)$ be two non-empty, smooth, complete real curves over a common real closed field $\Bbbk$. Then the rings R_K, R_L of functions regular on $\gamma(K)$, $\gamma(L)$ are Witt equivalent if and only if $\gamma(K)$ and $\gamma(L)$ have the same number of semi-algebraically connected components.*

PROOF. Let $\Phi\colon WR_K \to WR_L$ be an isomorphism. Then Φ maps the nil-radical of R_K onto the nil-radical of R_L and preserves the rank of free $\mathbb{Z}$-modules. Decomposing now IR_K and IR_L according to Proposition 3.4 we see that $\gamma(K)$ and $\gamma(L)$ must have the same number of semi-algebraically connected components.

To prove the opposite implication, suppose that $\gamma(K)$, $\gamma(L)$ have the same number N of semi-algebraically connected components. Every component is Nash-diffeomorphic to a circle, thus there is a homeomorphism $T\colon \gamma(K) \to \gamma(L)$ mapping components of $\gamma(K)$ onto the components of $\gamma(L)$. Using [7, Corollary 3.9] we may find a *tame Harrison map* $t\colon \dot{K}/\dot{K}^2 \to \dot{L}/\dot{L}^2$, i.e. an isomorphism of square class groups such that

(a) t preserves minus one: $t(-1) = -1$;
(b) for any square classes $f, g \in \dot{K}/\dot{K}^2$ the form $\langle f, g\rangle$ represents 1 over K if and only if $\langle \hat{t}f, \hat{t}g\rangle$ represents 1 over L;
(c) t maps 1-pt fans of orderings of K onto 1-pt fans of L.

Now [5, Proposition 3.4] asserts that the isomorphism $\Psi_t\colon WK \xrightarrow{\approx} WL$ of Witt rings of fields K, L, given by $\Psi_t\langle f_1, \ldots, f_n\rangle := \langle tf_1, \ldots, tf_n\rangle$, factors over WR_K:

$$\Psi_t\big|_{WR_K} WR_K \xrightarrow{\approx} WR_L. \qquad \square$$

It is worth to note that in the above proof we have shown that the isomorphism of Witt rings $WR_K \cong WR_L$ of rings of regular function on real curves implies the isomorphism of associated exact sequences. Indeed, consider Knebusch–Milnor exact sequences associated to WR_K and WR_L:

$$\mathcal{S}_K\colon \quad 0 \to WR_K \xrightarrow{i_K} WK \xrightarrow{\oplus\partial_K} \bigoplus WK_{\mathfrak{p}} \xrightarrow{\lambda_K} \mathbb{Z}^N \to 0,$$
$$\mathcal{S}_L\colon \quad 0 \to WR_L \xrightarrow{i_L} WL \xrightarrow{\oplus\partial_L} \bigoplus WL_{\mathfrak{q}} \xrightarrow{\lambda_L} \mathbb{Z}^M \to 0.$$

It follows that if either ends of above sequences are isomorphic (i.e. either $WR_K \cong WR_L$ or $N = M$) then the whole sequences are isomorphic, as well.

COROLLARY 4.2. *Under the above assumptions, the following conditions are equivalent:*
- $N = M$;
- $WR_K \cong WR_L$;
- $\mathcal{S}_K \cong \mathcal{S}_L$.

It is worth to stress the point that Witt equivalence of rings of regular functions on two real curves depends solely on the number of semi-algebraically connected components of these two curves and not on the relative positions
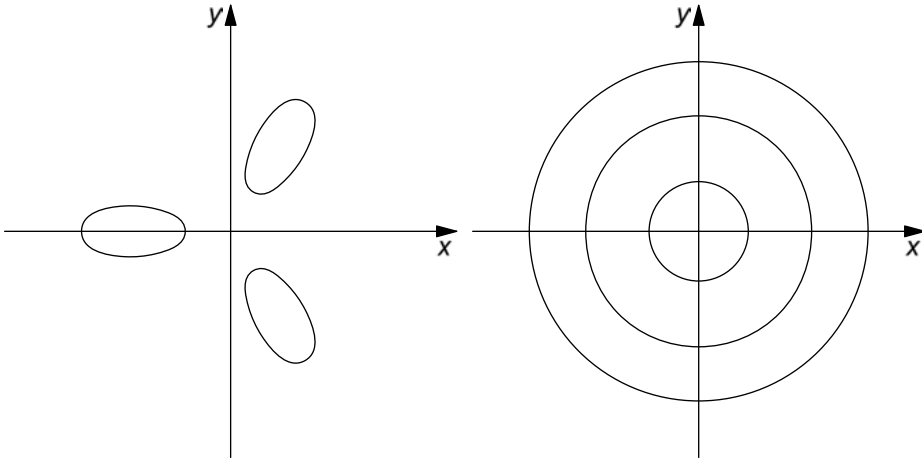
Figure 1. Curves with Witt equivalent rings of regular functions defined by Eq. (4.3)

of these components, neither directly on degrees of these curves nor on their genuses. To see this phenomenon clearly, consider two curves (illustrated in Figure 1) defined by the following polynomials (found using [10]):

$$C_0 = \big(y^2 + x(x+1)\big)\big(x^2 + y^2\big) - \frac{x^2 + y^2 - 2}{100} - 4xy^2,$$

$$(4.3) \qquad C_1 = \big(y^2 + x^2 - 9\big)\Big(\frac{y^2 + x^4 - 16}{20} + \big(y^2 + x^2 - 4\big)\big(y^2 + x^2 - 1\big)\Big)$$

$$+ \frac{y^2 + x^4 - 25}{100}.$$

Both curves consist of three components, hence Proposition 4.1 asserts that the rings of regular functions on these two curves are Witt equivalent. On the other hand, the configuration of the components of the first curve is $1 \amalg 1 \amalg 1$ while the second curve has a configuration $1\langle 1\langle 1 \rangle\rangle$. Likewise, the degrees of the two curves differ:

$$\deg C_0 = 4 \neq 6 = \deg C_1.$$

Finally, since both curves are smooth and their degrees differ by more than one, the genuses of these two curves are different, as well.

Proposition 4.1 is existential in nature—the Witt equivalence of rings of regular functions implies that the curves have the same number of semi-algebraically connected components and this in turn, as we have shown in the proof of the theorem, implies the existence of an isomorphism Ψ_t of Witt rings WK, WL of the fields, which factors over the Witt rings WR_K, WR_L of rings of regular functions. The proposition does not say, however, if the

restriction $\Psi_t \big|_{WR_K}$ is identical to the original isomorphism Φ or whether the two isomorphisms are at least correlated in any way. It is, thus, natural to ask the following question:

> *Can the isomorphism of the Witt rings WR_K, WR_L of rings of regular functions be extended to an isomorphism of Witt rings WK, WL of their quotient fields?*

We do not know the answer to this question in such a generality. However, if we assume that the isomorphism $WR_K \xrightarrow{\cong} WR_L$ is *strong* (in a sense which we will promptly define) the answer turns out to be affirmative.

Recall that the isomorphism of Witt rings of two *fields* is called strong if it maps classes of unary forms onto classes of unary forms. In our case, when we deal with projective modules, the notion of dimension of the form is a bit fuzzy. But the following notion seems to be justified by Corollary 2.3. We shall say that the isomorphism $\Phi : WR_K \xrightarrow{\cong} WR_L$ is *strong* if it maps $\langle \mathbb{E}_K \rangle$ onto $\langle \mathbb{E}_L \rangle$.

THEOREM 4.4. *If an isomorphism $\Phi : WR_K \xrightarrow{\cong} WR_L$ is strong, then there exists a strong isomorphism $\Psi_t : WK \xrightarrow{\cong} WL$ extending Φ (i.e. $\Psi_t \big|_{WR_K} \equiv \Phi$).*

4.1. Proof of Theorem 4.4

Let $\Phi \colon WR_K \xrightarrow{\cong} WR_L$ be a strong isomorphism. Using Proposition 4.1 we see that $\gamma(K)$ and $\gamma(L)$ have the same number of semi-algebraically connected components. Denote this number by N. The isomorphism Φ is assumed to map $\langle \mathbb{E}_K \rangle$ onto $\langle \mathbb{E}_L \rangle$ and so the condition

$$tf := g \iff \Phi\langle f \rangle = \langle g \rangle$$

defines a group isomorphism $t \colon \mathbb{E}_K \to \mathbb{E}_L$. Observe that Φ, being an isomorphism of rings, maps nilradical $\mathrm{Nil}\, WR_K$ onto the nilradical $\mathrm{Nil}\, WR_L$. It follows from Proposition 3.3 that t maps $\Sigma K^2 \subseteq \mathbb{E}_K$ onto $\Sigma L^2 \subseteq \mathbb{E}_L$. Let $z_1, \dots, z_N \in \mathbb{E}_K$ be fixed in the same way as in Proposition 3.4. The following lemma shows that they are mapped to their counterparts in $\mathbb{E}_L$.

LEMMA 4.5. *For every $1 \le i \le N$ there is $1 \le j \le N$ such that*

$$\mathrm{sgn}_k^L\, t z_i = \begin{cases} -1, & \text{if } k = j, \\ 1, & \text{if } k \ne j. \end{cases}$$

PROOF. Take a form $\langle 1, -z_i \rangle$. It belongs to IR_K and so $\Phi\langle 1, -z_i \rangle \in IR_L$. Thus, using Proposition 3.4, we have

$$\Phi\langle 1, -z_i \rangle = k_{i,1}\langle 1, -z_i'' \rangle + \cdots + k_{i,N}\langle 1, -z_N'' \rangle + \varepsilon\langle 1, -g \rangle,$$

where $j_{i,1}, \ldots, k_{i,N} \in \mathbb{Z}$, $z_1'', \ldots, z_N'' \in \mathbb{E}_L$ are fixed in the same way as in Proposition 3.4 but for the curve γ_L this time, i.e.:

$$\operatorname{sgn}_k^L z_j'' = \begin{cases} -1, & \text{if } k = j, \\ 1, & \text{if } k \neq j. \end{cases}$$

Further $\varepsilon \in \{0, 1\}$ and g is a sum of squares. First we show that every $k_{i,j}$ is either zero or one. Indeed, square the form $\langle 1, -z_i \rangle$ in WR_K. We have $\langle 1, -z_i \rangle^2 = 2\langle 1, -z_i \rangle$. Now Φ, as a *ring* homomorphism, preserves multiplication. Thus we obtain:

$$\begin{aligned} 2k_{i,1}\langle 1, -z_i'' \rangle + \cdots + 2k_{i,N}\langle 1, -z_N'' \rangle &= \Phi\big(2\langle 1, -z_i \rangle\big) \\ &= \Phi\big(\langle 1, -z_i \rangle^2\big) \\ &= 2k_{i,1}^2\langle 1, -z_i'' \rangle + \cdots + 2k_{i,N}^2\langle 1, -z_N'' \rangle. \end{aligned}$$

Therefore $k_{i,j}^2 = k_{i,j}$ for every $1 \leq j \leq N$ and so $k_{i,j} \in \{0, 1\}$.

Now, we show that all but one $k_{i,j}$'s are null. Suppose otherwise. Let for some $1 \leq i_1 \leq N$ the image $\Phi\langle 1, -z_{i_1} \rangle$ has two non-zero coordinates in the free $\mathbb{Z}$-module $\mathbb{Z}[\langle 1, -z_1'' \rangle] \oplus \cdots \oplus \mathbb{Z}[\langle 1, -z_N'' \rangle]$, then using Dirichlet's pigeonhole principle there is another element $\langle 1, -z_{i_2} \rangle$ such that $\Phi\langle 1, -z_{i_2} \rangle$ has at least one the same non-zero coordinate. Let for example

$$\begin{aligned} \Phi\langle 1, -z_{i_1} \rangle &= \langle 1, -z_a'' \rangle + \text{other terms}, \\ \Phi\langle 1, -z_{i_2} \rangle &= \langle 1, -z_a'' \rangle + \text{other terms}. \end{aligned}$$

Then

$$\begin{aligned} 0 = \Phi(0) = \Phi\big(\langle 1, -z_{i_1} \rangle \cdot \langle 1, -z_{i_2} \rangle\big) &= \Phi\langle 1, -z_{i_1} \rangle \cdot \Phi\langle 1, -z_{i_2} \rangle \\ &= 2\langle 1, -z_a'' \rangle + \text{other terms} \neq 0. \end{aligned}$$

This contradiction shows that indeed every $\langle 1, -z_i \rangle$ is mapped onto $\langle 1, -z_j'' \rangle + \langle 1, -g \rangle$ for some $1 \leq j \leq N$ and $g \in \Sigma L^2$ (and for different i's, the corresponding j's differ too). Now, we have

$$0 = \langle 1, -z_j'' \rangle \cdot \langle 1, -g \rangle = \langle 1, 1, -z_j'', -g \rangle - \langle 1, -z_j'' g \rangle.$$

Therefore

$$\langle 1, -z_j'' g\rangle = \langle 1, -z_j''\rangle + \langle 1, -g\rangle = \Phi\langle 1, -z_i\rangle,$$

and so, by the definition of t, we have $tz_i = z_j'' g$. $\square$

It follows from the above lemma that t induces a permutation $i \mapsto j =: \tau(i)$ of the set $\{1, \ldots, N\}$. We may treat it as the bijection of $\gamma_i^K \mapsto \gamma_{\tau(i)}^L$ of the sets of components of $\gamma(K)$ and $\gamma(L)$. We may find a homeomorphism $T: \gamma(K) \to \gamma(L)$ such that $T\gamma_i^K = \gamma_{\tau(i)}^L$. Fix a single point $\mathfrak{p}_i$ in every component γ_i^K of $\gamma(K)$ and let $\mathfrak{q}_{\tau(i)} := T\mathfrak{p}_i \in \gamma_{\tau(i)}^L$. We may now define an $\mathbb{F}_2$-linear isomorphism $\hat{t}: \dot{K}/\dot{K}^2 \xrightarrow{\cong} \dot{L}/\dot{L}^2$ of an $\mathbb{F}_2$-vector spaces that extends t. Recall that by Lemma 2.4 we have a decomposition

$$\dot{K}/\dot{K}^2 = \mathbb{E}_K \oplus \mathrm{lin}_{\mathbb{F}_2}\{\chi_{(\mathfrak{p},\mathfrak{p}_i)} \in \dot{K}/\dot{K}^2 : \mathfrak{p} \in \gamma_i^K, 1 \le i \le N\},$$

likewise for $\dot{L}/\dot{L}^2$:

$$\dot{L}/\dot{L}^2 = \mathbb{E}_L \oplus \mathrm{lin}_{\mathbb{F}_2}\{\chi_{(\mathfrak{p},\mathfrak{p}_i)} \in \dot{L}/\dot{L}^2 : \mathfrak{p} \in \gamma_i^L, 1 \le i \le N\}.$$

Let $\hat{t}\big|_{\mathbb{E}_K} := t$ and define $\hat{t}$ on the basis of the other summand by the condition:

$$\hat{t}\chi_{(\mathfrak{p},\mathfrak{p}_i)} := \chi_{(T\mathfrak{p},\mathfrak{q}_i)}.$$

By linearity this defines $\hat{t}$ on the whole $\mathbb{F}_2$-vector space $\dot{K}/\dot{K}^2$. Clearly $\hat{t}$ is a group isomorphism and $\hat{t}$ preserves local signatures in the sense that:

$$(4.6) \qquad\qquad \forall_{\mathfrak{p}\in\gamma(K)}\forall_{f\in\dot{K}/\dot{K}^2} \, \mathrm{sgn}_{\mathfrak{p}}^K f = \mathrm{sgn}_{T\mathfrak{p}}^L \hat{t}f.$$

Consequently, $\hat{t}$ preserves the parity of valuation, as well

$$\forall_{\mathfrak{p}\in\gamma(K)}\forall_{f\in\dot{K}/\dot{K}^2} \, \mathrm{ord}_{\mathfrak{p}}^K f \equiv \mathrm{ord}_{T\mathfrak{p}}^L \hat{t}f \pmod 2.$$

We claim that $\hat{t}$ is a Harrison map. Indeed, $\Phi\langle -1\rangle = \Phi(-\langle 1\rangle) = \langle -1\rangle$ and so $\hat{t}(-1) = t(-1) = -1$. For the second condition, take $f, g \in \dot{K}/\dot{K}^2$ such that 1 is represented by $\langle f, g\rangle$ over K. This means that the form $\varphi := \langle 1, -f, -g, fg\rangle$ is hyperbolic over K and so it is hyperbolic over every completion $K_{\mathfrak{p}}$ for $\mathfrak{p} \in \gamma(K)$. Take $\mathfrak{p} \in \gamma(K)$ such that neither f nor g changes sign at $\mathfrak{p}$-almost every $\mathfrak{p}$ will do. The form φ is hyperbolic in $K_{\mathfrak{p}}$ if and only if $\mathrm{sgn}_{\mathfrak{p}}^K f = -\mathrm{sgn}_{\mathfrak{p}}^K g$. It follows from Eq. (4.6) that $\mathrm{sgn}_{\mathfrak{q}}^L \hat{t}f = -\mathrm{sgn}_{\mathfrak{q}}^L \hat{t}g$ for almost every $\mathfrak{q} \in \gamma(L)$. Thus, by the means of the Witt theorem (cf. [4, Theorem 9.5]) the form

$\langle 1, -\hat{t}f, -\hat{t}g, \hat{t}f\hat{t}g \rangle$ is hyperbolic over L. Consequently $\langle \hat{t}f, \hat{t}g \rangle$ represents 1 over L. This proves the claim. It is well known that if $\hat{t}$ is a Harrison map, then the mapping $\langle f_1, \ldots, f_n \rangle \mapsto \langle \hat{t}f_1, \ldots, \hat{t}f_n \rangle$ is a strong isomorphism of Witt rings of fields. Denote it Ψ_t. It is clear from the construction, that $\Psi_t \big|_{WR_K} \equiv \Phi$. $\qquad\qquad\qquad\qquad\qquad\square$

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