

ON S -LENGTH OF GROUPS

OLGA MACEDOŃSKA, ALEKSANDRA POTYKA

Abstract. Let G be a group and S be a subsemigroup in G , generating G as a group. Every element in G is a product of elements from $S \cup S^{-1}$. An equality $G = S^{-1}S \cdots S^{-1}S$ allows to define an S -length $l(G)$ of the group G . The note concerns the problem posed by J. Krempa on possible values of $l(G)$. We show that for collapsing groups, supramenable groups and groups of a subexponential growth $l(G) \leq 2$. The S -length of a relatively free group can be equal to 1 or 2 or infinity, but it can not be equal to 3. The problem concerning other values is open.

1. Introduction

In the process of algebraic classifications in group theory there are used different parameters of groups defined by so called length or width functions. The notion of S -length of a group was suggested by J. Krempa in 1998 and some conjectures were made on its value.

Let S be a subsemigroup in a group G , generating G as a group. If G is a finite group, it is clear that $G = S = S^{-1}$. If G is an abelian group then $G = S^{-1}S = SS^{-1}$, that is G is a group of fractions of S (see e.g. [1]). A natural question arises whether for each $k > 2$ there is a group such that $G = S^{-1}S \cdots S^{-1}S$ with no less than k factors. The aim of this note is to describe the groups of S -length equal to 1 and 2. We also give an example of a group with S -length greater or equal to 3 and show that the S -length of a relatively free group can not be equal to 3.

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If G is a group and A is a subset in G then $\text{sgp}(A)$ will denote the sub-semigroup of G generated by A . By $|A|$ we denote the cardinality of A .

2. S -length of a group

Let G be a finitely generated group and n be the smallest number of generators in G . A subsemigroup $S \subseteq G$ generated by any n -element set of generators in G will be called a *base semigroup* of G .

The length of G with respect to a base semigroup S , denoted by $l(S, G)$ is the smallest natural k (if exists) such that

$$(1) \quad G = \underbrace{S^{-1}S S^{-1}S \dots S}_{k}^{(-1)^k} = \underbrace{S S^{-1}S S^{-1} \dots S}_{k}^{(-1)^{k-1}}.$$

If such a k does not exist we assume $l(S, G) = \infty$.

DEFINITION. Let $X(G)$ denote the set of all base semigroups in G . Then the S -length of G is defined as

$$l(G) = \sup\{l(S, G) : S \in X(G)\}.$$

EXAMPLE 1. Let $G = \langle a \rangle_2 * \langle b \rangle_3$ be the free product of finite cyclic groups of orders 2 and 3. Let $S_1 = \text{sgp}(a, b)$ and $S_2 = \text{sgp}(ab, ab^2)$. It is not difficult to see that $l(S_1, G) = 1$ (see also Example 4), while $l(S_2, G) \neq 1$, because each element in S_2 begins with a , so $S_2 \neq G$. We can see also that $a \notin S_2 S_2^{-1}$ and $a = (ab)(ab^2)^{-1}(ab) = (ab)^{-1}(ab^2)(ab)^{-1} \in S_2 S_2^{-1} S_2 \cap S_2^{-1} S_2 S_2^{-1}$, hence we can conclude that $l(G) \geq 3$.

EXAMPLE 2. If G is a periodic group (e.g. a finite group) then $l(G) = 1$.

EXAMPLE 3. If F is a free noncyclic group and S is generated by a set $\{a, b, \dots\}$ of free generators of F then for every k , $l(S, F) > k$, because the word $(ab^{-1})^k$ is not equal to any other word and needs $2k$ factors in (1). So we conclude that $l(F) = \infty$.

A word $w(x_1, \dots, x_n)$ is called *positive* if it is written without inverses of x_i 's. We give now necessary and sufficient conditions for a group G and a base semigroup $S \subseteq G$ to have $l(S, G) = 1$.

PROPOSITION 1. *Let $S = \text{sgp}(a_1, a_2, \dots, a_n)$ be a base semigroup in a group G . $l(S, G) = 1$ if and only if G has a defining relation $r(a_1, a_2, \dots, a_n) = 1$, where r is a positive word, containing each generator.*

PROOF. Let F be a free group and \mathcal{F} be a free subsemigroup in F , both generated by the set $X = \{x_1, x_2, \dots, x_n\}$. By assumption G has a presentation $F/N = \langle x_1, x_2, \dots, x_n \mid N \rangle$ for some $N \triangleleft F$, and the natural homomorphism $F \rightarrow G \cong F/N$ maps $\mathcal{F} \rightarrow S$.

If $l(S, G) = 1$, then $G = S = S^{-1}$, and $F = \mathcal{F}N = \mathcal{F}^{-1}N$. Hence $(x_1x_2 \cdots x_n)^{-1} \in F = \mathcal{F}N$. So the word $(x_1x_2 \cdots x_n)^{-1}$ is in a coset sN for some positive word $s = s(x_1, \dots, x_n) \in \mathcal{F}$. Thus the required defining relator is $r := x_1x_2 \cdots x_n \cdot s$ and the relation in G is $a_1a_2 \cdots a_n \cdot s(a_1, \dots, a_n) = 1$.

Conversely, let $G \cong F/N$. If $r \in N$ is a positive word containing each generator then for each fixed x_i we can write r as ux_iv for some $u, v \in \mathcal{F}$. Conjugation by u gives the defining relator $x_ivu \in N$, which implies that $x_i^{-1} \in vuN \subseteq \mathcal{F}N$. Hence $\mathcal{F}^{-1} \subseteq \mathcal{F}N$, $F = \mathcal{F}N$ and $G = S$. Since $G = G^{-1}$, we have $G = S = S^{-1}$ and $l(S, G) = 1$. □

EXAMPLE 4. In Example 1, the group $G = \langle a \rangle_2 * \langle b \rangle_3$, with a base semigroup $S_1 = \text{sgp}(a, b)$ has the defining relation $a^2b^3 = 1$. So $l(S_1, G) = 1$, however $l(G) \neq 1$.

We recall that a semigroup S satisfies Ore conditions if for every $g, h \in S$ there exist $g', h' \in S$ such that $gg' = hh'$ (right Ore condition) and there exist $g'', h'' \in S$ such that $g''g = h''h$ (left Ore condition) (for definition in different terminology see [1, §1.10]). We give now necessary and sufficient conditions for a group G and a base semigroup $S \subseteq G$ to have $l(S, G) \leq 2$.

PROPOSITION 2. *Let G be a group with a base semigroup S . $l(S, G) \leq 2$ if and only if S satisfies Ore conditions.*

PROOF. If $l(S, G) = 2$, then $G = SS^{-1} = S^{-1}S$ and for every $g, h \in S$, $h^{-1}g \in G = SS^{-1}$, which implies that there exist $g', h' \in S$ such that $h^{-1}g = h'g'^{-1}$ and the right Ore condition $gg' = hh'$ follows. The left Ore condition follows similarly from the equality $G = S^{-1}S$.

Conversely, the right Ore condition implies that for every $g, h \in S$ there exist $g', h' \in S$ such that $h^{-1}g = h'g'^{-1}$ which gives $S^{-1}S \subseteq SS^{-1}$. The opposite inclusion follows from the left Ore condition, so $SS^{-1} = S^{-1}S$. Each element of G is in a finite product of the form $SS^{-1}SS^{-1} \dots S^{(-1)^{k-1}}$ with k , say, factors. If $k = 3$ then by the equality $S^{-1}S = SS^{-1}$ we have $SS^{-1}S = S(SS^{-1}) = SS^{-1}$. Similarly, for any k we can show by induction that $G = S^{-1}S = SS^{-1}$. Hence $l(S, G) \leq 2$. □

THEOREM 1. *A group G without free noncyclic subsemigroup has $l(G) \leq 2$.*

PROOF. Let g, h be elements in G . Since by assumption the semigroup $sgp(g, h)$ is not free, there are two equal nontrivial words $u(g, h) = v(g, h)$. By using cancellation in G we can assume that $u(g, h)$ has g as the first letter and the first letter in $v(g, h)$ is h , so we obtain $g u_1(g, h) = h v_1(g, h)$, which implies the right Ore condition in every subsemigroup in G . For the left Ore condition we use the last letters in $u(g, h)$ and $v(g, h)$. Now the statement follows by Proposition 2. \square

COROLLARY 1. *All collapsing groups [7], supramenable groups [8] and groups of a subexponential growth [4],[3],[2] have $l(G) \leq 2$.*

To give an example of a group with S -length greater or equal to 3 we recall that the restricted wreath product (see [6, p.45]) $G = A wr \langle b \rangle$ of a group A and an infinite cyclic group $\langle b \rangle$ is a semidirect product of W and $\langle b \rangle$ where $W = \prod^\times A^{b^i}$ is the direct product of copies of A , numbered by elements of $\langle b \rangle$. Instead of A^{b^0} we write A , and $A^{b^i} = b^{-i} A b^i$. Every element $g \in G$ can be uniquely written as

$$g = b^j w,$$

where w is a product of commuting factors a^{b^i} , $a^{b^i} = b^{-i} a b^i$, $a^{b^i} b^j = b^j a^{b^{i+j}}$.

THEOREM 2. *If $G = \langle a \rangle_2 wr \langle b \rangle$ and $S = sgp(a, b)$, then $l(S, G) = 3$.*

PROOF. Note that $a^b a^{b^2} = b^{-1} a b^{-1} a b^2 = b^{-2} a b a b \notin S S^{-1}$. Thus $l(S, G) > 2$. To prove that $l(S, G) = 3$ it suffices to show that $G = \langle b \rangle S \langle b \rangle = \langle b \rangle S^{-1} \langle b \rangle$. Since $a = a^{-1}$, we have $a \in S \cap S^{-1}$. Let $w \in W$. Then w can be written as a product of, say k , commuting factors

$$(2) \quad w = a^{b^{i_1}} a^{b^{i_2}} \dots a^{b^{i_k}} = b^{-i_1} a b^{i_1 - i_2} a b^{i_2 - i_3} \dots b^{i_{k-1} - i_k} a b^{i_k}.$$

For example $a^{b^5} a^{b^3} a^{b^{-1}} = b^{-5} a b^{5-3} a b^{3+1} a b^{-1} = b^{-5} a b^2 a b^4 a b^{-1} \in \langle b \rangle S \langle b \rangle$. Since a^{b^i} commute, we can assume in (2) $i_1 > i_2 > \dots > i_k$, then $w \in b^{-i_1} S b^{i_k}$.

If we write the factors a^{b^i} in reverse order then $i_1 < i_2 < \dots < i_k$, and $w \in b^{-i_1} S^{-1} b^{i_k} \subseteq \langle b \rangle S^{-1} \langle b \rangle$. Thus $W \subseteq \langle b \rangle S \langle b \rangle \cap \langle b \rangle S^{-1} \langle b \rangle$ and multiplication by $\langle b \rangle$ implies $G = \langle b \rangle S \langle b \rangle = \langle b \rangle S^{-1} \langle b \rangle$. Hence $G = S^{-1} S S^{-1} = S S^{-1} S$ and $l(S, G) = 3$. \square

3. S -length of relatively free groups

A group G satisfies a law $u(x_1, \dots, x_m) \equiv v(x_1, \dots, x_m)$ if for every elements g_1, \dots, g_m in G the equality $u(g_1, \dots, g_m) = v(g_1, \dots, g_m)$ holds. A law $u(x_1, \dots, x_m) \equiv v(x_1, \dots, x_m)$ is called an m -variable positive law if the words u, v are positive words. The positive law is called *balanced* if for each variable x_i its exponent sum is the same in u and in v . The simplest example of the balanced positive law is the abelian law. Note that a non-balanced law implies a law $x^m \equiv 1$ for some $m \in \mathbb{N}$ and implies the balanced law $x^m y^m \equiv y^m x^m$.

A group G is called *relatively free of rank n* if it is free in some variety of groups, and has a set of free generators of cardinality n . We formulate the statements 13.11, 13.52, 13.53, 13.25 from [6] in the following

LEMMA 1. *Let $\{a_i\}$ be a set of free generators in a relatively free group G .*

- (i) *If $\{b_j\}$ is another set of free generators in G , then $|\{a_i\}| = |\{b_j\}|$.*
- (ii) *$|\{a_i\}|$ is the smallest number of generators in G .*
- (iii) *Every relator on the generators in $\{a_i\}$ is a law in the group G .*
- (iv) *Every mapping $\{a_i\} \rightarrow G$ can be extended to an endomorphism of G .*

We can show that for a relatively free group G with a base semigroup S the number $l(S, G)$ is an invariant of G , not depending on the choice of S .

LEMMA 2. *The S -length $l(G)$ of a relatively free group G is equal to $l(S, G)$ for any base semigroup S on a set of free generators.*

PROOF. Let G be a relatively free group. Let S_1 and S_2 be two base semigroups in G generated by the sets of free generators $\{a_i\}$ and $\{b_i\}$ respectively. We show that the equality $l(S_1, G) = l(S_2, G)$ holds. By (i) of Lemma 1, the sets $\{a_i\}$ and $\{b_i\}$ have the same cardinality and the map $a_i \rightarrow b_i$ defines by (iv) of Lemma 1, an automorphism α in G , such that $S_1^\alpha = S_2$. Thus α , applied to the expression (1) written with respect to S_1 , changes S_1 to S_2 and hence $l(S_1, G) \geq l(S_2, G)$. Similarly, by applying the automorphism α^{-1} to (1) for S_2 , we get $l(S_2, G) \geq l(S_1, G)$, which gives the equality $l(S_1, G) = l(S_2, G)$.

If a relatively free group G of rank n is a non-Hopf group [5] then there exists a set $\{b_i\}$ of n generators, which generate G not freely. Let S_2 be the base semigroups on the set $\{b_i\}$ and S_1 be the base semigroups on the set of free generators $\{a_i\}$. The map $a_i \rightarrow b_i$ defines by (iv) of Lemma 1, an epi-endomorphism in G which implies that $l(S_1, G) \geq l(S_2, G)$. Hence we conclude that $l(G) = l(S_1, G)$. \square

THEOREM 3. *The S -length $l(G)$ of a relatively free group G is less or equal to 2 if and only if G satisfies a positive law.*

PROOF. Let $l(G) \leq 2$ and let S be a base semigroup generated by a set of free generators $\{a_i\}$ in G . In view of Lemma 2, $l(S, G) \leq 2$ and by Proposition 2, S satisfies Ore conditions. So for generators a_1, a_2 there exist $s_1, s_2 \in S$ such that $a_1 s_1 = a_2 s_2$, then by (iii) of Lemma 1, G satisfies a positive law.

Conversely, let G satisfies a positive law $u'(x_1, \dots, x_n) = v'(x_1, \dots, x_n)$. Then G satisfies a binary positive law $u(x, y) = v(x, y)$, obtained by substitution $x_i \rightarrow xy^i$. It follows that G does not contain a free subsemigroup and the statement follows by Theorem 1. \square

COROLLARY 2. *Let G be a relatively free group.*

1. $l(G) = 1$ if and only if G satisfies a law $x^n = 1$ for some $n \in \mathbb{N}$.
2. $l(G) = 2$ if and only if G satisfies only balanced non-trivial positive laws.

There is a conjecture that for a relatively free group G the S -length $l(G)$ can be only 1, 2 or infinity. We prove that it can not be equal to 3.

THEOREM 4. *If G is a relatively free group, then $l(G) \neq 3$.*

PROOF. If G is a relatively free group and $l(G) = 3$ then the equality $G = S S^{-1} S = S^{-1} S S^{-1}$ implies by (iii) of Lemma 1, that G must satisfy a law

$$(3) \quad x y^{-1} z \equiv a^{-1} b c^{-1},$$

where a, b, c are positive words in a free group F generated by x, y, z, \dots

The law (3) can be written as

$$(4) \quad b \equiv a(x y^{-1} z)c.$$

We introduce three maps in the free group, defined by

$$\alpha: y \rightarrow 1, \quad \beta: y \rightarrow x, \quad \gamma: y \rightarrow z,$$

and each leaves other generators fixed. Each of these maps changes (4) into a positive law. If $l(G) = 3$ then, by Theorem 3, each of these positive laws must be trivial. We show that it is not possible.

If we apply α to (4), we obtain a positive law $b^\alpha \equiv a^\alpha x z c^\alpha$ which must be trivial that is we have the equality

$$b^\alpha \doteq a^\alpha x z c^\alpha,$$

where the symbol " \doteq " denotes the equality of words in a free group.

Since b is a positive word and is a pre-image of the word $a^\alpha x z c^\alpha$, it must have a form

$$b \doteq b_1(x y^k z) b_2,$$

where b_1, b_2 are positive words, $k \geq 0$, $b_1^\alpha \doteq a^\alpha$, $b_2^\alpha \doteq c^\alpha$. We use this form of b to rewrite the law (3) as

$$(5) \quad xy^{-1}z \equiv (a^{-1}b_1)xy^kz(b_2c^{-1}).$$

Since $b_1^\alpha \doteq a^\alpha$, the word $a^{-1}b_1$ becomes trivial under α replacing $y \rightarrow 1$, therefore it has the exponent sums of x and of z equal to zero, and similarly for b_2c^{-1} , which we denote as

$$(6) \quad \sigma_x(a^{-1}b_1) = 0, \quad \sigma_z(a^{-1}b_1) = 0, \quad \sigma_x(b_2c^{-1}) = 0, \quad \sigma_z(b_2c^{-1}) = 0.$$

If we apply β to (5), we get the law

$$z \equiv (a^{-1}b_1)^\beta (xx^kz)(b_2c^{-1})^\beta,$$

which must be trivial because otherwise it gives a nontrivial positive law. So we have the equality

$$z \doteq \underbrace{(a^{-1}b_1)^\beta xx^k}_{\underline{z}} \cdot \underbrace{(b_2c^{-1})^\beta}_{\underline{z}}.$$

Since $\sigma_z((a^{-1}b_1)^\beta) \stackrel{(6)}{=} 0$, the underlined z can not be cancelled from the left and similarly not from the right. Then the underbraced words must be trivial. In particular, $(a^{-1}b_1)^\beta xx^k \doteq 1$. It follows that $\sigma_x((a^{-1}b_1)^\beta) = -k - 1$. Since for any word w , $\sigma_x(w^\beta) = \sigma_x(w) + \sigma_y(w)$, we have

$$-k - 1 = \sigma_x((a^{-1}b_1)^\beta) = \sigma_x(a^{-1}b_1) + \sigma_y(a^{-1}b_1) \stackrel{(6)}{=} 0 + \sigma_y(a^{-1}b_1),$$

that is

$$\sigma_y(a^{-1}b_1) = -k - 1 < 0.$$

Now we apply γ to (5), and the law we get must again be trivial

$$x \doteq (a^{-1}b_1)^\gamma \cdot \underline{x} \cdot z^k z (b_2c^{-1})^\gamma.$$

Since $\sigma_x(a^{-1}b_1)^\gamma = \sigma_x(a^{-1}b_1) \stackrel{(6)}{=} 0$, the underlined x can not be canceled from the left and similarly not from the right. Hence $(a^{-1}b_1)^\gamma \doteq 1$. Then the exponent sum of z in $(a^{-1}b_1)^\gamma$ must be also zero and we have

$$0 = \sigma_z((a^{-1}b_1)^\gamma) = \sigma_z(a^{-1}b_1) + \sigma_y(a^{-1}b_1) = 0 - k - 1 < 0,$$

a contradiction. □

The following example illustrates the above reasoning for the specific law of the form $xy^{-1}z \equiv a^{-1}bc^{-1}$. We show how it implies a positive law.

EXAMPLE 5. Let $a = zy^3xy$, $b = zyxyxyzyx$, $c = xy$ then the law (3) is

$$(7) \quad xy^{-1}z \equiv (zy^3xy)^{-1}zyxyxyzyx(xy)^{-1},$$

If apply α to (7) we get $xz \equiv (zx)^{-1}b^\alpha(x)^{-1}$, which is trivial because

$$b^\alpha \doteq (zyxyxyzyx)^\alpha \doteq (zx)xz(x).$$

By applying β to (7) we get the law which again is trivial

$$z \doteq \underbrace{(zx^5)^{-1}(zx^3)x^2}_z \cdot \underbrace{(x^2)(x^2)^{-1}}_1.$$

By applying γ to (7) we get a nontrivial law

$$x \equiv (z^4xz)^{-1}z^2xzxz^2zx(xz)^{-1},$$

which implies a positive law $(z^4xz)x(xz) \equiv z^2xzxz^3x$, or shorter

$$z^2xzx^2z \equiv xzxz^3x.$$

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INSTITUTE OF MATHEMATICS
SILESIA UNIVERSITY OF TECHNOLOGY
KASZUBSKA 23
44-100 GLIWICE
POLAND
e-mail: O.Macedonska@polsl.pl
e-mail: A.Potyka@polsl.pl