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TWO FUNCTIONAL EQUATIONS ON GROUPS

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Abstract. In this note we give the general solution of the functional equation

$$f(x)\,f(x+y) = f(y)^2\,f(x-y)^2\,g(y), \quad x, y \in G,$$

and all the solutions of

$$f(x)\,f(x+y) = f(y)^2\,f(x-y)^2\,g(x), \quad x, y \in G,$$

with the additional supposition $g(x) \neq 0$ for all $x \in G$. In both cases G denotes an arbitrary group written additively and $f, g\colon G \to \mathbb{R}$ are the unknown functions.

1. Introduction

In his book [3] and also in [1], [2], Aczél investigated the functional equation

$$f(x)\,f(x+y) = f(y)^2\,f(x-y)^2\,a^{y+4}, \quad x, y \in \mathbb{R}, \tag{1}$$

where a is a fixed positive real number and $f\colon \mathbb{R} \to \mathbb{R}$ is the unknown function. He proved that the nowhere zero solutions of (1) are

$$f(x) = a^{x-2} \text{ and } f(x) = -a^{x-2}, \quad x \in \mathbb{R}. \tag{2}$$

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Motivated by this result and the observation that (1) has non-identically zero solutions different from (2) too, the authors of this paper created a sequence of problems connected with (1) for the fostering of talented students on different level of mathematical education and published it in [4] with solutions. On the other hand we found a possible way of the generalization that we intend to present in this paper.

2. Main results

First we deal with an obvious and natural generalization of (1) and prove the following

LEMMA 1. *Let G be a group and suppose that the functions $f, g\colon G \to \mathbb{R}$ satisfy the functional equation*

$$f(x) f(x+y) = f(y)^2 f(x-y)^2 g(y), \quad x, y \in G. \tag{3}$$

Then either f is identically zero, or there exists a subgroup A of G such that $g(x) \neq 0$ for all $x \in A$ and

$$f(x) = \begin{cases} \sqrt[3]{\frac{f(0)g(x)}{g(-x)^2}} & \text{if } x \in A, \\ 0 & \text{if } x \in G \setminus A. \end{cases} \tag{4}$$

PROOF. If f is not identically zero let $A = \{y \in G : f(y) \neq 0\}$. We prove that A is a group and g is different from zero on A. Indeed, $0 \in A$, otherwise, with the substitution $y = 0$, equation (3) would imply that f is identically zero. If $y \in A$ then, with the substitution $x = 0$, (3) implies that

$$f(0) f(y) = f(y)^2 f(-y)^2 g(y).$$

Thus $-y \in A$, and g does not vanish on A. Finally, if $x, y \in A$ then replacing x by $x + y$ in (3), we have

$$f(x+y) f(x+2y) = f(y)^2 f(x)^2 g(y),$$

which shows that $f(x+y) \neq 0$, that is, $x + y \in A$.

To prove (4) let $x = 0$ and $y \in A$ in (3). Then we get

$$f(0) = f(y) f(-y)^2 g(y), \quad y \in A, \tag{5}$$

whence, replacing y by $-y$

$$f(0) = f(-y)\, f(y)^2\, g(-y), \quad y \in A, \tag{6}$$

follows. From (5) and (6) we get

$$f(-y) = f(y)\,\frac{g(-y)}{g(y)}, \quad y \in A.$$

This and (5) imply that (4) holds (for y instead of x). □

In the following theorem we give the general solution of (3).

THEOREM 1. *Let G be group and $f, g\colon G \to \mathbb{R}$. Then f and g satisfy* (3) *if and only if either $f(x) = 0$ for all $x \in G$ and g is arbitrary, or there exist a subgroup A of G, a function $\varphi\colon A \to \mathbb{R}$ and real numbers α, β such that $\alpha^2\beta = 1$, $\varphi(0) = 1$,*

$$\varphi(x+y) = \varphi(x)\,\varphi(y), \quad x, y \in A, \tag{7}$$

and

$$f(x) = \begin{cases} \alpha\varphi(x) & \text{if } x \in A, \\ 0 & \text{if } x \in G \setminus A, \end{cases} \qquad g(x) = \begin{cases} \beta\varphi(x) & \text{if } x \in A, \\ \text{arbitrary} & \text{if } x \in G \setminus A. \end{cases} \tag{8}$$

PROOF. We only prove the necessity of the conditions because the sufficiency can easily be checked. We suppose also that f is not identically zero. According to Lemma 1 there exists a subgroup A of G such that $g(x) \neq 0$ for all $x \in A$ and (4) holds. Therefore (3) and (4) imply that

$$\frac{g(x)\, g(x+y)}{g(-x)^2\, g(-(x+y))^2} = f(0)^2\, \frac{g(y)^5}{g(-y)^4}\, \frac{g(x-y)^2}{g(-(x-y))^4}, \quad x, y \in A. \tag{9}$$

With the substitutions $y = x$ and $y = -x$ we get from (9) that

$$\frac{g(2x)}{g(-2x)^2} = \frac{f(0)^2}{g(0)^2}\, \frac{g(x)^4}{g(-x)^2}, \quad x \in A,$$

and

$$\left(\frac{g(2x)}{g(-2x)^2}\right)^2 = \frac{1}{g(0)\, f(0)^2}\, \frac{g(x)^5}{g(-x)^7}, \quad x \in A,$$

respectively. By the help of these two equations the quotient $\frac{g(2x)}{g(-2x)^2}$ can be eliminated and we obtain

$$g(-x) = \frac{g(0)}{f(0)^2} \frac{1}{g(x)}, \quad x \in A. \tag{10}$$

Therefore (9) can be written in the following simpler form

$$g(x)\, g(x+y) = \sqrt[3]{\frac{f(0)^{10}}{g(0)^4} g(y)^3\, g(x-y)^2}, \quad x, y \in A. \tag{11}$$

Write here $-y$ instead of y and use (10) to get

$$g(x)\, g(x-y) = \sqrt[3]{\frac{f(0)^{10}}{g(0)^4} \frac{g(0)^3}{f(0)^6} \frac{1}{g(y)^3} g(x+y)^2}, \quad x, y \in A.$$

Comparing this equation and (11) we find that

$$\sqrt[3]{\frac{g(0)^2}{f(0)^2}}\, g(x+y) = g(x)\, g(y), \quad x, y \in A. \tag{12}$$

With the substitution $x = y = 0$, this implies that

$$f(0)^2\, g(0) = 1. \tag{13}$$

Therefore, with the definitions $\beta = g(0)$ and $\varphi(x) = \frac{1}{\beta} g(x)$, $x \in A$, equation (12) implies (7) and $\varphi(0) = 1$. On the other hand, it follows from (4), (10), (13), and the known form of g on A that

$$f(x) = \sqrt[3]{\frac{f(0)^5}{g(0)^2}}\, g(x) = \sqrt[3]{f(0)^5\, g(0)}\, \varphi(x) = f(0)\, \varphi(x), \quad x \in A,$$

which, with the definition $\alpha = f(0)$, proves the first part of (8). $\alpha^2\beta = 1$ is obvious because of (13). The second part of (8) now follows from the definition of φ and equation (3). □

In what follows we deal with an other equation similar to (3), namely we consider the equation

$$f(x)\,f(x+y)=f(y)^2\,f(x-y)^2\,g(x)\,,\quad x,y\in G. \tag{14}$$

If we suppose that g is nowhere zero on G then the ideas, we used in the previous investigations, will work and we can prove the following

Theorem 2. *Let G be a group and $f\colon G\to\mathbb{R}$, $g\colon G\to\mathbb{R}\backslash\{0\}$ be functions. Then f and g satisfy (14) if and only if either $f(x)=0$ for all $x\in G$ and g is arbitrary nowhere zero function, or there exist a subgroup A of G and real numbers α,β such that $\alpha^2\beta=1$,*

$$f(x)=\begin{cases}\alpha & \textit{if}\quad x\in A,\\ 0 & \textit{if}\quad x\in G\setminus A,\end{cases} \tag{15}$$

and

$$g(x)=\begin{cases}\beta & \textit{if}\quad x\in A,\\ \textit{arbitrary nonzero} & \textit{if}\quad x\in G\setminus A.\end{cases} \tag{16}$$

Proof. We prove only the non-trivial part of the statement. Suppose that f is not identically zero. Then $f(0)\neq 0$ otherwise, with $y=0$, (14) would imply that f is identically zero. Let $A=\{y\in G:\ f(y)\neq 0\}$. We show that g is constant on A and A is group. Indeed, if $x\in A$ and $y=0$ in (14) then, with the definition $\beta=\frac{1}{f(0)^2}$, we have $f(x)^2=f(0)^2\,f(x)^2\,g(x)$ whence

$$g(x)=\beta,\quad x\in A \tag{17}$$

follows. On the other hand, $0\in A$, and if $y\in A$ then, with $x=0$, (14) implies that

$$f(0)f(y)=f(y)^2f(-y)^2g(0).$$

Thus $-y\in A$. Finally, if $x,y\in A$ then, replacing x by $x+y$ in (14), we have

$$f(x+y)f(x+2y)=f(y)^2f(x)^2g(x+y).$$

Since g is nowhere zero this implies that $x+y\in A$. Thus A is group.

Let now $x=0$ and $y\in A$ in (14). Then we obtain

$$f(0)^3=f(y)\,f(-y)^2\,. \tag{18}$$

Write here $-y$ instead of y to get

$$f(0)^3 = f(-y)\, f(y)^2 .$$

Comparing these two equations we get $f(-y) = f(y)$ for all $y \in A$. Thus, with the definition $\alpha = f(0)$, (18) implies (15). (16) and the validity of $\alpha^2\beta = 1$ are obvious. □

3. Remarks and examples

1. A common generalization of equation (3) and (14) is

$$f(x)\, f(x+y) = f(y)^2 f(x-y)^2 F(x,y), \quad x, y \in G, \tag{19}$$

where G is a group, $f\colon G \to \mathbb{R}$ and $F\colon G \times G \to \mathbb{R}$ are unknown functions. Supposing that F is nowhere zero and f is not identically zero, as in the proof of Theorem (2), the set $A = \{y \in G : f(y) \neq 0\}$ turns out to be a subgroup of G. Thus

$$f(x) = \begin{cases} \text{arbitrary non-zero} & \text{if } \ x \in A, \\ 0 & \text{if } \ x \in G \setminus A, \end{cases} \tag{20}$$

$$F(x,y) = \begin{cases} \frac{f(x) f(x+y)}{f(y)^2 f(x-y)^2} & \text{if } \ (x,y) \in A \times A, \\ \text{arbitrary non-zero} & \text{if } \ (x,y) \in (G \times G) \setminus (A \times A). \end{cases} \tag{21}$$

Conversely, if A is a subgroup of G then the functions f and F defined by (20) and (21) are solutions of (19). However, if $F\colon A \times A \to \mathbb{R}$ is given where A is a group one can ask the following: What is the necessary and sufficient condition for the equality

$$F(x,y) = \frac{f(x)\, f(x+y)}{f(y)^2 f(x-y)^2}, \quad x, y \in A,$$

to hold with some function $f\colon A \to \mathbb{R} \setminus \{0\}$? This problem is still open.

2. If $0 < a \in \mathbb{R}$ and $g(y) = a^{y+4}$, $y \in \mathbb{R}$ in (3) then we get equation (1). Furthermore, if $A = \mathbb{R}$ with the usual addition, $\varphi(x) = a^x$, $x \in \mathbb{R}$, $\beta = a^4$ and $\alpha^2 = a^{-4}$ in Theorem 1 then we have the nowhere zero solutions (2) of (1).

On the other hand, if $A = \mathbb{Q}$ (the set of all rational numbers) with the usual addition in Theorem 1 then the functions f and g given by

$$f(x) = \begin{cases} a^{x-2} & \text{if} \quad x \in \mathbb{Q}, \\ 0 & \text{if} \quad x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases} \quad \text{and} \quad g(x) = \begin{cases} a^{x+4} & \text{if} \quad x \in \mathbb{Q}, \\ \text{arbitrary} & \text{if} \quad x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

are nowhere continuous solutions of (3), in general.

3. If $G = \mathbb{R}$ with the usual addition and A is a proper subgroup of $\mathbb{R}$ then both (3) and (14) have solutions f and g of the form

$$f(x) = \begin{cases} 1 & \text{if} \quad x \in A, \\ 0 & \text{if} \quad x \in \mathbb{R} \setminus A, \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1 & \text{if} \quad x \in A, \\ 2 & \text{if} \quad x \in \mathbb{R} \setminus A. \end{cases}$$

It is obvious that these functions are discontinuous at least at the points of A. Indeed, if f or g were continuous at a point of A then A would contain an interval of positive length thus $A = \mathbb{R}$ would follow.

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