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Prace Naukowe Uniwersytetu Śląskiego nr 2617, Katowice

ON INJECTIVITY OF NATURAL HOMOMORPHISMS OF WITT RINGS

Marzena Ciemała, Kazimierz Szymiczek

Abstract. We study the homomorphism $W\mathcal{O} \to WK$ between the Witt ring of a domain $\mathcal{O}$ and the Witt ring of its field of fractions K in the case when $\mathcal{O}$ is not integrally closed. We give sufficient conditions for the noninjectivity of this homomorphism by constructing nonzero elements in the kernel. In particular, when K is an algebraic number field and $\mathcal{O}$ is a nonmaximal order in K with even conductor, then the ring homomorphism $W\mathcal{O} \to WK$ is not injective.

1. Introduction

It is known that, for a Dedekind domain $\mathcal{O}$ and its field of fractions K, the natural ring homomorphism

$$\varphi : W\mathcal{O} \to WK$$

between the Witt rings of $\mathcal{O}$ and K is injective. This was first proved by M. Knebusch in 1970 ([4, Satz 11.1.1]). T.C. Craven, A. Rosenberg and R. Ware investigated in [3] a more general situation and proved that when $\mathcal{O}$ is a regular noetherian domain of an arbitrary Krull dimension, then $\ker\varphi$ is a nilideal, that is, every element belonging to the kernel is a nilpotent element of the Witt ring $W\mathcal{O}$. They also gave a series of new examples where

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the kernel is actually zero and so φ is injective. In the opposite direction they mentioned that for the Gaussian field $K = \mathbb{Q}(i)$ and the order $\mathcal{O} = \mathbb{Z}[3i]$ the homomorphism φ is not injective. According to [3] this is *easy* with no further comments. We have tried to understand the simplicity of that statement and one explanation we have found, presumably the *easy* one, follows from the observation that for the unit element $\langle 1\rangle$ of the ring $W\mathbb{Z}[3i]$ we have $\varphi(2\langle 1\rangle) = 0 \in W\mathbb{Q}(i)$, while $2\langle 1\rangle \neq 0$ in $W\mathbb{Z}[3i]$, since the canonical homomorphism $\mathbb{Z}[3i] \to \mathbb{Z}[3i]/(3,3i) \cong \mathbb{F}_3$ induces ring homomorphism $W\mathbb{Z}[3i] \to W\mathbb{F}_3$ under which $2\langle 1\rangle$ goes into $2\langle 1\rangle \neq 0 \in W\mathbb{F}_3$. While this argument is generalizable it does not offer enough freedom in constructing nonzero elements in the kernel of the natural ring homomorphism $\varphi : W\mathcal{O} \to WK$ for a domain $\mathcal{O}$ and its field of fractions K.

In the first part of the paper (§§2,3) we study the bilinear space structure on free modules S of rank 2 over a domain $\mathcal{O}$ which become hyperbolic over the field of fractions of $\mathcal{O}$. We find necessary and sufficient conditions for metabolicity of S over $\mathcal{O}$ in terms of some ideals of $\mathcal{O}$ naturally related to the space S.

In §4 we prove the main theorem giving a practical condition for S to be a nonzero element in the kernel of the natural homomorphism $\varphi : W\mathcal{O} \to WK$. It is expressed in terms of integrality over $\mathcal{O}$ of the roots of the isotropy equation for S.

The simplest application shows that for each order $\mathbb{Z}[fi]$, $f > 1$, of the Gaussian field $\mathbb{Q}(i)$, the natural ring homomorphism $W\mathbb{Z}[fi] \to W\mathbb{Q}(i)$ is not injective, confirming (for $f = 3$) the assertion in [3].

We also prove that for each nonmaximal order $\mathcal{O}$ of any number field K with even conductor the homomorphism $W\mathcal{O} \to WK$ is not injective.

These results confirm in part the conjecture that for an algebraic number field K and its order $\mathcal{O}$ the natural ring homomorphism $W\mathcal{O} \to WK$ is injective if and only if $\mathcal{O}$ is the maximal order of K.

Some further results on the nature of the homomorphism are known. In [2] we have proved that each element in the kernel of the homomorphism $W\mathcal{O} \to WK$ is a nilpotent element in $W\mathcal{O}$. In [1] it is shown that, in the case of nonreal quadratic number fields, the homomorphism is surjective provided the conductor of $\mathcal{O}$ is coprime with the discriminant of K.

We use the notation and terminology of the J. Milnor and D. Husemoller's book [5]. The symbol $\langle E\rangle$ denotes the element of the Witt ring $W\mathcal{O}$ determined by the bilinear space E over $\mathcal{O}$. If P is a commutative ring and $\mathcal{O}$ is a subring of P, then by the *natural homomorphism* induced by the inclusion of $\mathcal{O}$ into P we mean the map

$$\varphi : W\mathcal{O} \to WP, \qquad \varphi\langle E\rangle = \langle E \otimes_{\mathcal{O}} P\rangle.$$

We also use the symbol $\langle E\rangle_P$ for $\varphi\langle E\rangle$.

2. Free modules of rank 2

Let $\mathcal{O}$ be an integral domain and K its field of fractions. We assume that $\operatorname{char} K \neq 2$. Let (S, β) be a nonsingular bilinear space over $\mathcal{O}$ with S a free module of rank 2. Let (u, v) be a basis for S and let

$$(S, \beta) \cong \begin{bmatrix} A & C \\ C & B \end{bmatrix} \tag{2.1}$$

be the matrix of β in the given basis. Then $A = \beta(u, u), B = \beta(v, v)$, $C = \beta(u, v) \in \mathcal{O}$. Since (S, β) is nonsingular, the determinant $AB - C^2$ is an invertible element of $\mathcal{O}$. We will analyze the conditions for A, B, C under which $\langle S, \beta \rangle$ is a nonzero element in the kernel of the natural ring homomorphism $\varphi : W\mathcal{O} \to WK$. Observe that $\varphi(\langle S, \beta \rangle) = 0$ if and only if $\langle S \rangle_K$ is a hyperbolic plane over K if and only if there is a nonzero $D \in K$ for which

$$AB - C^2 = -D^2.$$

It is easy to show that if $AB = 0$ then (S, β) is metabolic. Hence we always assume that $AB \neq 0$ and $C^2 \neq D^2$.

A nonzero element $s = xu + yv \in S$, $x, y \in \mathcal{O}$, is said to be isotropic if $\beta(s, s) = 0$. This is equivalent to

$$Ax^2 + 2Cxy + By^2 = 0.$$

Since $AB \neq 0$ and $x \neq 0$ or $y \neq 0$, we conclude that $xy \neq 0$ and so $\frac{y}{x}$ satisfies the *isotropy equation*

$$BX^2 + 2CX + A = 0.$$

We denote by d and d' the roots of the isotropy equation. Hence

$$d := \frac{-C + D}{B} = \frac{A}{-C - D}, \quad d' := \frac{-C - D}{B} = \frac{A}{-C + D}.$$

These are elements of K. The notation introduced above will be in force throughout the paper. In particular,

$$A, B, C \in \mathcal{O}, \quad AB \neq 0, \quad AB - C^2 = -D^2 \in U(\mathcal{O}), \quad D \in K \setminus \{0\},$$

where $U(\mathcal{O})$ is the group of invertible elements in $\mathcal{O}$.

Lemma 2.1. *A nonzero element $s = xu + yv \in S$, $x, y \in \mathcal{O}$ is isotropic if and only if* $y = dx$ *or* $y = d'x$.

Proof. $\beta(xu + yv, xu + yv) = 0$ if and only if $\frac{y}{x}$ satisfies the isotropy equation, hence

$$\frac{y}{x} = \frac{-C + D}{B} = d \qquad \text{or} \qquad \frac{y}{x} = \frac{-C - D}{B} = d',$$

as required. □

We now proceed to the analysis of the conditions under which the space (S, β) is not metabolic. For this we find all totally isotropic subspaces of S (see Lemma 2.4). Recall that the space (S, β) is said to be metabolic if there is a totally isotropic submodule $N \subset S$ which is a direct summand for S. And N is totally isotropic when $N = N^\perp$, where $N^\perp = \{s \in S : \beta(s, N) = 0\}$ is the orthogonal complement of N. We write $\mathrm{Is}(S)$ for the set of all isotropic elements in S,

$$\mathrm{Is}(S) := \{s \in S : \beta(s, s) = 0\},$$

and we also write

$$I = I(S) = \big\{x(u + dv) : x \in \mathcal{O},\ xd \in \mathcal{O}\big\},$$
$$I' = I'(S) = \big\{x(u + d'v) : x \in \mathcal{O},\ xd' \in \mathcal{O}\big\}.$$

Lemma 2.2. *For submodules M and N of S,*

$$\begin{aligned} N = N^\perp \quad &\Rightarrow \quad N \subseteq \mathrm{Is}(S), \\ N \subseteq M \quad &\Rightarrow \quad M^\perp \subseteq N^\perp, \\ N = N^\perp, \quad M = M^\perp, \quad N \subseteq M \quad &\Rightarrow \quad N = M. \end{aligned}$$

Proof. The first two properties are evident and the third follows from the second. □

Lemma 2.3. *$I(S)$ and $I'(S)$ are nonzero submodules of S and*

$$\mathrm{Is}(S) = I(S) \cup I'(S).$$

Moreover, $I(S) = I(S)^\perp$ *and* $I'(S) = I'(S)^\perp$.

PROOF. Clearly $I(S)$ and $I'(S)$ are submodules of S. To show they are *nonzero* submodules it suffices to point out a nonzero element $x \in \mathcal{O}$ such that $xd \in \mathcal{O}$ and $xd' \in \mathcal{O}$. If $D = \frac{c}{e}$, where $c, e \in \mathcal{O}$, one can take $x = eB$. From Lemma 2.1 it follows that $\mathrm{Is}(S) = I(S) \cup I'(S)$.

To prove that the two submodules are totally isotropic we first show that $I(S)^{\perp} \subseteq I(S)$. Take a nonzero element $s = au + bv \in S$, $a, b \in \mathcal{O}$, lying in $I(S)^{\perp}$. Then for all nonzero $x \in \mathcal{O}$ satisfying $xd \in \mathcal{O}$ we have

$$\beta(au + bv, x(u + dv)) = 0,$$

which is equivalent to

$$aA + bC + bdB + adC = 0.$$

From this it follows that $a \neq 0$ since otherwise $b \neq 0$ and $C + dB = 0$. But $C + dB = D \neq 0$, a contradiction. Hence we get

$$\frac{b}{a} = -\frac{A + dC}{C + dB} = d.$$

It follows that $s = au + adv \in I(S)$ since $b = ad$ belongs to $\mathcal{O}$. This shows $I(S)^{\perp} \subseteq I(S)$.

Now let $s_1, s_2 \in I(S)$ be of the form $s_1 = x(u + dv)$, $s_2 = y(u + dv)$ with nonzero x, y in $\mathcal{O}$ and $xd, yd \in \mathcal{O}$. Then

$$xy\beta(s_1, s_2) = \beta(xy(u + dv), xy(u + dv)) = 0,$$

hence $\beta(s_1, s_2) = 0$. This proves $I(S) \subseteq I(S)^{\perp}$. A similar argument proves that $I'(S)$ is totally isotropic. $\square$

LEMMA 2.4. *Let N be a totally isotropic submodule of S. Then*

$$N = I(S) \quad \text{or} \quad N = I'(S).$$

PROOF. From $N = N^{\perp}$ we get $N \subseteq \mathrm{Is}(S) = I(S) \cup I'(S)$. We show that actually

$$N \subseteq I(S) \quad \text{or} \quad N \subseteq I'(S).$$

If this is not the case, there are nonzero $s_1, s_2 \in N$ such that

$$s_1 \in I(S) \quad \text{and} \quad s_2 \in I'(S).$$

Let $s_1 = x(u+dv)$, $s_2 = y(u+d'v)$ with appropriate nonzero $x, y \in \mathcal{O}$. Since $\beta(s_1, s_2) = 0$, we get

$$xy(A + (d+d')C + dd'B) = 0.$$

Here $d + d' = -\frac{2C}{B}$ and $dd' = \frac{A}{B}$, hence it follows that

$$A - \frac{2C^2}{B} + A = 0,$$

that is, $-2D^2 = 2(AB - C^2) = 0$. Since characteristic of the field K is assumed not to be 2, this contradicts the nonsingularity of S. Thus we have proved that $N \subseteq I(S)$ or $N \subseteq I'(S)$. Suppose $N \subseteq I(S)$. Then according to Lemmas 2.2 and 2.3 we get

$$N \subseteq I(S) = I(S)^{\perp} \subseteq N^{\perp} = N.$$

Hence $N = I(S)$. If $N \subseteq I'(S)$ a similar argument shows that $N = I'(S)$. □

Corollary 2.5. *$I(S)$ and $I'(S)$ are the only totally isotropic submodules of S.*

Proof. This follows from Lemma 2.3 and Lemma 2.4. □

The modules $I(S), I'(S)$ have their counterparts in the ring $\mathcal{O}$:

$$\mathcal{J} = \mathcal{J}(S) = \{x \in \mathcal{O} : xd \in \mathcal{O}\} = \mathcal{O} \cap d^{-1}\mathcal{O},$$
$$\mathcal{J}' = \mathcal{J}'(S) = \{x \in \mathcal{O} : xd' \in \mathcal{O}\} = \mathcal{O} \cap d'^{-1}\mathcal{O}.$$

$\mathcal{J}(S)$ and $\mathcal{J}'(S)$ are ideals in $\mathcal{O}$ and as $\mathcal{O}-$modules they are isomorphic with $I(S)$ and $I'(S)$, respectively. If $\mathcal{O}$ is a UFD, these are principal ideals, hence invertible. In the general case we have the following lemma.

Lemma 2.6. *If the space (S, β) is metabolic, then $\mathcal{J}$ or $\mathcal{J}'$ is an invertible ideal in $\mathcal{O}$.*

Proof. If $\mathcal{J}(S)$ and $\mathcal{J}'(S)$ are not invertible, then they are not projective $\mathcal{O}-$modules (see [6, Prop. 1.15, p. 26]). Hence none of them can be a direct summand of the free module S. Since, by Corollary 2.5, the ideals $\mathcal{J}(S)$ and $\mathcal{J}'(S)$ are isomorphic with the only totally isotropic submodules of S, this implies that (S, β) is not metabolic. □

Lemma 2.7. *Let $\mathcal{O}$ be a domain and let K be the field of fractions of $\mathcal{O}$,* $\mathrm{char} K \neq 2$. *If $d \in \mathcal{O}$ or $d' \in \mathcal{O}$, then the $\mathcal{O}-$space (S, β) is metabolic.*

Proof. Suppose $d \in \mathcal{O}$. Then the totally isotropic subspace $I(S) = \{x(u+dv) : x \in \mathcal{O}\}$ is a free submodule of S with the basis element $u + dv$. We prove that $I(S)$ is a direct summand of S.

Let $s = xu + yv \in S$ for some $x, y \in \mathcal{O}$. Then

$$s = xu + yv = x(u + dv) + (y - xd)v,$$

hence $s \in I(S) + \mathcal{O}v$. On the other hand v is not isotropic, hence $I(S) \cap \mathcal{O}v = \{0\}$. Thus we get

$$S = I(S) \oplus \mathcal{O}v,$$

and since $I(S) = I(S)^{\perp}$, it follows that S is metabolic. If $d' \in \mathcal{O}$, the proof runs similarly. $\square$

3. Characterization of metabolic spaces

We assume that $\mathcal{O}$ is a domain with field of fractions K. We also continue to assume that d is a root of the *isotropy equation*

$$BX^2 + 2CX + A = 0,$$

where $A, B, C \in \mathcal{O}$ and $C^2 - AB$ is a unit in $\mathcal{O}$ and a square in K.

We write $d = \frac{b}{a}$, where $a, b \in \mathcal{O}$. If $\mathcal{O}$ is noetherian, we can assume that d is written in the lowest terms (that is, a and b do not have any common divisors which are non-invertible in $\mathcal{O}$), but we cannot expect any uniqueness of representation of d as a ratio of two elements of $\mathcal{O}$.

We have introduced earlier the ideal $\mathcal{J}$. Observe that

$$b\mathcal{J} = b(\mathcal{O} \cap d^{-1}\mathcal{O}) = b(\mathcal{O} \cap \tfrac{a}{b}\mathcal{O}) = a\mathcal{O} \cap b\mathcal{O}.$$

We also write

$$\mathcal{D} := a\mathcal{O} + b\mathcal{O}$$

for the ideal in $\mathcal{O}$ generated by a, b.

Lemma 3.1. *We have the following module isomorphism:*

$$b\mathcal{J} = a\mathcal{O} \cap b\mathcal{O} \cong I.$$

Proof. $a\mathcal{O} \cap b\mathcal{O} \cong I$ via $ay = bx \mapsto x(u + \frac{b}{a}v)$. □

Proposition 3.2. *The following sequence is exact*

$$0 \to I \to S \xrightarrow{\varphi} \mathcal{D} \to 0, \tag{3.1}$$

where $\varphi(xu + yv) = ay - bx$.

Proof. Observe that

$$xu + yv \in \ker\varphi \iff y = dx \in \mathcal{O} \iff xu + yv = x(u + dv) \in I$$

for $x, y \in \mathcal{O}$. □

We give now a characterization of metabolicity of S in terms of the ideals $\mathcal{D}$ and $\mathcal{J}$. Recall that by Corollary 2.5, S is metabolic iff I or I' is a direct summand of S. So it is sufficient to characterize the situation when S is metabolic and one of the subspaces I or I' is a direct summand of S. If I is a direct summand of S we say S is $I-$metabolic.

Theorem 3.3. *The following statements are equivalent.*
(a) *S is $I-$metabolic.*
(b) *The exact sequence* (3.1) *splits.*
(c) *$\mathcal{D}$ is a direct summand of S.*
(d) *$\mathcal{D}$ is an invertible ideal.*
(e) *$\mathcal{D}\mathcal{J}$ is a principal ideal.*
(f) *$\mathcal{D}\mathcal{J} = a\mathcal{O}$.*
(g) *$(a\mathcal{O} + b\mathcal{O})(a\mathcal{O} \cap b\mathcal{O}) = ab\mathcal{O}$.*

Proof. The equivalence of (a), (b), (c) follows from Proposition 3.2 and from standard properties of split exact sequences. If (c) holds, then $\mathcal{D}$ is a direct summand of the free module S, hence it is projective, hence invertible. Thus (c) implies (d). Conversely, if $\mathcal{D}$ is invertible, it is a projective module, and so the exact sequence (3.1) splits. Thus (d) implies (b).

Clearly, (e) implies (d) and we now prove that (a), (b), (c) and (d) imply (e). From (b) we get

$$S = I \oplus \mathcal{D} \quad \text{and} \quad I = I^{\perp},$$

the latter by Lemma 2.3. Hence (S,β) is a metabolic space. Let

$$\widehat{\beta}: S \to S^*, \quad \widehat{\beta}(s)(s') = \beta(s,s')$$

be the adjoint homomorphism. It is an isomorphism and $\widehat{\beta}(I) = \mathcal{D}^*$. Thus we have the module isomorphisms

$$\mathcal{J} \cong b\mathcal{J} \cong I \cong \widehat{\beta}(I) = \mathcal{D}^* \cong \mathcal{D}^{-1},$$

the latter isomorphism by (d). Hence (e) follows.

Now we show that (e) implies (f). If $\mathcal{D}\mathcal{J} = c\mathcal{O}$, then $\frac{1}{c}\mathcal{J}$ is the inverse of $\mathcal{D}$, hence

$$\begin{aligned}\tfrac{1}{c}\mathcal{J} &= [\mathcal{O} : \mathcal{D}] = \{x \in K : x\mathcal{D} \subseteq \mathcal{O}\} = \{x \in K : xa \in \mathcal{O} \quad \text{and} \quad xb \in \mathcal{O}\} \\ &= \tfrac{1}{a}\mathcal{O} \cap \tfrac{1}{b}\mathcal{O} = \tfrac{1}{ab}(a\mathcal{O} \cap b\mathcal{O}) = \tfrac{1}{a}\mathcal{J}.\end{aligned}$$

Hence $c\mathcal{O} = a\mathcal{O}$, as required. It remains to observe that (f) and (g) are equivalent since $b\mathcal{J} = a\mathcal{O} \cap b\mathcal{O}$. □

Remark 3.4. We give here a proof of a less efficient result than the equivalence of (a) and (e) in Theorem 3.3. Nevertheless it is of some interest. We claim that

S is $I-$metabolic if and only if $\mathcal{D}^2\mathcal{J}^2$ is a principal ideal.

If $\mathcal{D}^2\mathcal{J}^2$ is principal, then $\mathcal{D}$ is invertible and so S is metabolic. To prove the converse observe that (a) implies (b) so that we have the split exact sequence

$$0 \to \mathcal{J} \to S \to \mathcal{D} \to 0.$$

But then the following sequence is also split exact

$$0 \to \mathcal{D}^* \to S^* \to \mathcal{J}^* \to 0,$$

where the homomorphisms involved are the transposes of the corresponding homomorphisms in the first sequence. So it follows we have the following module isomorphisms

$$\mathcal{J} \oplus \mathcal{D} = S \cong S^* \cong \mathcal{J}^* \oplus \mathcal{D}^* \cong \mathcal{J}^{-1} \oplus \mathcal{D}^{-1},$$

the latter by the fact that both $\mathcal{J}$ and $\mathcal{D}$ are projective, hence invertible. Now by Steinitz's theorem we have the isomorphism

$$\mathcal{J}\mathcal{D} = c\mathcal{J}^{-1}\mathcal{D}^{-1}$$

for some $c \in K$. It follows that $\mathcal{D}^2\mathcal{J}^2$ is a principal ideal.

Proposition 3.5. *If $\mathcal{D}$ is invertible, then $\mathcal{J}$ is invertible.*

Proof. This follows from the implication (d) $\Rightarrow$ (e) in Theorem 3.3. $\square$

Remark 3.6. We do not know whether or not the converse statement to that in Proposition 3.5 holds true. If so it would give a satisfactory NSC for metabolicity of S: S is metabolic and I is a direct summand of S iff $\mathcal{J}$ is invertible iff I is a projective submodule of S.

4. The main theorem

For a ring R and its subring $\mathcal{O}$ let

$$\mathfrak{f} = \{x \in \mathcal{O} : xR \subseteq \mathcal{O}\}$$

be the conductor of the ring extension $\mathcal{O} \subset R$. Observe that $\mathfrak{f} = \mathcal{O} \iff R = \mathcal{O}$. The conductor is the largest ideal of $\mathcal{O}$ which is also an ideal in R. Indeed, if $\mathfrak{a}$ is an ideal in $\mathcal{O}$ and it is also an ideal in R, then $\mathfrak{a}R \subseteq \mathfrak{a} \subseteq \mathcal{O}$, hence $\mathfrak{a} \subseteq \mathfrak{f}$. The following example shows that an ideal $\mathfrak{a}$ of $\mathcal{O}$ contained in $\mathfrak{f}$ need not be an ideal of R.

Example 4.1. Let $\mathcal{O} = \mathbb{Z}[3i]$, $R = \mathbb{Z}[i]$. Then $\mathfrak{f} = 3R$. Consider the principal ideal $\mathfrak{a} = (3+3i)\mathcal{O}$. Clearly $\mathfrak{a} \subseteq \mathfrak{f}$. But $\mathfrak{a}$ is not an ideal in R. For $(3+3i)(1-i) = 6 \notin \mathfrak{a}$.

Now let $\mathcal{O}$ be a domain and K its field of fractions. For $a, b \in \mathcal{O}$, $ab \neq 0$, we introduce the following notation:

$$d = \tfrac{b}{a}, \quad \mathcal{D} = a\mathcal{O} + b\mathcal{O}, \quad \mathcal{J} = \tfrac{1}{b}(a\mathcal{O} \cap b\mathcal{O}).$$

We assume that $d \notin \mathcal{O}$ and consider the ring

$$R := \mathcal{O}[d] = \mathcal{O} + d\mathcal{O} + \cdots + d^n\mathcal{O} + \cdots .$$

Since $d \notin \mathcal{O}$ we have $\mathcal{O} \subsetneq R$ and so the conductor $\mathfrak{f}$ is a proper ideal in $\mathcal{O}$ (possibly 0). Observe that

$$\begin{aligned}\mathfrak{f} &= \{x \in \mathcal{O} : xd^n \in \mathcal{O} \quad \text{for all} \quad n \in \mathbb{N}\}\\ &= \mathcal{O} \cap d^{-1}\mathcal{O} \cap d^{-2}\mathcal{O} \cap \cdots\end{aligned}$$

and hence we always have

$$\mathfrak{f} \subseteq \mathcal{O} \cap d^{-1}\mathcal{O} = \mathcal{J}. \tag{4.1}$$

If d is integral over $\mathcal{O}$, that is, if there is an $n > 0$ such that

$$d^n \in \mathcal{O} + d\mathcal{O} + \cdots + d^{n-1}\mathcal{O},$$

then the ring R is a finitely generated $\mathcal{O}-$module and, in fact,

$$R = \mathcal{O} + d\mathcal{O} + \cdots + d^{n-1}\mathcal{O}.$$

We then say that d is integral over $\mathcal{O}$ of degree at most n. Observe that then

$$\mathfrak{f} = \mathcal{O} \cap d^{-1}\mathcal{O} \cap \cdots \cap d^{-n+1}\mathcal{O}. \tag{4.2}$$

PROPOSITION 4.2. *The following statements are equivalent.*
(a) $R = \mathcal{O} + d\mathcal{O} + \cdots + d^{n-1}\mathcal{O}$.
(b) $\mathcal{D}^{n-1}$ *is an ideal in the ring* R.
(c) d *is integral over* $\mathcal{O}$ *of degree at most* n.

PROOF. (a) $\Rightarrow$ (b) Consider the ideal $\mathcal{D} = a\mathcal{O} + b\mathcal{O}$ and its power

$$\mathcal{D}^{n-1} = a^{n-1}\mathcal{O} + a^{n-2}b\mathcal{O} + \cdots + b^{n-1}\mathcal{O}.$$

Observe that (a) implies $a^{n-1}R = \mathcal{D}^{n-1}$, hence (b) follows.

(b) $\Rightarrow$ (c) If $\mathcal{D}^{n-1}$ is an ideal in the ring R, then $R\mathcal{D}^{n-1} \subseteq \mathcal{D}^{n-1}$ and in particular $d\mathcal{D}^{n-1} \subseteq \mathcal{D}^{n-1}$. It follows that

$$\tfrac{b^n}{a} = d \cdot b^{n-1} \in \mathcal{D}^{n-1} = a^{n-1}(\mathcal{O} + d\mathcal{O} + \cdots + d^{n-1}\mathcal{O}).$$

Whence $d^n = \frac{b^n}{a^n} \in \mathcal{O} + d\mathcal{O} + \cdots + d^{n-1}\mathcal{O}$.

(c) obviously implies (a). $\square$

LEMMA 4.3. *Let R be a subring of the field K. Assume that R contains $\mathcal{O}$ and $\mathfrak{d}$ is an ideal of $\mathcal{O}$ which is also an ideal in R. If $\mathcal{O} \neq R$, then $\mathfrak{d}$ is not an invertible ideal in $\mathcal{O}$.*

PROOF. Suppose $\mathfrak{a}$ is a fractional ideal in $\mathcal{O}$ and $\mathfrak{d}\mathfrak{a} = \mathcal{O}$. Then

$$\mathcal{O} = \mathfrak{d}\mathfrak{a} = R\mathfrak{d}\mathfrak{a} = R\mathcal{O} = R,$$

a contradiction. $\square$

Our main result in this section is the following theorem.

Theorem 4.4. *Let $\mathcal{O}$ be a domain and let K be the field of fractions of $\mathcal{O}$,* $\mathrm{char} K \neq 2$. *Let (S, β) be a free bilinear space of rank 2 over $\mathcal{O}$ and let d and d' be the roots of the isotropy equation for S. If d and d' are integral over $\mathcal{O}$ each of degree at least 2, then the $\mathcal{O}-$space (S, β) is not metabolic. Hence the class $\langle S \rangle$ is a nonzero element in the kernel of the natural ring homomorphism $W\mathcal{O} \to WK$.*

Proof. Consider the rings $R := \mathcal{O}[d]$ and $R' := \mathcal{O}[d']$. If d is integral over $\mathcal{O}$ of degree $n \geq 2$, $\mathcal{D}^{n-1}$ is an ideal in the ring R by Prop. 4.2. Hence by Lemma 4.3, $\mathcal{D}^{n-1}$ is not an invertible ideal in $\mathcal{O}$, and since $n - 1 \geq 1$, also $\mathcal{D}$ is not invertible. Hence by Theorem 3.3 the space S is not $I-$metabolic.

A parallel argument shows that also the ideal $\mathcal{D}' = a'\mathcal{O} + b'\mathcal{O}$ is not invertible and so S is not $I'-$metabolic. Hence S is not metabolic. □

Example 4.5. Let $f > 1$ be a positive integer and let $\mathcal{O} = \mathbb{Z}[fi]$ be the corresponding order in the field $\mathbb{Q}(i)$. Set

$$A = B = 1, \quad C = 0.$$

Then $D^2 = C^2 - AB = -1$. Taking $D = i$ we get

$$d = D = i, \quad d' = -D = -i.$$

Here both d and d' are integral over $\mathcal{O}$ of degree 2. Thus the conditions of Theorem 4.4 are satisfied and hence S is not metabolic. It follows that $\langle S \rangle$ is a nonzero element in the kernel of

$$W\mathbb{Z}[fi] \to W\mathbb{Q}(i).$$

In other words, if $\mathcal{O}$ is an arbitrary non-maximal order of the field $\mathbb{Q}(i)$, then the natural homomorphism $W\mathcal{O} \to W\mathbb{Q}(i)$ is not injective and

$$0 \neq \langle 1, 1 \rangle \in \ker(W\mathcal{O} \to W\mathbb{Q}(i)).$$

Remark 4.6. If d is integral over $\mathcal{O}$ of degree at most n, then

$$\mathcal{D}^{n-1} \subseteq \mathfrak{f} \subseteq \mathcal{J}.$$

Indeed, by Prop. 4.2, we have $a^{n-1}R = \mathcal{D}^{n-1}$ so that $\mathcal{D}^{n-1}$ is a (principal) ideal in R, and hence is contained in the conductor $\mathfrak{f}$. On the other hand $\mathfrak{f} \subseteq \mathcal{J}$ holds even without the assumption about integrality of d as we have observed in (4.1).

Recall that a subring $\mathcal{O}$ of a Dedekind domain $\mathcal{O}_K$ with the field of fractions K is said to be an order in $\mathcal{O}_K$ when $\mathcal{O}$ is a one-dimensional noetherian domain, $\mathcal{O}_K$ is the integral closure of $\mathcal{O}$ in K, and $\mathcal{O}_K$ is a finitely generated $\mathcal{O}-$module.

Proposition 4.7. *Let K be the field of fractions of a Dedekind domain $\mathcal{O}_K$ and let $\mathcal{O}$ be a nonmaximal order in $\mathcal{O}_K$. For $d \in K$ let $\mathfrak{f}$ be the conductor of the ring extension $\mathcal{O} \subset R = \mathcal{O}[d]$. Then $\mathfrak{f}$ is a nonzero ideal if and only if d is integral over $\mathcal{O}$.*

Proof. Fix $x \in \mathfrak{f}$, $x \neq 0$. Then $xd^n \in \mathcal{O}$ for all $n \in \mathbb{N}$. Since $d = \frac{b}{a}$, we have

$$a^n \mid b^n x \quad \text{in} \quad \mathcal{O} \quad \text{for all } n \in \mathbb{N}.$$

If $a \mid b$ in $\mathcal{O}_K$, then $d \in \mathcal{O}_K$ and so d is integral over $\mathcal{O}$ since $\mathcal{O}_K$ is integrally closed. We show that $a \nmid b$ in $\mathcal{O}_K$ leads to a contradiction. So suppose $a \nmid b$ in $\mathcal{O}_K$ and $a^n \mid b^n x$ in $\mathcal{O}$ for all $n \in \mathbb{N}$. Hence also $a \nmid b$ in $\mathcal{O}_K$ and $a^n \mid b^n x$ in $\mathcal{O}_K$ for all $n \in \mathbb{N}$. Since $\mathcal{O}_K$ is a Dedekind domain there exists a prime ideal $\mathfrak{p}$ in $\mathcal{O}_K$ such that

$$\mathfrak{p}^s \parallel a, \quad \mathfrak{p}^t \parallel b \quad \text{and} \quad s > t$$

for some nonnegative integers s, t. Let also $\mathfrak{p}^r \parallel x$ for some $r \geq 0$. Then $a^n \mid b^n x$ implies

$$sn \leq tn + r \quad \text{for all } n \in \mathbb{N}.$$

Hence $s \leq t + \frac{r}{n} < t + 1$ for large n. It follows $s \leq t$, a contradiction.

Conversely, if d is integral over $\mathcal{O}$ of degree at most n, then by Remark 4.6 the nonzero ideal $\mathcal{D}^{n-1}$ is contained in $\mathfrak{f}$, hence $\mathfrak{f} \neq 0$. □

Proposition 4.8. *Let K be the field of fractions of a Dedekind domain $\mathcal{O}_K$ and let $\mathcal{O}$ be a nonmaximal order in $\mathcal{O}_K$. For $d \in K$ let $\mathfrak{f}$ be the conductor of the ring extension $\mathcal{O} \subset R = \mathcal{O}[d]$. The following statements are equivalent.*
(a) *d is integral over $\mathcal{O}$ of degree at most n.*
(b) *$\mathcal{D}^{n-1} \subseteq \mathfrak{f}$.*

Proof. (a) $\Rightarrow$ (b) has already been noticed in Remark 4.6.

(b) $\Rightarrow$ (a) If $\mathcal{D}^{n-1} \subseteq \mathfrak{f}$, then $a^{n-1}, b^{n-1} \in \mathfrak{f}$, that is

$$a^{n-1}\mathcal{O}[d] \subseteq \mathcal{O} \quad \text{and} \quad b^{n-1}\mathcal{O}[d] \subseteq \mathcal{O}.$$

Hence $a^{n-1}d^m, b^{n-1}d^m \in \mathcal{O}$ for all $m \in \mathbb{N}$ and this is equivalent to

$$a^m \mid b^{m+n-1} \quad \text{in} \quad \mathcal{O} \quad \text{for all } m \in \mathbb{N}.$$

If $a \mid b$ in $\mathcal{O}_K$, then as above $d \in \mathcal{O}_K$ is integral over $\mathcal{O}$. We show that $a \nmid b$ in $\mathcal{O}_K$ leads to a contradiction. Since $\mathcal{O}_K$ is a Dedekind domain and $a \nmid b$, there exists a prime ideal $\mathfrak{p}$ in $\mathcal{O}_K$ such that

$$\mathfrak{p}^s \parallel a, \quad \mathfrak{p}^t \parallel b \quad \text{and} \quad s > t$$

for some nonnegative integers s, t. On the other hand $a^m \mid b^{m+n-1}$ implies $sm \leq t(m+n-1)$, that is

$$s \leq t + \tfrac{t(n-1)}{m} < t + 1$$

for large m. It follows $s \leq t$, a contradiction. □

5. Finite extensions of $\mathbb{Q}$

We now point out a special case of Theorem 4.4.

THEOREM 5.1. *Let $\mathcal{O}$ be a domain and let K be the field of fractions of $\mathcal{O}$,* char$K \neq 2$. *Let $\mathcal{O}_K$ be the integral closure of $\mathcal{O}$ in K. Suppose there exists an element $t \in \mathcal{O}_K$ such that*

$$2t, 2t^2 \in \mathcal{O} \quad \textit{and} \quad t \notin \mathcal{O}.$$

Then the natural ring homomorphism $W\mathcal{O} \to WK$ is not injective.

PROOF. We take a free $\mathcal{O}-$module S of rank 2 and define a bilinear form β on S with the matrix (2.1) where

$$A = 2(t-1)t, \quad B = 2, \quad C = 2t - 1.$$

Then $AB - C^2 = -1 = -D^2$ with $D = 1$ and hence S becomes hyperbolic over K. In order that (S, β) be a nonsingular bilinear space over $\mathcal{O}$ we have to assure that $A, B, C \in \mathcal{O}$ so that we require that

$$2(t-1)t, 2t - 1 \in \mathcal{O},$$

and these conditions are satisfied since $2t, 2t^2 \in \mathcal{O}$. Further, we compute

$$d = -t + 1, \quad d' = -t,$$

and these do not belong to $\mathcal{O}$ but do belong to $\mathcal{O}_K$ by hypothesis. Hence, according to Theorem 4.4, (S, β) is a nonmetabolic space. $\square$

The assumptions in Theorem 5.1 can be satisfied whenever $\mathcal{O}_K$ is noetherian. For simplicity, we switch to orders in number fields.

Theorem 5.2. *Let $\mathcal{O}$ be an order in an algebraic number field K and let $\mathcal{O}_K$ be the maximal order in K. Suppose the conductor $\mathfrak{f} = \mathfrak{f}_{\mathcal{O}_K/\mathcal{O}}$ is even in the sense that $\mathfrak{f} \subseteq 2\mathcal{O}_K$. Then the natural ring homomorphism $W\mathcal{O} \to WK$ is not injective.*

Proof. Take an element $2t_1 \in \mathfrak{f}$. If $t_1 \in \mathfrak{f}$, we have $t_1 = 2t_2$ with some $t_2 \in \mathcal{O}_K$. If again $t_2 \in \mathfrak{f}$, we have $t_2 = 2t_3$ with some $t_3 \in \mathcal{O}_K$. Hence $t_1\mathcal{O}_K \subsetneq t_2\mathcal{O}_K \subsetneq t_3\mathcal{O}_K$. Since $\mathcal{O}_K$ is noetherian, this process terminates and there exists $t_0 \in \mathcal{O}_K$ such that $2t_0 \in \mathfrak{f}$ and $t_0 \notin \mathfrak{f}$.

Now observe that there exists $u \in \mathcal{O}_K$ such that $ut_0 \notin \mathcal{O}$. If not, then for any element $u \in \mathcal{O}_K$ we have $ut_0 \in \mathcal{O}$, whence $t_0 \in \mathfrak{f}$, a contradiction. It follows that $2ut_0 \in \mathfrak{f}$, since $\mathfrak{f}$ is an ideal in $\mathcal{O}_K$, and also $2(ut_0)^2 \in \mathfrak{f}$. Thus $t = ut_0$ satisfies the conditions of Theorem 5.1 and the result follows. $\square$

Let $K = \mathbb{Q}(\sqrt{d})$, where d is a square-free integer. We set $\omega = \sqrt{d}$ when $d \equiv 2$ or $3 \pmod 4$, and $\omega = \frac{1}{2}(1 + \sqrt{d})$ when $d \equiv 1 \pmod 4$. Then $\mathbb{Z}[\omega]$ is the maximal order in K and each order is of the form $\mathcal{O} = \mathbb{Z}[f\omega]$ for an integer $f > 1$.

Corollary 5.3. *Let K be an arbitrary quadratic number field with maximal order $\mathbb{Z}[\omega]$ and let $\mathcal{O} = \mathbb{Z}[f\omega]$ be an order in K. If f is an even integer, then the natural ring homomorphism $W\mathcal{O} \to WK$ is not injective.*

Proof. The conductor $\mathfrak{f} = \mathfrak{f}_{\mathcal{O}_K/\mathcal{O}} = f\mathcal{O}_K$ is even, hence Theorem 5.2 applies. Actually we can take $t = \frac{1}{2}f\omega$ in Theorem 5.1. $\square$

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Institute of Mathematics
Silesian University
Bankowa 14
40-007 Katowice
Poland
e-mail: szymiczek@math.us.edu.pl