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SEPARATION THEOREMS FOR CONDITIONAL FUNCTIONAL EQUATIONS

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Abstract. We prove two separation theorems for solutions of conditional Cauchy and Jensen equations.

1. Introduction

Separation (or sandwich) theorems have been widely investigated by several authors, let us mention here only a few papers. A classical result is the Mazur–Orlicz Theorem [6], which was later generalized by R. Kaufman [3] and then further developed by P. Kranz [5]. In 1978 G. Rodé [9] proved his famous result which is a far reaching generalization of the Hahn–Banach Theorem. Three years later P. Volkmann and H. Weigel [11] further generalized this theorem. H. König [4] presented a simpler proof of the Rodé's Theorem and Z. Páles [8] proved a geometric version of this theorem. K. Nikodem, Z. Páles and S. Wąsowicz [7] generalized several older results and also provided necessary (not only sufficient) conditions for the separation.

In spite of great flexibility of the above-mentioned theorems, we do not know if it is possible to apply them to obtain separation theorems for conditional functional equations. In the present paper we use much simpler techniques to prove such statements for the Cauchy and Jensen equation on spheres and its generalizations.

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In 1994 C. Alsina and J.L. Garcia-Roig [1] investigated the following conditional functional equation:

$$\|x\| = \|y\| \implies f(x+y) = f(x) + f(y), \tag{A}$$

where $f\colon X \to Y$ is a map defined on a real inner product space X with $\dim X \geq 2$. They have proved that if $Y = \mathbb{R}^n$, then each solution of (A) is additive. Moreover, if Y is a real linear topological space and $f\colon X \to Y$ is a continuous solution of (A), then f is a continuous linear transformation.

A more general result obtained Gy. Szabó in 1993 [10]. He proved that if X is a real normed linear space with $\dim X \geq 3$ and Y is an abelian group, then each solution of (A) is additive.

In 1997 R. Ger and J. Sikorska investigated the following generalization of equation (A):

$$\varphi(x) = \varphi(y) \implies f(x+y) = f(x) + f(y), \tag{B}$$

where φ is a given map satisfying certain axioms, which are in particular fulfilled by $\varphi = \|\cdot\|$ on a normed linear space. They have found conditions sufficient for an arbitrary solution of (B) to be additive [2, Theorem 1 and Theorem 2]. Moreover, they proved the Hyers–Ulam stability of (B) [2, Theorem 3 and Theorem 4].

In 2001 M. Ziółkowski [12] investigated the following conditional Jensen equation:

$$\varphi(x) = \varphi(y) \implies f\Big(\frac{x+y}{2}\Big) = \frac{f(x)+f(y)}{2}, \tag{C}$$

and the following equation

$$\varphi(x+y) = \varphi(x-y) \implies f\Big(\frac{x+y}{2}\Big) = \frac{f(x)+f(y)}{2}. \tag{D}$$

Under analogous assumptions upon φ to these from the paper of R. Ger and J. Sikorska [2] he proved that the general solution of (C) and of (D) is of the form $f(x) = a(x) + c$, where a is an additive mapping and c is a constant [12, Theorems 1, 2 and 3]. Moreover, he provided some stability results for equations (C) and (D) [12, Theorems 4, 5 and 6].

The purpose of the present paper is to obtain some separation theorems for equations (C) and (D). First, we will prove two general statements and then we will join them with the above-mentioned stability results of R. Ger, J. Sikorska and of M. Ziółkowski. Finally, we make use of their results connected with the solutions of equations (B) and (D) to obtain separation by additive mappings.

2. Main results

In the first theorem we will provide conditions sufficient for the separation for a general conditional Cauchy equation. Assume that $(S,+)$ is an arbitrary semi-group (not necessarily abelian) and $H \subset S \times S$. We will formulate our assumptions in terms of the Hyers–Ulam stability of the general conditional functional equation:

$$(x,y) \in H \implies f(x+y) = f(x) + f(y) \tag{1}$$

for maps $f\colon S \to \mathbb{R}$. We say that the equation (1) is *stable in the sense of Hyers–Ulam* on H, or *stable* on H for short, if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for each $F\colon S \to \mathbb{R}$ satisfying

$$(x,y) \in H \implies |F(x+y) - F(x) - F(y)| \leq \varepsilon, \tag{2}$$

there exists a solution $f\colon S \to \mathbb{R}$ of (1) such that $\|F - f\|_{\sup} \leq \delta$ (where $\|\cdot\|_{\sup}$ denotes the standard supremum norm). Analogically we understand the stability of the conditional Jensen equation:

$$(x,y) \in H \implies f\Big(\frac{x+y}{2}\Big) = \frac{f(x)+f(y)}{2}. \tag{3}$$

Theorem 1. *Assume that $(S,+)$ is a semi-group, $H \subset S \times S$, $p\colon S \to \mathbb{R}$ and $q\colon S \to \mathbb{R}$ satisfy*

$$(x,y) \in H \implies p(x+y) \leq p(x) + p(y), \tag{4}$$

$$(x,y) \in H \implies q(x+y) \geq q(x) + q(y), \tag{5}$$

$q \leq p$ and $\|p - q\|_{\sup} < +\infty$. If

$$x \in S \implies (x,x) \in H, \tag{6}$$

and the conditional Cauchy equation (1) is stable on H, then there exists a solution $f\colon S \to \mathbb{R}$ of (1) such that $q \leq f \leq p$.

Proof. Fix $(x,y)\in H$ arbitrarily and check that

$$p(x+y)-p(x)-p(y)\le 0$$

and

$$\begin{aligned}p(x+y)-p(x)-p(y) &\ge q(x+y)-p(x)-p(y)\\ &\ge q(x+y)-q(x)-q(y)-2\|p-q\|_{\sup}.\end{aligned}$$

Thus, after letting $\varepsilon := 2\|p-q\|_{\sup}$ we arrive at

$$(x,y)\in H \implies |p(x+y)-p(x)-p(y)|\le\varepsilon.$$

From our assumptions it follows that there exist a $\delta>0$ and a solution $f\colon S\to\mathbb{R}$ of (1) such that $\|p-f\|_{\sup}\le\delta$.

Now, by the use of (6) jointly with (1), (4) and (5) we obtain

$$f(2x)=2f(x),\quad p(2x)\le 2p(x),\quad q(2x)\ge 2q(x),\quad x\in S.$$

On the other hand, we have

$$q(x)-\delta\le p(x)-\delta\le f(x)\le p(x)+\delta,\quad x\in S,$$

and thus

$$\begin{aligned}2^nq(x)-\delta\le q(2^nx)-\delta\le f(2^nx)=2^nf(x)&\le p(2^nx)+\delta\\ &\le 2^np(x)+\delta,\quad x\in S.\end{aligned}$$

Divide this estimations side-by-side by 2^n to get

$$q(x)-\frac{1}{2^n}\delta\le f(x)\le p(x)+\frac{1}{2^n}\delta,\quad x\in S.$$

Now, tend with n to $+\infty$ to deduce that

$$q(x)\le f(x)\le p(x),\quad x\in S. \qquad\square$$

Now, we will apply a result of R. Ger and J. Sikorska [2, Theorem 4] which provides a sufficient condition for equation (B) to be stable.

Corollary 2. *Assume that* $(S,+)$ *is abelian semigroup,* Z *is a nonempty set, and* $\varphi\colon S\to Z$ *is a given function which admits a "duplication formula", i.e. there exists a* $\Phi\colon Z\to Z$ *such that*

$$\varphi(2x)=\Phi(\varphi(x)),\quad x\in S. \tag{7}$$

Further, let $p\colon S\to\mathbb{R}$ *and* $q\colon S\to\mathbb{R}$ *satisfy*

$$\varphi(x)=\varphi(y)\implies p(x+y)\le p(x)+p(y), \tag{8}$$

$$\varphi(x)=\varphi(y)\implies q(x+y)\ge q(x)+q(y), \tag{9}$$

$q\le p$ *and* $\|p-q\|_{\sup}<+\infty$. *Then there exists a solution* $f\colon S\to\mathbb{R}$ *of* (B) *such that* $q\le f\le p$.

Proof. It is enough to define

$$H:=\{(x,y)\in S\times S:\varphi(x)=\varphi(y)\}$$

and observe that in this case equations (B) and (1) are equivalent. Thus all assumptions of Theorem 1 are satisfied. □

Now, we will quote two sets of conditions from the paper of R. Ger and J. Sikorska [2] which ensures that each solution of (B) is additive.

(i) for any two linearly independent vectors $x,y\in X$ there exist linearly independent vectors $u,v\in\operatorname{Lin}\{x,y\}$ such that $\varphi(u+v)=\varphi(u-v)$;
(ii) if $x,y\in X$, $\varphi(x+y)=\varphi(x-y)$, then $\varphi(\alpha x+y)=\varphi(\alpha x-y)$ for all $\alpha\in\mathbb{R}$;
(iii) for all $x\in X$ and $\lambda>0$ there exists an $y\in X$ such that $\varphi(x+y)=\varphi(x-y)$ and $\varphi((\lambda+1)x)=\varphi((\lambda-1)x-2y)$.

The second set of assumptions involves consideration of a binary relation $\prec$ on a topological group Z:

(a) for every $x\in Z$ the relationship $0\prec x$ implies that $-x\prec 0$;
(b) the half-lines $\{x\in Z:x\prec 0\}$ and $\{x\in Z:0\prec x\}$ are disjoint and open in Z.

Corollary 3. *Assume that* X *is a real linear space with* $\dim X\ge 2$, Z *is a given nonempty set and* $\varphi\colon X\to Z$ *is an even mapping satisfying conditions* (i), (ii) *and* (iii). *Further, let* $p\colon X\to\mathbb{R}$ *and* $q\colon X\to\mathbb{R}$ *satisfy* (8), (9), $q\le p$ *and* $\|p-q\|_{\sup}<+\infty$. *Then there exists an additive mapping* $f\colon S\to\mathbb{R}$ *such that* $q\le f\le p$.

Corollary 4. *Assume that* $(X,+)$ *and* $(Z,+)$ *are topological groups,* $(X,+)$ *is commutative,* $(Z,+)$ *is equipped with a connected binary relation* $\prec \subset Z \times Z$ *satisfying conditions* (a) *and* (b) *and* $\varphi\colon X \to Z$ *is a continuous mapping such that for every* $x, y \in X$ *the set*

$$\{t \in X : \varphi(x+t) = \varphi(x-t) = \varphi(y)\} \tag{10}$$

is nonempty and connected provided that $\varphi(x) \prec \varphi(y)$*. Further, let* $p\colon X \to \mathbb{R}$ *and* $q\colon X \to \mathbb{R}$ *satisfy* (8), (9), $q \le p$ *and* $\|p-q\|_{\sup} < +\infty$*. Then there exists an additive mapping* $f\colon X \to \mathbb{R}$ *such that* $q \le f \le p$*.*

A special case of equation (B) is equation (A). Therefore, we have the following corollary.

Corollary 5. *Assume that* X *is a real normed linear space with* $\dim X \ge 2$ *and let* $p\colon X \to \mathbb{R}$ *and* $q\colon S \to \mathbb{R}$ *satisfy*

$$\|x\| = \|y\| \implies p(x+y) \le p(x) + p(y),$$
$$\|x\| = \|y\| \implies q(x+y) \ge q(x) + q(y),$$

$q \le p$ *and* $\|p-q\|_{\sup} < +\infty$*. Then there exists an additive mapping* $f\colon X \to \mathbb{R}$ *such that* $q \le f \le p$*.*

Now, we will provide an analogue to Theorem 1 for the conditional Jensen equation.

Theorem 6. *Assume that* $(S,+)$ *is a uniquely* 2*-divisible semi-group with the neutral element* 0*,* $H \subset S \times S$*,* $p\colon S \to \mathbb{R}$ *and* $q\colon S \to \mathbb{R}$ *satisfy*

$$(x,y) \in H \implies p\Big(\frac{x+y}{2}\Big) \le \frac{p(x)+p(y)}{2}, \tag{11}$$

$$(x,y) \in H \implies q\Big(\frac{x+y}{2}\Big) \ge \frac{q(x)+q(y)}{2}, \tag{12}$$

$p \le q$*,* $p(0) = q(0) = 0$ *and* $\|p-q\|_{\sup} < +\infty$*. If*

$$x \in S \implies (x,0) \in H, \tag{13}$$

and the conditional Jensen equation (3) *is stable on* H*, then there exists a solution* $f\colon S \to \mathbb{R}$ *of* (3) *such that* $p \le f \le q$*.*

Proof. The proof is analogous to the proof of Theorem 1. For arbitrarily fixed $(x,y)\in H$ we have

$$p\Big(\frac{x+y}{2}\Big)-\frac{p(x)+p(y)}{2}\le 0$$

and

$$\begin{aligned}p\Big(\frac{x+y}{2}\Big)-\frac{p(x)+p(y)}{2}&\ge p\Big(\frac{x+y}{2}\Big)-\frac{q(x)+q(y)}{2}\\&\ge q\Big(\frac{x+y}{2}\Big)-\frac{q(x)+q(y)}{2}-\|p-q\|_{\sup}.\end{aligned}$$

Put $\varepsilon:=\|p-q\|_{\sup}$ to get

$$(x,y)\in H\implies\Big|p\Big(\frac{x+y}{2}\Big)-\frac{p(x)+p(y)}{2}\Big|\le\varepsilon.$$

From the assumptions it follows that there exist a $\delta>0$ and a solution $f\colon S\to\mathbb{R}$ of (3) such that $\|p-f\|_{\sup}\le\delta$.

Now, use of (13) jointly with (3), (11), (12) and apply the equality $p(0)=q(0)=0$ to obtain

$$f(2x)=2f(x)-f(0),\quad p(2x)\ge 2p(x),\quad q(2x)\le 2q(x),\quad x\in S.$$

We have

$$p(x)-\delta\le f(x)\le q(x)+\delta,\quad x\in S,$$

and thus

$$\begin{aligned}2^np(x)-\delta\le p(2^nx)-\delta\le f(2^nx)=2^nf(x)-(2^n-1)f(0)\le q(2^nx)+\delta\\\le 2^nq(x)+\delta,\quad x\in S.\end{aligned}$$

From this one may deduce the estimate

$$p(x)\le f(x)-f(0)\le q(x),\quad x\in S.$$

To finish the proof it remains to replace f by $f-f(0)$. $\square$

Remark 7. If $p\colon S \to \mathbb{R}$ and $q\colon S \to \mathbb{R}$ are arbitrary solutions of (11) and (12) and condition (13) is fulfilled, then the mappings $p'\colon S \to \mathbb{R}$ and $q'\colon S \to \mathbb{R}$ given by

$$p'(x) := p(x) - p(0), \quad q'(x) := q(x) - q(0), \quad x \in S$$

are solutions of (11) and (12) which satisfy $p'(0) = q'(0) = 0$. However, in general $p \le q$ does not imply that $p' \le q'$. Therefore, the assumption that p and q vanish at zero in the previous theorem cannot be dropped in that way but it is enough to assume that $p(0) \ge q(0)$ only. We do not know whether this assumption can be omitted completely.

Now, make use of a result of M. Ziółkowski [12, Theorem 6] which provides a sufficient condition for equation (D) to be stable.

Corollary 8. *Assume that $(S,+)$ is uniquely 2-divisible abelian group, Z is a nonempty set, a function $\varphi\colon S \to Z$ satisfies*

$$\varphi(x) = \varphi(y) \implies \varphi(2x) = \varphi(2y), \quad x \in S. \tag{14}$$

Further, let $p\colon S \to \mathbb{R}$ and $q\colon S \to \mathbb{R}$ satisfy

$$\varphi(x+y) = \varphi(x-y) \implies p\Big(\frac{x+y}{2}\Big) \le \frac{p(x)+p(y)}{2}, \tag{15}$$

$$\varphi(x+y) = \varphi(x-y) \implies q\Big(\frac{x+y}{2}\Big) \ge \frac{q(x)+q(y)}{2}, \tag{16}$$

$p \le q$, $p(0) = q(0) = 0$ and $\|p-q\|_{\sup} < +\infty$. Then there exists a solution $f\colon S \to \mathbb{R}$ of (D) *such that $p \le f \le q$.*

Proof. Define

$$H := \{(x,y) \in S \times S : \varphi(x+y) = \varphi(x-y)\}$$

and observe that in this case (3) is equivalent to (D). Thus our Theorem 6 is applicable. $\square$

Now, we will state three results, analogous to Corollaries 3, 4 and 5, which provides separation by additive mappings.

Corollary 9. *Assume that* X *is a real linear space with* $\dim X \geq 2$*,* Z *is a given nonempty set and* $\varphi\colon X \to Z$ *is an even mapping satisfying conditions* (i), (ii) *and* (iii)*. Further, let* $p\colon X \to \mathbb{R}$ *and* $q\colon X \to \mathbb{R}$ *satisfy* (15), (16), $p \leq q$, $p(0) = q(0) = 0$ *and* $\|p - q\|_{\sup} < +\infty$*. Then there exists an additive mapping* $f\colon X \to \mathbb{R}$ *such that* $p \leq f \leq q$.

Corollary 10. *Assume that* $(X, +)$ *and* $(Z, +)$ *are topological groups,* $(X, +)$ *is commutative and uniquely* 2*-divisible,* $(Z, +)$ *is equipped with a connected binary relation* $\prec \subset Z \times Z$ *satisfying conditions* (a) *and* (b) *and* $\varphi\colon X \to Z$ *is a continuous mapping such that for every* $x, y \in X$ *the set* (10) *is nonempty and connected provided that* $\varphi(x) \prec \varphi(y)$ *and such that for all* $x \in X$ *we have* $\varphi(0) = \varphi(x)$ *or* $\varphi(0) \prec \varphi(x)$*. Further, let* $p\colon S \to \mathbb{R}$ *and* $q\colon S \to \mathbb{R}$ *satisfy* (15), (16), $p \leq q$, $p(0) = q(0) = 0$ *and* $\|p - q\|_{\sup} < +\infty$*. Then there exists an additive mapping* $f\colon S \to \mathbb{R}$ *such that* $p \leq f \leq q$.

Corollary 11. *Assume that* X *is a real normed linear space with* $\dim X \geq 2$*,* $p\colon X \to \mathbb{R}$ *and* $q\colon X \to \mathbb{R}$ *satisfy*

$$\|x + y\| = \|x - y\| \implies p\Big(\frac{x + y}{2}\Big) \leq \frac{p(x) + p(y)}{2},$$
$$\|x + y\| = \|x - y\| \implies q\Big(\frac{x + y}{2}\Big) \geq \frac{q(x) + q(y)}{2},$$

$q \leq p$, $p(0) = q(0) = 0$ *and* $\|p - q\|_{\sup} < +\infty$*. Then there exists an additive mapping* $f\colon X \to \mathbb{R}$ *such that* $q \leq f \leq p$.

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