Annales Mathematicae Silesianae **21** (2007), 41–48
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# INFINITE PRODUCT FOR $e^{6\zeta(3)}$

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**Abstract.** The author uses the summation of rational series using the properties of the digamma function $\Psi(x)$ and the methods of the residue calculus to evaluate the function $H_\alpha(x)$ for $\alpha = 1$ and $x = a^{-1}(N)$, $N \in \mathbb{N}$ (see Theorem 1) which is called the function generating the generalized harmonic numbers of order 1 (see Definition 1). The relation between the functions $H_1(x)$, $x > 0$, and $\Psi(x)$ is used to find the approximations of the constant $e^{6\zeta(3)}$ in the form of the infinite product which contains only the numbers $e$, $\pi$ and the roots of unity, where $\zeta(3)$ is the Apéry constant.

## 1. Introduction

The study of the arithmetical nature of the values of the Riemann function $\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}$ at integers $s > 1$ is one of the most attractive topics of the modern number theory. Euler's formula

$$\zeta(s) = -\frac{(2\pi \mathrm{i})^s B_s}{2s!}, \quad s = 2, 4, 6, \ldots$$

marked the first progress in this area. Some criteria for irrationality of such kind of factorial series can be found in [4] and [5]. In 1882 F. Lindemann proved that $\pi$ is transcendental, which implies that $\zeta(s)$ is transcendental if $s$ is even. The problem of the irrationality of the values of $\zeta(s)$ at odd integers is not solved yet, except the case $s = 3$, which was proved by Apéry in 1978.

*Received: 2.07.2007. Revised: 30.11.2007.*
(2000) Mathematics Subject Classification: 40C10, 33B15.
*Key words and phrases:* summation methods, infinite series, infinite products.

There are many papers concerning the results involving the value $\zeta(3)$. In this paper we investigate the possibility to express the value $e^{6\zeta(3)}$ using the numbers $\pi$, $e$ and the roots of unity as the infinite product. J. Sondow and J. Guillera (see [3] and [6]) found new infinite products of many classical constants such as $\gamma$, $\pi/e$, $e^{\gamma}$ or $e^{7\zeta(3)/4\pi^2}$ in 2003 and 2005. Certain interesting infinite product expansions were obtained by Gosper in 1996:

$$\prod_{n=1}^{\infty}\begin{pmatrix} \frac{(n+1)^4}{4\,096\left(n+\frac{5}{4}\right)^2\left(n+\frac{7}{4}\right)^2} & \frac{24\,570n^4+64\,161n^3+62\,152n^2+26\,427n+4\,154}{31\,104\left(n+\frac{1}{3}\right)\left(n+\frac{1}{2}\right)\left(n+\frac{2}{3}\right)} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \zeta(3) \\ 0 & 1 \end{pmatrix}.$$

Transcendence criteria of special infinite products are investigated by J. Hančl and P. Corvaja in [2].

## 2. Main result

Theorem 1. *Let $N \in \mathbb{N}$, $\zeta_{1,N} := -1$ for all $N \in \mathbb{N}$, $\zeta_{j,N} := \varepsilon_{j,N} - 1$, $j = 2, \dots, N$, where $\varepsilon_{j,N} \neq 1$ are the $(N-1)$ solutions of the equation $\varepsilon_{j,N}^N = 1$ and*

$$Z_{n,N} := \frac{(\zeta_{n,N}+1)^{N-1}-1}{\zeta_{n,N}\prod_{j=1,\ j\neq n}^{N}(\zeta_{n,N}-\zeta_{j,N})}, \quad 1 \leq n \leq N.$$

*Then*

$$e^{6\zeta(3)} = \lim_{N\to+\infty}\prod_{j=1}^{a(N)}\frac{e^{\pi^2-6a^2(N)}}{\zeta_{j,a(N)}^{6a^3(N)Z_{j,a(N)}}},$$

*where $a(N)\colon \mathbb{N} \to \mathbb{N}$ is an increasing function.*

For the proof of Theorem 1 it is convenient to introduce the function $H_\alpha(x)$ generating the generalized harmonic numbers of order 1.

Definition 1. Let $\alpha \in \mathbb{R}^+$ and $x \in \mathbb{R}^+$. Then the function

$$H_\alpha(x) := \int_{1-\frac{1}{\alpha}}^{1}\frac{1-(1-t)^x}{t}\,dt$$

is called the function generating the generalized harmonic numbers of order 1.

COROLLARY 1. *Let $N \in \mathbb{N}$. Then*

$$H_1(N) = \sum_{j=1}^{N} \frac{1}{j}.$$

LEMMA 1. *Let $N \in \mathbb{N}$, $N > 2$ and $a := \frac{1}{M}$, $M \in \mathbb{N}$, $M > 2$. Then*

$$0 \le \Delta\Psi(N,a) \le \frac{1}{\sqrt{(2(N+a)-1)(1-2a)}},$$

*where $\Delta\Psi(N,a) := \Psi(N+a) - \Psi(N)$.*

PROOF. Let $N \in \mathbb{N}$, $N > 2$ and $a := \frac{1}{M}$, $M \in \mathbb{N}$, $M > 2$. Using the formula

$$\Psi(z) = \int_0^{+\infty} \left(e^{-t} - \frac{1}{(1+t)^z}\right) \frac{dt}{t}, \quad \mathfrak{Re}(z) > 0, \tag{1}$$

we get

$$\begin{aligned}
\Delta\Psi(N,a) &= \int_0^{+\infty} \left(e^{-t} - \frac{1}{(1+t)^{N+a}} - e^{-t} + \frac{1}{(1+t)^N}\right) \frac{dt}{t} \\
&= \int_0^{+\infty} \left(\frac{1}{(1+t)^N} - \frac{1}{(1+t)^{N+a}}\right) \frac{dt}{t} \\
&= \int_0^{+\infty} \frac{(1+t)^a - 1}{t} \frac{dt}{(1+t)^{N+a}} \\
&\le \sqrt{\int_0^{+\infty} \left(\frac{(1+t)^a - 1}{t}\right)^2 dt \int_0^{+\infty} \left(\frac{1}{(1+t)^{N+a}}\right)^2 dt} \\
&= \sqrt{\frac{1}{2(N+a)-1} \int_0^{+\infty} \left(\frac{(1+t)^a - 1}{t}\right)^2 dt}.
\end{aligned}$$

For the last integral we obtain the estimation

$$\int_0^{+\infty}\left(\frac{(1+t)^a-1}{t}\right)^2 dt=\int_0^1\left(\frac{1-\zeta^{-a}}{1-\zeta}\right)^2 d\zeta=\int_0^1\left(\frac{1-\zeta^{-\frac{1}{M}}}{1-\zeta}\right)^2 d\zeta$$

$$=M\int_0^1\left(\frac{1-\frac{1}{\tau}}{1-\tau^M}\right)^2\tau^{M-1}\,d\tau$$

$$=M\int_0^1\frac{\tau^{M-3}}{\left(\sum_{j=1}^M\tau^{j-1}\right)^2}\,d\tau$$

$$\leq M\int_0^1\tau^{M-3}\,d\tau=\frac{1}{1-2a}.$$

This implies that

$$\Delta\Psi(N,a)\leq\frac{1}{\sqrt{(2(N+a)-1)(1-2a)}}.$$

The inequality $0\leq\Delta\Psi(N,a)$ follows easily from (1), which completes the proof of the lemma. □

Proof of Theorem 1. Set

$$\phi(a):=\sum_{\substack{n=1\\a>0}}^{+\infty}\frac{1}{n^3+an^2}. \tag{2}$$

The infinite series on the right hand side of (2) represents for $a\in\mathbb{R}$ the absolute convergent infinite series and thus the value $\phi(a)$ is defined for all $a\in\mathbb{R}$. Using the definition of the Riemann $\zeta$-function, we get the fact that $\lim_{a\to0}\phi(a)=\zeta(3)$.

Now, for the function $\phi(a)$ we obtain $(a>0)$

$$\phi(a)=\sum_{n=1}^{+\infty}\left(\frac{1}{an^2}-\frac{1}{a^2}\left(\frac{1}{n}-\frac{1}{n+a}\right)\right)$$

$$=\frac{1}{a}\sum_{n=1}^{+\infty}\frac{1}{n^2}-\frac{1}{a^2}\lim_{\substack{N\to+\infty\\N\in\mathbb{N}}}\sum_{n=1}^{N}\left(\frac{1}{n}-\frac{1}{n+a}\right).$$

Using the recursion formula $\Psi(z+1) = \Psi(z) + \frac{1}{z}$, we have

$$\sum_{n=1}^{N} \frac{1}{n} = \sum_{n=1}^{N} \left(\Psi(n+1) - \Psi(n)\right) = \Psi(N+1) - \Psi(1) = \Psi(N+1) + \gamma,$$

and

$$\sum_{\substack{n=1\\a>0}}^{N} \frac{1}{n+a} = \sum_{\substack{n=1\\a>0}}^{N} \left(\Psi(n+a+1) - \Psi(n+a)\right) = \Psi(N+a+1) - \Psi(a+1).$$

This and Lemma 1 implies that

$$\lim_{\substack{N\to+\infty\\N\in\mathbb{N}}} \sum_{\substack{n=1\\a>0}}^{N} \left(\frac{1}{n} - \frac{1}{n+a}\right) = -\lim_{\substack{N\to+\infty\\N\in\mathbb{N}}} \Big( \left(\Psi(N+a+1) - \Psi(N+1)\right) + \left(\Psi(a+1) - \Psi(1)\right) \Big) = \Psi(a+1) - \Psi(1),$$

which yields the fact that

$$\phi(a) = \frac{\pi^2}{6a} - \frac{1}{a^2}\left(\Psi(a+1) - \Psi(1)\right).$$

Note that $-\Psi(1) = \gamma$ is the Euler–Mascheroni constant. Using the integral representation for the term $\Psi(a+1) - \Psi(1)$, we obtain the formula

$$\phi(a) = \frac{\pi^2}{6a} - \frac{1}{a^2} \int_0^1 \frac{1-t^a}{1-t}\, dt. \tag{3}$$

From the fact that $\lim_{a\to 0} \phi(a) = \zeta(3)$ we have the relation

$$\lim_{N\to+\infty} \phi\left(a^{-1}(N)\right) = \zeta(3),$$

where $N \in \mathbb{N}$. The integral in (3) can be treated as the value of the generalized harmonic number function $H_\alpha(x)$ for $\alpha = 1$ and $x := a = a^{-1}(N)$.

The value $H_1(a^{-1}(N)) =: H\left(a^{-1}(N)\right)$, $N \in \mathbb{N}$, can be computed as follows:

$$\begin{aligned} H\left(a^{-1}(N)\right) &= \int_0^1 \frac{1-t^{a^{-1}(N)}}{1-t}\,dt = a(N)\int_0^1 \frac{1-\tau}{1-\tau^{a(N)}}\tau^{a(N)-1}\,d\tau \\ &= a(N)\int_0^1 \frac{\tau^{a(N)-1}}{\sum_{j=1}^{a(N)}\tau^{j-1}}\,d\tau \\ &= a(N)\int_0^1 \frac{\sum_{j=1}^{a(N)}\tau^{j-1}-\sum_{j=1}^{a(N)-1}\tau^{j-1}}{\sum_{j=1}^{a(N)}\tau^{j-1}}\,d\tau \\ &= a(N)-a(N)\int_0^1 \frac{\sum_{j=1}^{a(N)-1}\tau^{j-1}}{\sum_{j=1}^{a(N)}\tau^{j-1}}\,d\tau \\ &= a(N)-a(N)\int_0^{+\infty} \frac{\sum_{j=1}^{a(N)-1}(\zeta+1)^{1-j}}{\sum_{j=1}^{a(N)}(\zeta+1)^{1-j}}\frac{d\zeta}{(\zeta+1)^2} \\ &= a(N)-a(N)\int_0^{+\infty} \frac{\sum_{j=1}^{a(N)-1}(\zeta+1)^{j-1}}{\sum_{j=1}^{a(N)}(\zeta+1)^{j-1}}\frac{d\zeta}{\zeta+1} \\ &= a(N)-a(N)\int_0^{+\infty} \frac{(\zeta+1)^{a(N)-1}-1}{(\zeta+1)^{a(N)}-1}\frac{d\zeta}{\zeta+1}. \end{aligned}$$

Using the fact that

$$\deg\Big((\zeta+1)\cdot((\zeta+1)^{a(N)}-1)\Big) = \deg\Big((\zeta+1)^{a(N)-1}-1\Big)+2, \quad \forall N\in\mathbb{N},$$

we can compute the last integral for $H\left(a^{-1}(N)\right)$ with the help of the residue calculus.

For the brevity we write

$$h_{a(N)}(\zeta) = \frac{(\zeta+1)^{a(N)-1}-1}{(\zeta+1)^{a(N)}-1}\cdot\frac{1}{\zeta+1}.$$

Let $\zeta_{1,a(N)} = -1$ and $\zeta_{j,a(N)} = \varepsilon_{j,a(N)}-1$, $j = 2,\dots,a(N)$, where $\varepsilon_{j,a(N)}$ are the solutions of the equation $\varepsilon_{j,a(N)}^{a(N)} = 1$, except the trivial solution 1. Then

$$h_{a(N)}(\zeta) = \frac{(\zeta+1)^{a(N)-1}-1}{\zeta\prod_{j=2}^{a(N)}\left(\zeta-\zeta_{j,a(N)}\right)}\cdot\frac{1}{\zeta+1} = \frac{(\zeta+1)^{a(N)-1}-1}{\zeta\prod_{j=1}^{a(N)}\left(\zeta-\zeta_{j,a(N)}\right)}, \quad \zeta\neq 0.$$

This implies that

$$\begin{aligned}\int_0^{+\infty} h_{a(N)}(\zeta)\,d\zeta &= -\sum_{n=1}^{a(N)} \operatorname*{Res}_{\zeta=\zeta_{n,a(N)}} h_{a(N)}(\zeta)\ln\zeta \\ &= -\sum_{n=1}^{a(N)} \lim_{\zeta\to\zeta_{n,a(N)}} \left( \frac{(\zeta+1)^{a(N)-1}-1}{\zeta\prod_{j=1,\ j\neq n}^{a(N)}(\zeta-\zeta_{j,a(N)})} \ln\zeta \right) \\ &= -\sum_{n=1}^{a(N)} \left( \frac{(\zeta_{n,a(N)}+1)^{a(N)-1}-1}{\zeta_{n,a(N)}\prod_{j=1,\ j\neq n}^{a(N)}(\zeta_{n,a(N)}-\zeta_{j,a(N)})} \ln\zeta_{n,a(N)} \right).\end{aligned}$$

Setting

$$Z_{n,a(N)} := \frac{(\zeta_{n,a(N)}+1)^{a(N)-1}-1}{\zeta_{n,a(N)}\prod_{j=1,\ j\neq n}^{a(N)}(\zeta_{n,a(N)}-\zeta_{j,a(N)})}$$

we obtain finally

$$H\left(a^{-1}(N)\right) = a(N) - a(N)\ln\prod_{j=1}^{a(N)} \zeta_{j,a(N)}^{-Z_{j,a(N)}}.$$

Thus

$$\phi\left(a^{-1}(N)\right) = \frac{1}{6}\ln\prod_{j=1}^{a(N)} \frac{e^{\pi^2-6\cdot a^2(N)}}{\zeta_{j,a(N)}^{6\cdot a^3(N)Z_{j,a(N)}}}.$$

Since $\lim\limits_{N\to+\infty}\phi\left(a^{-1}(N)\right)=\zeta(3)$, we obtain

$$e^{6\zeta(3)} = \lim_{N\to+\infty}\prod_{j=1}^{a(N)} \frac{e^{\pi^2-6a^2(N)}}{\zeta_{j,a(N)}^{6a^3(N)Z_{j,a(N)}}},$$

which completes the proof of Theorem 1. □

Remark 1. Note that the result in Theorem 1 must be taken in the sense that there exists a branch in the complex plain of the number

$$A_N := \prod_{j=1}^{a(N)} \frac{e^{\pi^2-6a^2(N)}}{\zeta_{j,a(N)}^{6a^3(N)Z_{j,a(N)}}}$$

such that $A_N \in \mathbb{R}$ for every $N \in \mathbb{N}$.

Open problem. Is the number $e^{\zeta(3)}$ transcendental?

**Acknowledgments.** I would like to thank professor Mišík for the help with applications in the complex analysis and professor Hančl for language corrections.

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