THE CONTINUITY OF MULTILINEAR INTEGRAL OPERATORS ON MORREY SPACES

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Abstract. In this paper, the continuity of some multilinear integral operators on Morrey spaces are obtained. The operators contain singular integral operators, Littlewood–Paley operators, Marcinkiewicz operators and Bochner–Riesz operators.

1. Introduction

As the development of singular integral operators, their commutators and multilinear operators have been well studied (see [1–6]). From [5] [6], we know that the commutators and multilinear operators are bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. As the Morrey spaces may be considered as an extension of the Lebesgue spaces, and their have played an important role in studying the local behaviour of solutions for the partial differential equations(see [7–9]), it is natural and important to study the boundedness for the operators on the Morrey spaces. In [7], the authors obtain the boundedness for a large class of sublinear operators and commutators on the Morrey spaces. The purpose of this paper is to study the continuity of some multilinear integral operators on Morrey spaces, which contain singular integral operators, Littlewood–Paley operators, Marcinkiewicz operators and Bochner–Riesz operators.

Received: 16. 06. 2005. Revised: 30. 10. 2005.
(1991) Mathematics Subject Classification: 42B20, 42B25.
Key words and phrases: Multilinear operator, Singular integral operator, Littlewood–Paley operator, Marcinkiewicz operator, Bochner–Riesz operator, Morrey space, BMO.
Supported by the NNSF (Grant: 10271071).

2. Preliminaries and Theorems

Throughout this paper, denote p' by $1/p+1/p'=1$ for $1<p<\infty$. Let φ be a positive, increasing function on $\mathbb{R}^+$ and there exists a constant $D>0$ such that

$$\varphi(2t) \le D\varphi(t), \quad \text{for all} \quad t \ge 0.$$

Let f be a locally integrable function on $\mathbb{R}^n$, we define that, for $1 \le p < \infty$,

$$||f||_{L^{p,\varphi}} = \sup_{x\in\mathbb{R}^n,\ r>0} \left(\frac{1}{\varphi(r)} \int_{B(x,r)} |f(y)|^p dy \right)^{1/p},$$

where, and in what follows, $B = B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$ is a ball in $\mathbb{R}^n$. The Morrey spaces are defined by

$$L^{p,\varphi}(\mathbb{R}^n) = \{f \in L^1_{loc}(\mathbb{R}^n) : ||f||_{L^{p,\varphi}} < \infty\}.$$

If $\varphi(r) = r^\delta$ with $\delta \ge 0$, then $L^{p,\varphi} = L^{p,\delta}$, which is the classical Morrey spaces (see [17] [18]).

For a set E and a locally integrable function f, let $f(E) = \int_E f(x)dx$ and $f_E = |E|^{-1}\int_E f(x)dx$. For any locally integrable function f, the sharp function of f is defined by

$$f^\#(x) = \sup_{x\in B} \frac{1}{|B|} \int_B |f(y) - f_B| dy.$$

It is well-known that (see [19])

$$f^\#(x) = \sup_{x\in B} \inf_{c\in C} \frac{1}{|B|} \int_B |f(y) - c| dy.$$

We say that f belongs to $BMO(\mathbb{R}^n)$ if $f^\#$ belongs to $L^\infty(\mathbb{R}^n)$ and $||f||_{BMO} = ||f^\#||_{L^\infty}$.

Let $M(f)$ be the Hardy–Littlewood maximal operator, we define that, for $1 \le p < \infty$,

$$M_p(f) = (M|f|^p)^{1/p}, \quad f^\#(x) = \sup_{x\in B} \frac{1}{|B|} \int_B |f(y) - f_B| dy.$$

In this paper, we will study some multilinear integral operators as following.

Let m_j be the positive integers ($j = 1, \cdots, l$), $m_1 + \cdots + m_l = m$ and A_j be the functions on $\mathbb{R}^n$ ($j = 1, \cdots, l$). Set

$$R_{m_j+1}(A_j; x, y) = A_j(x) - \sum_{|\alpha|\le m_j} \frac{1}{\alpha!} D^\alpha A_j(y)(x-y)^\alpha.$$

DEFINITION 1. Let S and S' be Schwartz space and its dual and $T: S \to S'$ be the linear operator. The multilinear integral operator associated to T is defined by

$$T^A(f)(x) = T\left(|x - \cdot|^{-m} \prod_{j=1}^{l} R_{m_j+1}(A_j; x, \cdot) f(\cdot) \right)(x).$$

DEFINITION 2. Fix $t > 0$. Let $F_t : S \to S'$ be the linear operator. Set

$$F_t^A(f)(x) = F_t\left(|x - \cdot|^{-m} \prod_{j=1}^{l} R_{m_j+1}(A_j; x, \cdot) f(\cdot)\right)(x).$$

Let H be the Banach space $H = \{h : ||h|| < \infty\}$ such that, for each fixed $x \in \mathbb{R}^n$, $F_t(f)(x)$ and $F_t^A(f)(x)$ may be viewed as a mapping from $[0, +\infty)$ to H. Then, the multilinear integral operator related to F_t is defined by

$$S^A(f)(x) = ||F_t^A(f)(x)||,$$

We define that $S(f)(x) = ||F_t(f)(x)||$.

Note that, when $m = 0$,

$$T^A(f)(x) = A(x)T(f)(x) - T(Af)(x)$$

and

$$S^A(f)(x) = ||A(x)F_t(f)(x) - A(x)F_t(Af)(x)||,$$

which are the commutators generated by T, S and A.

Now we state our results as following.

THEOREM 1. *Let $1 < p < \infty$ and $0 < D < 2^n$. If the multilinear integral operator T^A is bounded on $L^p(\mathbb{R}^n)$ and satisfies the following size condition:*

$$|T(f)(x)| \le C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} dy$$

for any $f \in L^1(\mathbb{R}^n)$ with compact support and $x \notin \operatorname{supp} f$, then

$$||T^A(f)||_{L^{p,\varphi}} \le C||f||_{L^{p,\varphi}}.$$

THEOREM 2. *Let $1 < p < \infty$ and $0 < D < 2^n$. If the multilinear integral operator S^A is bounded on $L^p(\mathbb{R}^n)$ and satisfies the following size condition:*

$$S(f)(x) \le C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} dy$$

for any $f \in L^1(\mathbb{R}^n)$ with compact support and $x \notin \operatorname{supp} f$, then

$$||S^A(f)||_{L^{p,\varphi}} \le C||f||_{L^{p,\varphi}}.$$

REMARK 1. The size conditions in **Theorem 1 and 2** are satisfied by many operators. Now we give some examples.

EXAMPLE 1. Singular integral operators.

Let T be the singular integral operators (see [4], [19–20]) such that

$$T(f)(x) = \int_{\mathbb{R}^n} K(x,y) f(y) dy$$

for every bounded and compactly supported function f, where K satisfies: for $\varepsilon > 0$,

$$|K(x,y)| \le C|x-y|^{-n}$$

and

$$|K(y,x) - K(z,x)| + |K(x,y) - K(x,z)| \le C|y-z|^{\varepsilon}|x-z|^{-n-\varepsilon}$$

if $2|y-z| \le |x-z|$. The multilinear operator related to T is defined by

$$T^A(f)(x) = \int_{\mathbb{R}^n} \frac{\prod_{j=1}^{l} R_{m_j+1}(A_j;x,y)}{|x-y|^m} K(x,y)f(y)dy.$$

Then, it is easy to see that T^A satisfies the conditions of Theorem 1 (see [4], [6]), thus the conclusion of Theorem 1 holds for T^A.

EXAMPLE 2. Littlewood–Paley operators.

Fixed $\varepsilon > 0$ and $\mu > (3n+2)/n$. Let ψ be a fixed function which satisfies:

(1) $\int_{\mathbb{R}^n} \psi(x)dx = 0$,

(2) $|\psi(x)| \le C(1+|x|)^{-(n+1)}$,

(3) $|\psi(x+y) - \psi(x)| \le C|y|^{\varepsilon}(1+|x|)^{-(n+1+\varepsilon)}$ when $2|y| < |x|$;

We denote that $\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. The multilinear Littlewood–Paley operators are defined by

$$g_\psi^A(f)(x) = \left(\int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

$$S_\psi^A(f)(x) = \left[\int\int_{\Gamma(x)} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2}$$

and

$$g_\mu^A(f)(x) = \left[\int\int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{n\mu} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$F_t^A(f)(x) = \int_{\mathbb{R}^n} \frac{\prod_{j=1}^{l} R_{m_j+1}(A_j;x,y)}{|x-y|^m} \psi_t(x-y)f(y)dy,$$

$$F_t^A(f)(x,y) = \int_{\mathbb{R}^n} \frac{\prod_{j=1}^{l} R_{m_j+1}(A_j;x,z)}{|x-z|^m} f(z)\psi_t(y-z)dz$$

and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. Set $F_t(f)(y) = f * \psi_t(y)$. We also define that

$$g_\psi(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

$$S_\psi(f)(x) = \left(\int\int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}$$

and

$$g_\mu(f)(x) = \left(\int\int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{n\mu} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2},$$

which are the Littlewood–Paley operators (see [20]). Let H be the space

$$H = \left\{h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t\right)^{1/2} < \infty\right\}$$

or

$$H = \left\{h : ||h|| = \left(\int\int_{\mathbb{R}^{n+1}_+} |h(y,t)|^2 dydt/t^{n+1}\right)^{1/2} < \infty\right\},$$

then, for each fixed $x \in \mathbb{R}^n$, $F_t^A(f)(x)$ and $F_t^A(f)(x,y)$ may be viewed as the mapping from $[0,+\infty)$ to H, and it is clear that

$$g_\psi^A(f)(x) = ||F_t^A(f)(x)||, \quad g_\psi(f)(x) = ||F_t(f)(x)||,$$

$$S_\psi^A(f)(x) = \left|\left|\chi_{\Gamma(x)} F_t^A(f)(x,y)\right|\right|, \quad S_\psi(f)(x) = \left|\left|\chi_{\Gamma(x)} F_t(f)(y)\right|\right|$$

and

$$g_\mu^A(f)(x) = \left|\left|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2} F_t^A(f)(x,y)\right|\right|,$$

$$g_\mu(f)(x) = \left|\left|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2} F_t(f)(y)\right|\right|.$$

It is easy to see that g_ψ^A, S_ψ^A and g_μ^A satisfy the conditions of Theorem 2 (see [10–12], [14]), thus the conclusion of Theorem 2 holds for g_ψ^A, S_ψ^A and g_μ^A.

EXAMPLE 3. Marcinkiewicz operators.

Fixed $\lambda > 1$ and $0 < \varepsilon \leq 1$. Let Ω be homogeneous of degree zero on $\mathbb{R}^n$ with $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$. Assume that $\Omega \in Lip_\varepsilon(S^{n-1})$. The Marcinkiewicz multilinear operators are defined by

$$\mu_\Omega^A(f)(x) = \left(\int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

$$\mu_S^A(f)(x) = \left[\int\int_{\Gamma(x)} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}}\right]^{1/2}$$

and

$$\mu_\lambda^A(f)(x) = \left[\int\int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+3}}\right]^{1/2},$$

where

$$F_t^A(f)(x) = \int_{|x-y|\le t} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;x,y)}{|x-y|^m} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy$$

and

$$F_t^A(f)(x,y) = \int_{|y-z|\le t} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;y,z)}{|y-z|^m} \frac{\Omega(y-z)}{|y-z|^{n-1}} f(z)dz.$$

Set

$$F_t(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

We also define that

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

$$\mu_S(f)(x) = \left(\int\int_{\Gamma(x)} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}}\right)^{1/2}$$

and

$$\mu_\lambda(f)(x) = \left(\int\int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} |F_t(f)(y)|^2 \frac{dydt}{t^{n+3}}\right)^{1/2},$$

which are the Marcinkiewicz operators (see [21]). Let H be the space

$$H = \left\{h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t^3\right)^{1/2} < \infty\right\}$$

or

$$H = \left\{h : ||h|| = \left(\int\int_{\mathbb{R}^{n+1}_+} |h(y,t)|^2 dydt/t^{n+3}\right)^{1/2} < \infty\right\}.$$

Then, it is clear that

$$\mu_\Omega^A(f)(x) = ||F_t^A(f)(x)||, \quad \mu_\Omega(f)(x) = ||F_t(f)(x)||,$$

$$\mu_S^A(f)(x) = \left|\left|\chi_{\Gamma(x)} F_t^A(f)(x,y)\right|\right|, \quad \mu_S(f)(x) = \left|\left|\chi_{\Gamma(x)} F_t(f)(y)\right|\right|$$

and

$$\mu_\lambda^A(f)(x) = \left|\left|\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} F_t^A(f)(x,y)\right|\right|,$$

$$\mu_\lambda(f)(x) = \left\| \left(\frac{t}{t+|x-y|} \right)^{n\lambda/2} F_t(f)(y) \right\|.$$

It is easy to see that μ_Ω^A, μ_S^A and μ_λ^A satisfy the conditions of Theorem 2 (see [13], [21]), thus Theorem 2 holds for μ_Ω^A, μ_S^A and μ_λ^A.

EXAMPLE 4. Bochner–Riesz operator.

Let $\delta > (n-1)/2$, $B_t^\delta(f)\hat{}(\xi) = (1-t^2|\xi|^2)_+^\delta \hat{f}(\xi)$ and $B_t^\delta(z) = t^{-n}B^\delta(z/t)$ for $t > 0$. Set

$$F_{\delta,t}^A(f)(x) = \int_{\mathbb{R}^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;x,y)}{|x-y|^m} B_t^\delta(x-y)f(y)dy,$$

The maximal Bochner–Riesz multilinear operator are defined by

$$B_{\delta,*}^A(f)(x) = \sup_{t>0} |B_{\delta,t}^A(f)(x)|.$$

We also define that

$$B_{\delta,*}(f)(x) = \sup_{t>0} |B_t^\delta(f)(x)|$$

which is the maximal Bochner–Riesz operator (see [16]). Let H be the space $H = \{h : ||h|| = \sup_{t>0} |h(t)| < \infty\}$, then

$$B_{\delta,*}^A(f)(x) = ||B_{\delta,t}^A(f)(x)||, \quad B_*^\delta(f)(x) = ||B_t^\delta(f)(x)||.$$

It is easy to see that $B_{\delta,*}^A$ satisfies the conditions of Theorem 2 (see [15], [22]), thus Theorem 2 holds for $B_{\delta,*}^A$.

3. Proofs of Theorems

To prove the theorem, we need the following lemmas.

LEMMA. (see [3]). *Let A be a function on $\mathbb{R}^n$ and $D^\alpha A \in L^q(\mathbb{R}^n)$ for all α with $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(A;x,y)| \le C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|Q(x,y)|} \int_{Q(x,y)} |D^\alpha A(z)|^q dz \right)^{1/q},$$

where $Q(x,y)$ is the cube centered at x and having side of length $5\sqrt{n}|x-y|$.

PROOF OF THEOREM 1. Let $f \in L^{p,\varphi}(\mathbb{R}^n)$. For a ball $Q = Q(x_0, r) \subset \mathbb{R}^n$, set

$$f(y) = (f\chi_Q)(y) + \sum_{k=1}^{\infty}(f\chi_{2^{k+1}Q\setminus 2^kQ})(y) = f_0(y) + \sum_{k=1}^{\infty} f_k(y),$$

then

$$\int_Q |T^A(f)(x)|^p dx \le \int_Q |T^A(f_0)(x)|^p dx + \sum_{k=1}^{\infty}\int_Q |T^A(f_k)(x)|^p dx.$$

By the L^p-boundedness of T^A, we get

$$\int_Q |T^A(f_0)(x)|^p dx \le C\int_{\mathbb{R}^n} |f_0(x)|^p dx = C\int_Q |f(x)|^p dx \le C||f||^p_{L^{p,\varphi}}\varphi(r).$$

Without loss of generality, it may be assumed $l = 2$. We have, by the size condition of T, for $k \ge 1$,

$$\sum_{k=1}^{\infty}\int_Q |T^A(f_k)(x)|^p dx \le \sum_{k=1}^{\infty}\int_Q \left(\int_{\mathbb{R}^n} \frac{\prod_{j=1}^2 R_{m_j+1}(A_j; x, y)|f_k(y)|}{|x-y|^{m+n}} dy\right)^p dx.$$

Let $\tilde{A}_j(x) = A_j(x) - \sum_{|\alpha|=m_j} \frac{1}{\alpha!}(D^\alpha A_j)_{2Q}x^\alpha$, then $R_{m_j}(A_j; x, y) = R_{m_j}(\tilde{A}_j; x, y)$ and $D^\alpha \tilde{A}_j = D^\alpha A_j - (D^\alpha A_j)_{2Q}$ for $|\alpha| = m_j$. We write

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{\prod_{j=1}^2 R_{m_j+1}(A_j; x, y)}{|x-y|^{m+n}} f_k(y)dy \\
&\quad = \int_{\mathbb{R}^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y)}{|x-y|^{m+n}} f_k(y)dy \\
&\quad - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!}\int_{\mathbb{R}^n} \frac{R_{m_2}(\tilde{A}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^{m+n}} D^{\alpha_1}\tilde{A}_1(y) f_k(y)dy \\
&\quad - \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!}\int_{\mathbb{R}^n} \frac{R_{m_1}(\tilde{A}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^{m+n}} D^{\alpha_2}\tilde{A}_2(y) f_k(y)dy \\
&\quad + \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!}\int_{\mathbb{R}^n} \frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^{m+n}} D^{\alpha_1}\tilde{A}_1(y) D^{\alpha_2}\tilde{A}_2(y) f_k(y)dy \\
&\quad = I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

By Lemma and the following inequality (see [19])

$$|b_{Q_1} - b_{Q_2}| \le C\log(|Q_2|/|Q_1|)||b||_{BMO} \text{ for } Q_1 \subset Q_2,$$

we know that, for $x \in Q$ and $y \in 2^{k+1}Q \setminus 2^kQ$,

$$\begin{aligned}
|R_m(\tilde{A}; x, y)| &\le C|x-y|^m \sum_{|\alpha|=m} (||D^\alpha A||_{BMO} + |(D^\alpha A)_{2Q(x,y)} - (D^\alpha A)_{2Q}|) \\
&\le Ck|x-y|^m \sum_{|\alpha|=m} ||D^\alpha A||_{BMO}.
\end{aligned}$$

Note that $|x-y| \sim |x_0-y|$ for $x \in Q$ and $y \in \mathbb{R}^n \setminus 2Q$, we obtain

$$\begin{aligned}|I_1| &\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)k^2\int_{2^{k+1}Q\setminus 2^kQ}\frac{|f(y)|}{|x-y|^n}dy\\ &\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)k^2\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(y)|^p dy\right)^{1/p}\\ &\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)k^2|2^{k+1}Q|^{-1/p}\varphi(2^k r)^{1/p}||f||_{L^{p,\varphi}}.\end{aligned}$$

For I_2, we obtain, by Hölder's inequality,

$$\begin{aligned}|I_2| &\le C\sum_{|\alpha|=m_2}||D^\alpha A_2||_{BMO}\sum_{|\alpha_1|=m_1}k\int_{2^{k+1}Q\setminus 2^kQ}\frac{|D^{\alpha_1}\tilde{A}_1(y)||f(y)|}{|x-y|^n}dy\\ &\le C\sum_{|\alpha|=m_2}||D^\alpha A_2||_{BMO}\frac{k}{|2^{k+1}Q|}\left(\int_{2^{k+1}Q}|f(y)|^p dy\right)^{1/p}\\ &\quad\times\left(\int_{2^{k+1}Q}|D^{\alpha_1}A_1(y)-(D^\alpha A_1)_{2Q}|^{p'}dy\right)^{1/p'}\\ &\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)k^2|2^{k+1}Q|^{-1/p}\varphi(2^k r)^{1/p}||f||_{L^{p,\varphi}}.\end{aligned}$$

Similarly,

$$|I_3| \le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)k^2|2^{k+1}Q|^{-1/p}\varphi(2^k r)^{1/p}||f||_{L^{p,\varphi}}.$$

For I_4, taking $q_1, q_2 > 1$ such that $1/p+1/q_1+1/q_2=1$, then

$$\begin{aligned}|I_4| &\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\int_{2^{k+1}Q\setminus 2^kQ}\frac{|D^{\alpha_1}\tilde{A}_1(y)||D^{\alpha_2}\tilde{A}_2(y)|}{|x-y|^n}|f_2(y)|dy\\ &\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\frac{1}{|2^{k+1}Q|}\left(\int_{2^{k+1}Q}|f(y)|^p dy\right)^{1/p}\\ &\quad\times\left(\int_{2^{k+1}Q}|D^{\alpha_1}A_1(y)-(D^\alpha A_1)_{2Q}|^{q_1}dy\right)^{1/q_1}\\ &\quad\times\left(\int_{2^{k+1}Q}|D^{\alpha_2}A_2(y)-(D^\alpha A_2)_{2Q}|^{q_2}dy\right)^{1/q_2}\\ &\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{BMO}\right)k^2|2^{k+1}Q|^{-1/p}\varphi(2^k r)^{1/p}||f||_{L^{p,\varphi}}.\end{aligned}$$

Thus

$$\sum_{k=1}^{\infty}\int_Q |T^A(f_k)(x)|^p dx \le C\sum_{k=1}^{\infty}|Q|k^{2p}|2^{k+1}Q|^{-1}\varphi(2^k r)||f||^p_{L^{p,\varphi}}$$
$$\le C\sum_{k=1}^{\infty}k^{2p}(2^{-n}D)^k\varphi(r)||f||^p_{L^{p,\varphi}}$$
$$\le C\varphi(r)||f||^p_{L^{p,\varphi}}$$

and

$$||T^A(f)||_{L^{p,\varphi}} \le C||f||_{L^{p,\varphi}}.$$

This completes the proof of Theorem 1. □

Proof of Theorem 2. Let $f \in L^{p,\varphi}(\mathbb{R}^n)$. Similarly to the proof of Theorem 1, for a ball $Q = Q(x_0, r) \subset \mathbb{R}^n$, set

$$f(y) = (f\chi_Q)(y) + \sum_{k=1}^{\infty}(f\chi_{2^{k+1}Q\setminus 2^k Q})(y) = f_0(y) + \sum_{k=1}^{\infty} f_k(y),$$

then

$$\int_Q |S^A(f)(x)|^p dx \le \int_Q |S^A(f_0)(x)|^p dx + \sum_{k=1}^{\infty}\int_Q |S^A(f_k)(x)|^p dx.$$

By the L^p-boundedness of S^A, we get

$$\int_Q |S^A(f_0)(x)|^p dx \le C\int_{\mathbb{R}^n}|f_0(x)|^p dx = C\int_Q |f(x)|^p dx \le C||f||^p_{L^{p,\varphi}}\varphi(r).$$

Without loss of generality, it may be assume $l = 2$. We have, by the size condition of S, for $k \ge 1$,

$$\sum_{k=1}^{\infty}\int_Q |S^A(f_k)(x)|^p dx \le \sum_{k=1}^{\infty}\int_Q \left(\int_{\mathbb{R}^n}\frac{\prod_{j=1}^2 R_{m_j+1}(A_j;x,y)|f_k(y)|}{|x-y|^{m+n}}dy\right)^p dx.$$

Let $\tilde{A}_j(x) = A_j(x) - \sum_{|\alpha|=m_j}\frac{1}{\alpha!}(D^\alpha A_j)_{2Q}x^\alpha$. Similar to the proof of Theorem 1, we get

$$\sum_{k=1}^{\infty}\int_Q |S^A(f_k)(x)|^p dx \le C\sum_{k=1}^{\infty}|Q|k^{2p}|2^{k+1}Q|^{-1}\varphi(2^k r)||f||^p_{L^{p,\varphi}}$$
$$\le C\sum_{k=1}^{\infty}k^{2p}(2^{-n}D)^k\varphi(r)||f||^p_{L^{p,\varphi}}$$
$$\le C\varphi(r)||f||^p_{L^{p,\varphi}}$$

and

$$||S^A(f)||_{L^{p,\varphi}} \le C||f||_{L^{p,\varphi}}.$$

This completes the proof of Theorem 2. □

Acknowledgement. The author would like to express his gratitude to the referee for his comments and suggestions.

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