WEIGHTED SHARP BOUNDEDNESS FOR MULTILINEAR COMMUTATORS

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Abstract. In this paper, the sharp estimates for some multilinear commutators related to certain sublinear integral operators are obtained. The operators include Littlewood-Paley operator and Marcinkiewicz operator. As application, we obtain the weighted $L^p(p > 1)$ inequalities and $L\log L$ type estimate for the multilinear commutators.

1. Introduction

Let $b \in BMO(\mathbb{R}^n)$ and T be the Calderón–Zygmund operator. The commutator $[b,T]$ generated by b and T is defined by $[b,T]f(x) = b(x)Tf(x) - T(bf)(x)$. By using a classical result of Coifman, Rochberg and Weiss[2], we know that the commutator $[b,T]$ is bounded on $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$. However, it was observed that $[b,T]$ is not bounded, in general, from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$. In [11], the sharp inequalities for some multilinear commutators of the Calderón–Zygmund singular integral operators are obtained. The main purpose of this paper is to prove the sharp estimates for some multilinear commutators related to certain sublinear integral operators. In fact, we shall establish the sharp estimates for the multilinear commutators only under certain conditions on the size of the operators. The operators include Littlewood–Paley operator and Marcinkiewicz operator. As the applications, we obtain the weighted norm inequalities and $L\log L$ type estimate for these multilinear commutators. In Section 2, we will give some concepts and Theorems of this paper, whose proofs will appear in Section 3.

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2. Preliminaries and Theorems

First, let us introduce some notations (see [4], [8], [10], [11]). Throughout this paper, $Q = Q(x_0, d)$ will denote a cube of $\mathbb{R}^n$ with sides parallel to the axes centered at x_0 and having side length d. For $a > 0$ and a cube Q, aQ will denote a cube with the same center as Q and a times edges of Q and $Q^c = \{x \in \mathbb{R}^n : x \notin Q\}$. For any locally integrable function f, the sharp function of f is defined by

$$f^{\#}(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where the supremum is taken over all cubes Q cintaining x, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x)dx$. It is well-known that (see [4])

$$f^{\#}(x) = \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy,$$

where the infimum is taken over all numbers. We say that f belongs to $BMO(\mathbb{R}^n)$ if $f^{\#}$ belongs to $L^\infty(\mathbb{R}^n)$ and $||f||_{BMO} = ||f^{\#}||_{L^\infty}$. For $0 < r < \infty$, we denote $f_r^{\#}$ by

$$f_r^{\#}(x) = [(|f|^r)^{\#}(x)]^{1/r}.$$

Let M be the Hardy-Littlewood maximal operator, that is that $M(f)(x) = \sup_{Q \ni x} |Q|^{-1} \int_Q |f(y)| dy$, we write that $M_p(f) = (M(f^p))^{1/p}$. For $k \in N$, we denote by M^k the operator M iterated k times, i.e., $M^1(f)(x) = M(f)(x)$ and $M^k(f)(x) = M(M^{k-1}(f))(x)$ for $k \geq 2$.

Let Φ be a Young function and $\tilde{\Phi}$ be the complementary associated to Φ, we denote that the Φ-average by, for a function f

$$||f||_{\Phi,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \Phi \left(\frac{|f(y)|}{\lambda} \right) dy \leq 1 \right\}$$

and the maximal function associated to Φ by

$$M_\Phi(f)(x) = \sup_{Q \ni x} ||f||_{\Phi,Q}.$$

The main Young functions which will be used in this paper are $\Phi(t) = \exp(t^r) - 1$ and $\Psi(t) = t\log^r(t + e)$, the corresponding Φ-average and maximal functions are denoted by $||\cdot||_{\exp L^r, Q}$, $M_{\exp L^r}$ and $||\cdot||_{L(\log L)^r, Q}$, $M_{L(\log L)^r}$. We have the following inequalities, for any $r > 0$ and $m \in N$

$$M(f) \leq M_{L(\log L)^r}(f), \quad M_{L(\log L)^m}(f) \leq CM^{m+1}(f).$$

For $r \geq 1$, we denote

$$||b||_{osc_{\exp L^r}} = \sup_Q ||b - b_Q||_{\exp L^r, Q}.$$

The spaces $Osc_{\exp L^r}$ is defined by

$$Osc_{\exp L^r} = \{b \in L^1_{\log}(\mathbb{R}^n) : ||b||_{osc_{\exp L^r}} < \infty\}.$$

It has been known that (see [11])

$$||b - b_{2^kQ}||_{\exp L^r, 2^kQ} \le Ck||b||_{Osc_{\exp L^r}}.$$

It is obvious that $Osc_{\exp L^r}$ coincides with the BMO space if $r = 1$. For $r_j > 0$ and $b_j \in Osc_{\exp L^{r_j}}$ for $j = 1, \cdots, m$, we denote that $1/r = 1/r_1 + \cdots + 1/r_m$ and $||\vec{b}|| = \prod_{j=1}^m ||b_j||_{Osc_{\exp L^{r_j}}}$. Given a positive integer m and $1 \le j \le m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C_j^m$, denote that $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, denote $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{Osc_{\exp L^{r_\sigma}}} = ||b_{\sigma(1)}||_{Osc_{\exp L^{r_{\sigma(1)}}}} \cdots ||b_{\sigma(j)}||_{Osc_{\exp L^{r_{\sigma(j)}}}}$.

We denote the Muckenhoupt weights by A_p for $1 \le p < \infty$, that is (see [4])

$$A_p = \left\{ w : \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)}dx \right)^{p-1} < \infty \right\}, \quad 1 < p < \infty$$

and

$$A_1 = \{w : M(w)(x) \le Cw(x), a.e.\}.$$

We are going to consider some integral operators as following.

Let $b_j (j = 1, ..., m)$ be the fixed locally integral functions on $\mathbb{R}^n$.

DEFINITION 1. Let $\lambda > 3 + 2/n$, $\varepsilon > 0$ and ψ be a fixed integrable function defined on $\mathbb{R}^n$, which satisfies the following properties:

(1) $\int_{\mathbb{R}^n} \psi(x)dx = 0$,

(2) $|\psi(x)| \le C(1 + |x|)^{-(n+1)}$,

(3) $|\psi(x + y) - \psi(x)| \le C|y|^\varepsilon (1 + |x|)^{-(n+1+\varepsilon)}$ when $2|y| < |x|$.

Set $\mathbb{R}^{n+1}_+ = \{(x,t) : x \in \mathbb{R}^n, t \ge 0\}$. Let f be a integrable function on $\mathbb{R}^n$ with compact support. The Littlewood-Paley multilinear commutator is defined by

$$g_\lambda^{\vec{b}}(f)(x) = \left[\int\int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t + |x - y|} \right)^{n\lambda} |F_{1t}^{\vec{b}}(f)(x,y)|^2 \frac{dydt}{t^{n+1}} \right]^{1/2}$$

where

$$F_{1t}^{\vec{b}}(f)(x,y) = \int_{\mathbb{R}^n} \left[\prod_{j=1}^m (b_j(x) - b_j(z)) \right] \psi_t(y - z)f(z)dz$$

and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. Set $F_{1t}(f)(y) = \int_{\mathbb{R}^n} \psi_t(z)f(y - z)dz$. We also define that

$$g_\lambda(f)(x) = \left(\int\int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t + |x - y|} \right)^{n\lambda} |F_{1t}(f)(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}$$

which is the Littlewood–Paley function (see [13]).

Let H_1 be the Hilbert space

$$H_1 = \left\{ h : ||h||_{H_1} = \left(\int\int_{\mathbb{R}^{n+1}_+} |h(y,t)|^2 dydt/t^{n+1} \right)^{1/2} < \infty \right\}.$$

Then for each fixed $x \in \mathbb{R}^n$, $F_t^A(f)(x,y)$ may be viewed as a mapping from $(0, +\infty)$ to H_1, and it is clear that

$$g_\lambda^{\tilde{b}}(f)(x) = \left\| \left(\frac{t}{t + |x-y|} \right)^{n\lambda/2} F_{1t}^{\tilde{b}}(f)(x,y) \right\|_{H_1}$$

and

$$g_\lambda(f)(x) = \left\| \left(\frac{t}{t + |x-y|} \right)^{n\lambda/2} F_{1t}(f)(y) \right\|_{H_1}.$$

Definition 2. Let Fix $\lambda > \max(1, 2n/(n+2))$, $0 < \gamma \leq 1$ and Ω be homogeneous of degree zero on $\mathbb{R}^n$ such that $\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0$. Assume that $\Omega \in Lip_\gamma(S^{n-1})$, that is there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \leq M|x-y|^\gamma$. We denote $\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. Let f be a integrable function on $\mathbb{R}^n$ with compact support. The Marcinkiewicz multilinear commutator is defined by

$$\mu_\lambda^{\tilde{b}}(f)(x) = \left[\int\int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t + |x-y|} \right)^{n\lambda} |F_{2t}^{\tilde{b}}(f)(x,y)|^2 \frac{dydt}{t^{n+3}} \right]^{1/2}$$

where

$$F_{2t}^{\tilde{b}}(f)(x,y) = \int_{|y-z| \leq t} \frac{\Omega(y-z)}{|y-z|^{n-1}} \left[\prod_{j=1}^m (b_j(x) - b_j(z)) \right] f(z) dz.$$

Set

$$F_{2t}(f)(y) = \int_{|y-z| \leq t} \frac{\Omega(y-z)}{|y-z|^{n-1}} f(z) dz.$$

We also define that

$$\mu_\lambda(f)(x) = \left(\int\int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t + |x-y|} \right)^{n\lambda} |F_{2t}(f)(y)|^2 \frac{dydt}{t^{n+3}} \right)^{1/2}$$

which is the Marcinkiewicz integral (see [14]).

Let H_2 be the space

$$H_2 = \left\{ h : ||h||_{H_2} = \left(\int\int_{\mathbb{R}^{n+1}_+} |h(y,t)|^2 dydt/t^{n+3} \right)^{1/2} < \infty \right\}.$$

Then, it is clear that

$$\mu_\lambda^A(f)(x) = \left\| \left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} F_{2t}^{\tilde{b}}(f)(x,y) \right\|_{H_2}$$

and

$$\mu_\lambda(f)(x) = \left\| \left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} F_{2t}(f)(y) \right\|_{H_2}.$$

More generally, we define the following multilinear commutator related to certain integral operators.

DEFINITION 3. Let f be a integrable function on $\mathbb{R}^n$ with compactly supported and $F(x,y,t)$ be a function define on $\mathbb{R}^n \times \mathbb{R}^n \times [0,+\infty)$, we denote

$$F_t(f)(x) = \int_{\mathbb{R}^n} F(x,y,t) f(y) dy$$

and

$$F_t^{\tilde{b}}(f)(x) = \int_{\mathbb{R}^n} \left[\prod_{j=1}^m (b_j(x) - b_j(y))\right] F(x,y,t) f(y) dy.$$

Let H be the Banach space $H = \{h : ||h||_H < \infty\}$ such that, for each fixed $x \in \mathbb{R}^n$, $F_t(f)(x)$ and $F_t^{\tilde{b}}(f)(x)$ may be viewed as a mapping from $[0,+\infty)$ to H. Then, the multilinear commutator related to $F_t^{\tilde{b}}$ is defined by

$$T_{\tilde{b}}(f)(x) = ||F_t^{\tilde{b}}(f)(x)||_H.$$

We also denote that

$$T(f)(x) = ||F_t(f)(x)||_H.$$

Note that when $b_1 = \cdots = b_m$, $T_{\tilde{b}}$ is just the m order commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [1–3], [5], [6], [8–11]). Our main purpose is to establish the sharp inequalities for the multilinear commutator operators.

The following theorems are our main results.

THEOREM 1. *Let $r_j \geq 1$ and $b_j \in Osc_{\exp L^{r_j}}$ for $j = 1, \cdots, m$. Denote that $1/r = 1/r_1 + \cdots + 1/r_m$.*

(1) *Then for any $0 < p < q < 1$, there exists a constant $C > 0$ such that for any $f \in C_0^\infty(\mathbb{R}^n)$ and any $x \in \mathbb{R}^n$,*

$$(g_\lambda^{\tilde{b}}(f))_p^\#(x) \leq C \left(||b|| M_{L(\log L)^{1/r}}(f)(x) + \sum_{j=1}^m \sum_{\sigma \in C_j^m} M_q(g_\lambda^{\tilde{b}_{\sigma^c}}(f))(x) \right);$$

(**2**) *If* $1 < p < \infty$ *and* $w \in A_p$, *then*

$$||g_\lambda^{\tilde{b}}(f)||_{L^p(w)} \leq C||\tilde{b}||||f||_{L^p(w)};$$

(**3**) *If* $w \in A_1$. *Denote that* $\Phi(t) = t\log^{1/r}(t+e)$. *Then there exists a constant* $C > 0$ *such that for all* $\lambda > 0$,

$$w(\{x \in \mathbb{R}^n : g_\lambda^{\tilde{b}}(f)(x) > \lambda\}) \leq C \int_{\mathbb{R}^n} \Phi\left(\frac{||\tilde{b}|||f(x)|}{\lambda}\right) w(x)dx.$$

THEOREM 2. *Let* $r_j \geq 1$ *and* $b_j \in Osc_{\exp L^{r_j}}$ *for* $j = 1, \cdots, m$. *Denote that* $1/r = 1/r_1 + \cdots + 1/r_m$.

(**1**) *Then for any* $0 < p < q < 1$, *there exists a constant* $C > 0$ *such that for any* $f \in C_0^\infty(\mathbb{R}^n)$ *and any* $x \in \mathbb{R}^n$,

$$(\mu_\lambda^{\tilde{b}}(f))_p^\#(x) \leq C\left(||b||M_{L(\log L)^{1/r}}(f)(x) + \sum_{j=1}^m \sum_{\sigma \in C_j^m} M_q(\mu_\lambda^{\tilde{b}_{\sigma^c}}(f))(x)\right);$$

(**2**) *If* $1 < p < \infty$ *and* $w \in A_p$, *then*

$$||\mu_\lambda^{\tilde{b}}(f)||_{L^p(w)} \leq C||\tilde{b}||||f||_{L^p(w)};$$

(**3**) *If* $w \in A_1$. *Denote that* $\Phi(t) = t\log^{1/r}(t+e)$. *Then there exists a constant* $C > 0$ *such that for all* $\lambda > 0$,

$$w(\{x \in \mathbb{R}^n : \mu_\lambda^{\tilde{b}}(f)(x) > \lambda\}) \leq C \int_{\mathbb{R}^n} \Phi\left(\frac{||\tilde{b}|||f(x)|}{\lambda}\right) w(x)dx.$$

3. Proofs of Theorems

We begin with a general theorem.

MAIN THEOREM. *Let* $r_j \geq 1$ *and* $b_j \in Osc_{\exp L^{r_j}}$ *for* $j = 1, \cdots, m$. *Denote that* $1/r = 1/r_1 + \cdots + 1/r_m$. *Suppose that* T *is the same as in Definition 1 such that* T *is bounded on* $L^p(w)$ *for all* $w \in A_p$, $1 < p < \infty$ *and weak bounded of* $(L^1(w), L^1(w))$ *for all* $w \in A_1$. *If* T *satisfies the following size condition:*

$$||F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f)(x) - F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f)(x_0)|| \leq CM_{L(\log L)^{1/r}}(f)(\tilde{x})$$

for any cube $Q = Q(x_0, d)$ *with supp*$f \subset (2Q)^c$ *and* $x, \tilde{x} \in Q = Q(x_0, d)$. *Then for any* $0 < p < q < 1$, *there exists a constant* $C_0 > 0$ *such that for any* $f \in C_0^\infty(\mathbb{R}^n)$ *and any* $x \in \mathbb{R}^n$,

$$(T_{\tilde{b}}(f))_p^\#(x) \leq C_0\left(||b||M_{L(\log L)^{1/r}}(f)(x) + \sum_{j=1}^m \sum_{\sigma \in C_j^m} M_q(T_{\tilde{b}_{\sigma^c}}(f)(x))\right)$$

To prove the theorem, we need the following lemmas.

LEMMA 1 (Kolmogorov, [4, p. 485]). *Let $0 < p < q < \infty$ and for any function $f \geq 0$. We define that, for $1/r = 1/p - 1/q$,*

$$||f||_{WL^q} = \sup_{\lambda>0} \lambda|\{x \in \mathbb{R}^n : f(x) > \lambda\}|^{1/q}, N_{p,q}(f) = \sup_{E} ||f\chi_E||_{L^p}/||\chi_E||_{L^r},$$

where the sup is taken for all measurable sets E with $0 < |E| < \infty$. Then

$$||f||_{WL^q} \leq N_{p,q}(f) \leq (q/(q-p))^{1/p}||f||_{WL^q}.$$

LEMMA 2 ([11]). *Let $r_j \geq 1$ for $j = 1, \cdots, m$, we denote that $1/r = 1/r_1 + \cdots + 1/r_m$. Then*

$$\frac{1}{|Q|}\int_Q |f_1(x)\cdots f_m(x)g(x)|dx \leq ||f||_{\exp L^{r_1},Q}\cdots||f||_{\exp L^{r_m},Q}||g||_{L(\log L)^{1/r},Q}.$$

PROOF OF MAIN THEOREM. It suffices to prove for $f \in C_0^\infty(\mathbb{R}^n)$ and some constant C_0, the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q |T_{\vec{b}}(f)(x) - C_0|^p dx\right)^{1/p}$$
$$\leq C\left(||b||M_{L(\log L)^{1/r}}(f)(\tilde{x}) + \sum_{j=1}^m \sum_{\sigma \in C_j^m} M_q(T_{\vec{b}_{\sigma^c}}(f))(\tilde{x})\right).$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. We first consider the case $m = 1$. We write, for $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{\mathbb{R}^n\setminus 2Q}$,

$$F_t^{b_1}(f)(x) = (b_1(x) - (b_1)_{2Q})F_t(f)(x) - F_t((b_1 - (b_1)_{2Q})f_1)(x) - F_t((b_1 - (b_1)_{2Q})f_2)(x),$$

then

$$\begin{aligned}
&|T_{b_1}(f)(x) - T(((b_1)_{2Q} - b_1)f_2)(x_0)| \\
&\leq ||F_t^{b_1}(f)(x) - F_t(((b_1)_{2Q} - b_1)f_2)(x_0)||_H \\
&\leq ||(b_1(y) - (b_1)_{2Q})F_t(f)(x)||_H + ||F_t((b_1 - (b_1)_{2Q})f_1)(x)||_H \\
&\quad + ||F_t((b_1 - (b_1)_{2Q})f_2)(x) - F_t((b_1 - (b_1)_{2Q})f_2)(x_0)||_H \\
&= I(x) + II(x) + III(x).
\end{aligned}$$

For $I(x)$, by Hölder's inequality for the exponent $1/l + 1/l' = 1$ with $1 < l < q/p$ and $pl = q$, we have

$$\left(\frac{1}{|Q|}\int_Q |I(x)|^p dx\right)^{1/p}$$
$$= \left(\frac{1}{|Q|}\int_Q |b_1(x)-(b_1)_{2Q}|^p|T(f)(x)|^p dx\right)^{1/p}$$
$$\leq \left(\frac{C}{|2Q|}\int_{2Q} |b_1(x)-(b_1)_{2Q}|^{pl'}\right)^{1/pl'} \left(\frac{1}{|Q|}\int_Q |T(f)(x)|^{pl} dx\right)^{1/pl}$$
$$\leq C||b_1||_{Osc_{\exp L^r}} M_{pl}(T(f))(\tilde{x})$$
$$\leq C||b_1||_{Osc_{\exp L^r}} M_q(T(f))(\tilde{x}).$$

For $II(x)$, by Lemma 1 with $q=1$, the weak bounded of $(L^1(w), L^1(w))$ for T and Lemma 2, we have

$$\left(\frac{1}{|Q|}\int_Q |II(x)|^p dx\right)^{1/p} = \left(\frac{1}{|Q|}\int_Q |T((b_1-(b_1)_{2Q})f_1)(x)|^p dx\right)^{1/p}$$
$$= |Q|^{-1}\frac{||T((b_1-(b_1)_{2Q})f_1)\chi_Q||_{L^p}}{|Q|^{1/p-1}}$$
$$\leq C|Q|^{-1}||T((b_1-(b_1)_{2Q})f\chi_{2Q})||_{WL^1}$$
$$\leq C|2Q|^{-1}\int_{2Q}|b_1(x)-(b_1)_{2Q}||f(x)|dx$$
$$\leq C||b_1-(b_1)_{2Q}||_{\exp L^r,2Q}||f||_{L(\log L)^{1/r},2Q}$$
$$\leq C||b_1||_{Osc_{\exp L^r}} M_{L(\log L)^{1/r}}(f)(\tilde{x}).$$

For $III(x)$, using the size condition of T, we have

$$\left(\frac{1}{|Q|}\int_Q |III(x)|^p dx\right)^{1/p} \leq CM_{L(\log L)^{1/r}}(f)(\tilde{x}).$$

Now, we consider the case $m \geq 2$. We write, for $b=(b_1,...,b_m)$,

$$F_t^{\tilde{b}}(f)(x) = \int_{\mathbb{R}^n}\left[\prod_{j=1}^m (b_j(x)-b_j(y))\right] F(x,y,t)f(y)dy$$
$$= \int_{\mathbb{R}^n}(b_1(x)-(b_1)_{2Q})-(b_1(y)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})$$
$$-(b_m(y)-(b_m)_{2Q})F(x,y,t)f(y)dy$$
$$= \sum_{j=0}^m \sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2Q})_\sigma \int_{\mathbb{R}^n}(b(y)-(b)_{2Q})_{\sigma}F(x,y,t)f(y)dy$$
$$= (b_1(x)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})F_t(f)(x)$$
$$+(-1)^m F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f)(x)$$

$$+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2Q})_\sigma\int_{\mathbb{R}^n}(b(y)-b(x))_{\sigma^c}F(x,y,t)f(y)dy$$

$$=(b_1(x)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})F_t(f)(x)$$

$$+(-1)^m F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f)(x)$$

$$+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}c_{m,j}(b(x)-(b)_{2Q})_\sigma F_t^{\tilde{b}_{\sigma^c}}(f)(x),$$

thus

$$|T_{\tilde{b}}(f)(x)-(-1)^m T((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q}))f_2)(x_0)|$$

$$\le ||F_{\tilde{b}}(f)(x)-(-1)^m F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q}))f_2)(x_0)||_H$$

$$\le ||(b_1(x)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})F_t(f)(x)||_H$$

$$+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||(b(x)-(b)_{2Q})_\sigma F_t^{\tilde{b}_{\sigma^c}}(f)(x)||_H$$

$$+||F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_1)(x)||_H$$

$$+||F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_2)(x)$$

$$-F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_2)(x_0)||_H$$

$$=I_1(x)+I_2(x)+I_3(x)+I_4(x).$$

For $I_1(x)$ and $I_2(x)$, similar to the proof of the Case $m=1$, we get, for $1/l_1+\cdots+1/l_m+1/l=1$ with $1<l<q/p$ and $pl=q$,

$$\left(\frac{1}{|Q|}\int_Q|I_1(x)|^p dx\right)^{1/p}$$

$$\le C\prod_{j=1}^m\left(\frac{1}{|2Q|}\int_{2Q}|b_j(x)-(b_j)_{2Q}|^{pl_j}\right)^{1/pl_j}\left(\frac{1}{|Q|}\int_Q|T(f)(x)|^{pl}dx\right)^{1/pl}$$

$$\le C\prod_{j=1}^m||b_j||_{Osc_{\exp L^{r_j}}}M_{pl}(T(f))(\tilde{x})\le CM_q(T(f))(\tilde{x})$$

and

$$\left(\frac{1}{|Q|}\int_Q|I_2(x)|^p dx\right)^{1/p}$$

$$\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\frac{1}{|2Q|}\int_{2Q}|(b(x)-(b)_{2Q})_\sigma|^{pl_\sigma}\right)^{1/pl_\sigma}\left(\frac{1}{|Q|}\int_Q|T_{\tilde{b}_{\sigma^c}}(f)(x)|^{pl}dx\right)^{1/pl}$$

$$\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||b_\sigma||_{Osc_{\exp L^{r_\sigma}}}M_{pl}(T_{\tilde{b}_{\sigma^c}}(f))(\tilde{x})$$

$$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} M_q(T_{\tilde{b}_{\sigma^c}}(f))(\tilde{x}).$$

For $I_3(x)$, by the weak bounded of $(L^1(w), L^1(w))$ for T and Lemma 1 and 2, we obtain

$$\begin{aligned}\left(\frac{1}{|Q|}\int_Q (I_3(x))^p dx\right)^{1/p} &\leq \frac{C}{|2Q|}\int_{2Q} |b_1(x)-(b_1)_{2Q}|\cdots|b_m(x)-(b_m)_{2Q}||f(x)|dx\\ &\leq C||b_1-(b_1)_{2Q}||_{\exp L^{r_1},2Q}\cdots||b_m-(b_m)_{2Q}||_{\exp L^{r_m},2Q}||f||_{L(\log L)^{1/r},2Q}\\ &\leq C||b||M_{L(\log L)^{1/r}}(f)(\tilde{x}).\end{aligned}$$

For I_4, using the size condition of T, we have

$$\left(\frac{1}{|Q|}\int_Q (I_4(x))^p dx\right)^{1/p} \leq CM_{L(\log L)^{1/r}}(f)(\tilde{x}).$$

This completes the proof of the main theorem. □

To prove Theorem 1 and 2, it suffices to verify that $g_\lambda^{\tilde{b}}$ and $\mu_\lambda^{\tilde{b}}$ satisfy the size condition in Main Theorem, that is, for $j = 1, 2$,

$$\left\|\left[\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2}-\left(\frac{t}{t+|x_0-y|}\right)^{n\lambda/2}\right]F_{jt}((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f)(y)\right\|_{H_j} \leq CM_{L(\log L)^{1/r}}(f)(\tilde{x}).$$

Suppose supp$f \subset Q^c$ and $x \in Q = Q(x_0, d)$. Note that $|x_0 - z| \approx |x - z|$ for $z \in Q^c$.

For $g_\lambda^{\tilde{b}}$, by the condition of ψ and the inequality: $a^{1/2} - b^{1/2} \leq (a - b)^{1/2}$ for $a \geq b > 0$, we get

$$\begin{aligned}&\left\|\left[\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2}-\left(\frac{t}{t+|x_0-y|}\right)^{n\lambda/2}\right]F_{1t}((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f)(y)\right\|_{H_1}\\ &\leq \left\|\left|\left(\frac{t}{t+|x-y|}\right)^{n\lambda}-\left(\frac{t}{t+|x_0-y|}\right)^{n\lambda}\right|^{1/2} F_{1t}((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f)(y)\right\|_{H_1}\\ &\leq \left[\iint_{\mathbb{R}_+^{n+1}}\left[\int_{(2Q)^c}\frac{t^{n\lambda/2}|x_0-x|^{1/2}}{(t+|x_0-y|)^{(n\lambda+1)/2}}\prod_{j=1}^m|b_j(z)-(b_j)_{2Q}||f(z)||\psi_t(y-z)|dz\right]^2\frac{dydt}{t^{n+1}}\right]^{1/2}\\ &\leq C\int_{(2Q)^c}\prod_{j=1}^m|b_j(z)-(b_j)_{2Q}||f(z)|\left(\iint_{\mathbb{R}_+^{n+1}}\frac{t^{1-n+n\lambda}|x_0-x|dydt}{(t+|x_0-y|)^{n\lambda+1}(t+|y-z|)^{2n+2}}\right)^{1/2}dz\end{aligned}$$

$$\le C\int_{(2Q)^c}\prod_{j=1}^m |b_j(z)-(b_j)_{2Q}||f(z)||x_0-x|^{1/2}$$
$$\times\left[\int_0^\infty t^{-n}\left(\int_{\mathbb{R}^n}\left(\frac{t}{t+|x_0-y|}\right)^{n\lambda}\frac{dy}{(t+|y-z|)^{2n+2}}\right)dt\right]^{1/2}dz;$$

noting that $2t+|y-z|\ge 2t+|x_0-z|-|x_0-y|\ge t+|x_0-z|$ when $|x_0-y|\le t$ and $2^{k+1}t+|y-z|\ge 2^{k+1}t+|x_0-z|-|x_0-y|\ge |x_0-z|$ when $|x_0-y|\le 2^{k+1}t$, we get, recall that $\lambda>(3n+2)/n$,

$$t^{-n}\int_{\mathbb{R}^n}\left(\frac{t}{t+|x_0-y|}\right)^{n\lambda}\frac{dy}{(t+|y-z|)^{2n+2}}$$
$$=t^{-n}\int_{|x_0-y|\le t}\left(\frac{t}{t+|x_0-y|}\right)^{n\lambda}\frac{dy}{(t+|y-z|)^{2n+2}}$$
$$+t^{-n}\sum_{k=0}^\infty\int_{2^kt<|x_0-y|\le 2^{k+1}t}\left(\frac{t}{t+|x_0-y|}\right)^{n\lambda}\frac{dy}{(t+|y-z|)^{2n+2}}$$
$$\le t^{-n}\left[\int_{|x_0-y|\le t}\frac{2^{2n+2}dy}{(2t+2|y-z|)^{2n+2}}+\sum_{k=0}^\infty\int_{|x_0-y|\le 2^{k+1}t}2^{-kn\lambda}\frac{2^{(k+2)(2n+2)}dy}{(2^{k+2}t+2^{k+2}|y-z|)^{2n+2}}\right]$$
$$\le Ct^{-n}\left[\int_{|x_0-y|\le t}\frac{dy}{(2t+|y-z|)^{2n+2}}+\sum_{k=0}^\infty\int_{|x_0-y|\le 2^{k+1}t}2^{-kn\lambda}\frac{2^{k(2n+2)}dy}{(t+2^{k+1}t+|y-z|)^{2n+2}}\right]$$
$$\le Ct^{-n}\left[\int_{|x_0-y|\le t}\frac{dy}{(t+|x_0-z|)^{2n+2}}+\sum_{k=0}^\infty\int_{|x_0-y|\le 2^{k+1}t}2^{-kn\lambda}\frac{2^{k(2n+2)}dy}{(t+|x_0-z|)^{2n+2}}\right]$$
$$\le Ct^{-n}\left[\frac{t^n}{(t+|x_0-z|)^{2n+2}}+\sum_{k=0}^\infty 2^{k(3n+2-n\lambda)}\frac{t^n}{(t+|x_0-z|)^{2n+2}}\right]$$
$$\le\frac{C}{(t+|x_0-z|)^{2n+2}},$$

since

$$\int_0^\infty\frac{dt}{(t+|x_0-z|)^{2n+2}}=C|x_0-z|^{-2n-1},$$

we obtain

$$\left\|\left[\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2}-\left(\frac{t}{t+|x_0-y|}\right)^{n\lambda/2}\right]F_{1t}((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f)(y)\right\|_{H_1}$$
$$\le C\int_{(2Q)^c}\prod_{j=1}^m|b_j(z)-(b_j)_{2Q}||f(z)|\frac{|x_0-x|^{1/2}}{|x_0-z|^{n+1/2}}dz$$
$$\le C\sum_{k=1}^\infty\int_{2^{k+1}Q\setminus 2^kQ}|x_0-x|^{1/2}|x_0-z|^{-(n+1/2)}\left|\prod_{j=1}^m(b_j(z)-(b_j)_{2Q})\right||f(z)|dz$$

$$\leq C\sum_{k=1}^{\infty}2^{-k/2}|2^{k+1}Q|^{-1}\int_{2^{k+1}Q}\left|\prod_{j=1}^{m}(b_j(z)-(b_j)_{2Q})\right||f(z)|dz$$
$$\leq C\sum_{k=1}^{\infty}2^{-k/2}\prod_{j=1}^{m}||b_j-(b_j)_{2Q}||_{\exp L^{r_j},2^{k+1}Q}||f||_{L(\log L)^{1/r},2^{k+1}Q}$$
$$\leq C\sum_{k=1}^{\infty}k^m 2^{-k/2}\prod_{j=1}^{m}||b_j||_{Osc_{\exp L^{r_j}}}M_{L(\log L)^{1/r}}(f)(\tilde{x})$$
$$\leq C\prod_{j=1}^{m}||b_j||_{Osc_{\exp L^{r_j}}}M_{L(\log L)^{1/r}}(f)(\tilde{x}).$$

For $\mu_\lambda^{\vec{b}}$, by the condition of Ω, we get

$$\left|\left|\left[\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2}-\left(\frac{t}{t+|x_0-y|}\right)^{n\lambda/2}\right]F_{2t}((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f)(y)\right|\right|_{H_2}$$
$$\leq C\left[\int\int_{\mathbb{R}_+^{n+1}}\left[\int_{(2Q)^c}\frac{\chi_{\Gamma(z)}(y,t)t^{n\lambda/2}|x_0-x|^{1/2}}{(t+|x-y|)^{(n\lambda+1)/2}|y-z|^{n-1}}\right.\right.$$
$$\left.\left.\times\prod_{j=1}^{m}|b_j(z)-(b_j)_{2Q}||f(z)|dz\right]^2\frac{dydt}{t^{n+3}}\right]^{1/2}$$
$$\leq C\int_{(2Q)^c}\prod_{j=1}^{m}|b_j(z)-(b_j)_{2Q}||f(z)|\left(\int\int_{\mathbb{R}_+^{n+1}}\frac{\chi_{\Gamma(z)}(y,t)t^{n\lambda-n-3}|x_0-x|dydt}{(t+|x-y|)^{n\lambda+1}|y-z|^{2n-2}}\right)^{1/2}dz;$$

note that $|x-z|\leq 2t$, $|y-z|\geq|x-z|-|x-y|\geq|x-z|-t$ when $|x-y|\leq t$, $|y-z|\leq t$, and $|x-z|\leq t(1+2^{k+1})\leq 2^{k+2}t$, $|y-z|\geq|x-z|-2^{k+3}t$ when $|x-y|\leq 2^{k+1}t$, $|y-z|\leq t$, we obtain

$$\left|\left|\left[\left(\frac{t}{t+|x-y|}\right)^{n\lambda/2}-\left(\frac{t}{t+|x_0-y|}\right)^{n\lambda/2}\right]F_{2t}((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f)(y)\right|\right|_{H_2}$$
$$\leq C\int_{(2Q)^c}\prod_{j=1}^{m}|b_j(z)-(b_j)_{2Q}||f(z)||x_0-x|^{1/2}$$
$$\times\left[\int_0^{\infty}\int_{|x-y|\leq t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda+1}\frac{\chi_{\Gamma(z)}(y,t)t^{-n}dydt}{(|x-z|-t)^{2n+2}}\right]^{1/2}dz$$
$$+C\int_{(2Q)^c}\prod_{j=1}^{m}|b_j(z)-(b_j)_{2Q}||f(z)||x_0-x|^{1/2}$$
$$\times\left[\int_0^{\infty}\sum_{k=0}^{\infty}\int_{2^kt<|x-y|\leq 2^{k+1}t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda+1}\frac{\chi_{\Gamma(z)}(y,t)t^{-n}dydt}{(|x-z|-2^{k+3}t)^{2n+2}}\right]^{1/2}dz$$

$$\leq C\int_{(2Q)^c}\prod_{j=1}^m|b_j(z)-(b_j)_{2Q}||f(z)||x_0-x|^{1/2}\left[\int_{|x-z|/2}^{\infty}\frac{dt}{(|x-z|-t)^{2n+2}}\right]^{1/2}dz$$

$$+C\int_{(2Q)^c}\prod_{j=1}^m|b_j(z)-(b_j)_{2Q}||f(z)||x_0-x|^{1/2}$$

$$\times\left[\sum_{k=0}^{\infty}\int_{2^{-2-k}|x-z|}^{\infty}\frac{2^{-k(n\lambda+2)}(2^kt)^nt^{-n}2^kdt}{(|x-z|-2^{k+3}t)^{2n+2}}\right]^{1/2}dz$$

$$\leq C\int_{(2Q)^c}\prod_{j=1}^m|b_j(z)-(b_j)_{2Q}||f(z)||x_0-x|^{1/2}|x-z|^{-n-1/2}dz$$

$$+C\int_{(2Q)^c}\prod_{j=1}^m|b_j(z)-(b_j)_{2Q}||f(z)||x_0-x|^{1/2}\sum_{k=0}^{\infty}2^{k(n-n\lambda-2)/2}|x-z|^{-n-1/2}dz$$

$$\leq C\int_{(2Q)^c}\prod_{j=1}^m|b_j(z)-(b_j)_{2Q}||f(z)|\frac{|x_0-x|^{1/2}}{|x_0-z|^{n+1/2}}dz$$

$$\leq C\prod_{j=1}^m||b_j||_{Osc_{\exp L^{r_j}}}M_{L(\log L)^{1/r}}(f)(\tilde{x}).$$

These yields the desired results.

By (1) and the boundedness of g_λ, μ_λ and $M_{L(\log L)^{1/r}}$, we may obtain the conclusions (2) (3) of Theorem 1 and 2. This completes the proof of Theorem 1 and 2.

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