## Report of Meeting

# The Sixth Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities February 1–4, 2006 Berekfürdő, Hungary

The Sixth Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities was held in Berekfürdő, Hungary from February 1 to February 4, 2006, at Thermal Hotel Szivek. It was organized by the Institute of Mathematics of the University of Debrecen, with the financial support of the Hungarian Scientific Research Fund OTKA T-043080.

24 participants came from the University of Debrecen (Hungary) and the Silesian University of Katowice (Poland) at 12 from each of both cities.

Professor László Losonczi opened the Seminar and welcomed the participants to Berekfürdő.

The scientific talks presented at the Seminar focused on the following topics: equations in a single variable and in several variables, iteration theory, equations on algebraic structures, regularity properties of the solutions of certain functional equations, functional inequalities, Hyers-Ulam stability, functional equations and inequalities involving mean values, generalized convexity. Interesting discussions were generated by the talks.

There were three very profitable Problem Sessions.

The social program included a guided tour in the village Berekfürdő involving a visit to its popular open air bath, a bowling competition, and a banquet. The mayor of Berekfürdő, Dr. Lajos Hajdu kindly assisted us as our guide in the village. He also warmly welcomed the participants of the meeting at the beginning of the banquet.

The closing address was given by Professor Roman Ger. His invitation to the Seventh Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities in February 2007 in Poland was gratefully accepted.

Summaries of the talks in alphabetic order of the authors follow in section 1, problems and remarks in chronological order in section 2, and the list of participants in the final section.

## 1. Abstracts of talks

Roman Badora: *Remarks on the stability of functional equations in single variable*

In this talk we present an application of the Banach limit to the study of the Hyers-Ulam stability for the following gamma-type functional equation:

$$\phi(f(x)) = g(x)\phi(x).$$

We also present the stability result for the following nonlinear equation:

$$\phi(x) = F(x, \phi(f(x))).$$

Mihály Bessenyei: *Hadamard's inequality on simplices*

The classical Hermite–Hadamard inequality gives a lower and an upper estimation for the integral average of convex functions defined on intervals, involving the midpoint and the endpoints of the domain. Applying Choquet's Theory, analogous inequalities can be obtained for convex functions of several variables defined on compact and convex sets. The aim of the talk is to verify Hadamard's inequality in a particular case, for convex functions defined on simplices, via an elementary approach and independently of Choquet's Theory.

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[9] Niculescu C. P., Persson L. -E., *Old and new on the Hermite–Hadamard inequality*, Real Anal. Exchange **29** (2003/2004), vol. 2, 619–623.

ZOLTÁN BOROS: *Monotonicity of the $\mathbb{Q}$-subdifferential of Jensen-convex functions* (Joint work with Zsolt Páles)

Let $K$ denote a subfield of the real number field $\mathbb{R}$ and $X$ be a vector space over $K$. Let $\mathcal{A}_K$ denote the set of all $K$-linear mappings $A : X \to \mathbb{R}$, and let $\mathcal{P}_0(\mathcal{A}_K)$ denote the family of all non-empty subsets of $\mathcal{A}_K$. Let $D$ be a non-void, $K$-algebraically open, $K$-convex subset of $X$.

DEFINITION 1. Let $f : D \to \mathbb{R}$ and $x_0 \in D$. The set

$$\partial_K f(x_0) = \{\, A \in \mathcal{A}_K \mid f(x_0) + A(x - x_0) \le f(x) \text{ for every } x \in D \,\}$$

is called the *$K$-subdifferential of $f$ at $x_0$*. If $A \in \partial_K f(x_0)$, we say that $A$ is a *$K$-subgradient* of the function $f$ at the point $x_0$.

Following the ideas of Minty [1] and Rockafellar [2], [3], we show that the $K$-subdifferential $\partial_K f : D \to \mathcal{P}_0(\mathcal{A}_K)$ of a $K$-convex function $f$ fulfils

$$\sum_{j=0}^{n} A_j(x_{j+1} - x_j) \le 0$$

for every $n \in \mathbb{N}$, $x_j \in D$ $(j = 0, 1, \ldots, n, n+1)$ with $x_{n+1} = x_0$, and $A_j \in \partial_K f(x_j)$ $(j = 0, 1, \ldots, n)$. We characterize $K$-subdifferentials of $K$-convex functions as maximal mappings from $D$ into $\mathcal{P}_0(\mathcal{A}_K)$ with this property. Various concepts of monotonicity and maximality are considered and compared.

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PÁL BURAI: *A Matkowski–Sutô type equation*

In this presentation, we deal with the following equation

$$\varphi^{-1}\left(\alpha\varphi(x) + (1-\alpha)\varphi(y)\right) + \psi^{-1}\left((1-\alpha)\psi(x) + \alpha\psi(y)\right) = x + y\,, \tag{1}$$

where $\varphi$ and $\psi$ are strictly monotone and continuous functions on the same interval. We give the continuously differentiable solutions.

DEFINITION 1. Let $I \subset \mathbb{R}$ a nonempty open interval and let $\mathrm{CM}(I)$ denote the class of all continuous, strictly monotone functions defined on $I$.

DEFINITION 2. A continuous function $M : I^2 \to I$ is called a mean on $I$ if

$$\min\{x, y\} \le M(x, y) \le \max\{x, y\},$$

for all $x, y \in I$.

DEFINITION 3. Let $p \in \mathbb{R}$ be a real constant and $I \subset \mathbb{R}$ be a nonempty interval. Let us define the following function on $I$:

$$\chi_p(x) := \begin{cases} x & \text{if } p = 0 \\ e^{px} & \text{if } p \neq 0 \end{cases} \qquad (x \in I).$$

DEFINITION 4. Let $\varphi, \Phi \in \mathrm{CM}(I)$. We say that $\varphi$ and $\Phi$ are equivalent on $I$, if there exist real constants $a, b$ $(a \neq 0)$ so that

$$\varphi(x) = a\Phi(x) + b$$

for every $x \in I$. In notation: $\varphi \sim \Phi$ on $I$.

DEFINITION 5. The mean $M : I^2 \to I$ is called weighted-quasi-arithmetic mean on $I$ if there exists $\varphi \in \mathrm{CM}(I)$ and $\alpha \in ]0, 1[$ such that

$$M(x, y) = \varphi^{-1}\big(\alpha\varphi(x) + (1 - \alpha)\varphi(y)\big) =: A_\varphi(x, y; \alpha)$$

for every $x, y \in I$.

THEOREM 1. *Let $I \subset \mathbb{R}$ be a nonempty open interval $\varphi, \psi \in \mathrm{CM}(I)$. Assume that $\varphi$ and $\psi$ are solutions of the functional equation* (1). *If there exists a nonempty open interval $J \subset I$ such that $\varphi$ and $\psi$ are continuously differentiable on $J$, then there is a constant $p \in \mathbb{R}$ so that*

$$\varphi \sim \chi_p \qquad \textit{and} \qquad \psi \sim \chi_{-p}$$

*on $I$.*

ZOLTÁN DARÓCZY: *On the equality of means*
(Joint work with Zsolt Páles)

In the theory of means the equality of means belonging to the same class is a natural question. For symmetrized weighted quasi-arithmetic means it can be formulated as follows.

PROBLEM. Let $0 < \alpha < 1$ and $0 < \beta < 1$, furthermore let $\varphi, \psi \in \mathrm{CM}(I)$. The following question is considered: what is the necessary and sufficient condition for the equation

$$\begin{aligned} \varphi^{-1}\left(\alpha\varphi(x) + (1 - \alpha)\varphi(y)\right) + \varphi^{-1}\left((1 - \alpha)\varphi(x) + \alpha\varphi(y)\right) = \\ \psi^{-1}\left(\beta\psi(x) + (1 - \beta)\psi(y)\right) + \psi^{-1}\left((1 - \beta)\psi(x) + \beta\psi(y)\right) \end{aligned} \tag{1}$$

to hold for all $x, y \in I$.

THEOREM (DARÓCZY–PÁLES). *For $\alpha \in \{\beta, 1-\beta\}$ (1) holds if and only if $\varphi \sim \psi$ on $I$.*

CONJECTURE. For $\alpha \notin \{\beta, 1-\beta\}$ (1) holds if and only if $\varphi \sim \mathrm{id}$ and $\psi \sim \mathrm{id}$ on $I$.

In the talk we give nontrivial proofs for some interesting special cases of the conjecture.

WŁODZIMIERZ FECHNER: *Functional inequalities connected with quadratic functionals*

Let $(X, +)$ be an abelian group. We are interested in the functional inequality

$$D(f)(x,y) \geq \phi(x,y), \quad x, y \in X, \tag{1}$$

where $f\colon X \to \mathbb{R}$ and $\phi\colon X \times X \to \mathbb{R}$ are unknown and $D$ is given and has the following two properties:

- for each $x, y \in X$ the map $D(\cdot)(x,y)\colon \mathbb{R}^X \to \mathbb{R}$ is a linear functional;
- if $\phi$ is biadditive and symmetric and $f(x) = \frac{1}{2}\phi(x,x)$ for $x \in X$, then $D(f) = \phi$.

We are looking for interesting conditions under which the following representation holds:

$$f(x) = f_0(x) + \frac{1}{2}\phi(x,x), \qquad x \in X,$$

where $f_0$ satisfies $D(f_0)(x,y) \geq 0$ for each $x, y \in X$, and $\phi$ is biadditive and symmetric.

Three following cases of inequality (1) are of our special interest:

$$D(f)(x,y) = f(x+y) - f(x) - f(y), \qquad x, y \in X,$$

$$D(f)(x,y) = f(x+y) - \frac{f(2x)+f(2y)}{4}, \quad x, y \in X,$$

$$D(f)(x,y) = \frac{f(2x+2y)}{4} - f(x) - f(y), \quad x, y \in X.$$

ROMAN GER: *Logarithmic mean and Daróczy's equation*

We are looking for possibly mild regularity conditions under which the functional equation

$$f\left(L(x,y)\right) = \frac{f(x)+f(y)}{2}, \quad x, y \in (0,\infty),$$

where $L$ stands for the logarithmic mean, admits merely constant solutions. These studies lead, in particular, to the functional equation

$$f(2x^2) + f(2y^2) = f(x(x+y)) + f(y(x+y)), \quad x, y \in (0,\infty),$$

considered by Z. Daróczy during the Second DKWS held in Hajdúszoboszló in 2002 (cf. Problem 3. in the *Report of Meeting*, Annales Mathematicae Silesianae **16** (2003), pp. 95–96, and my Remark 4, *ibidem*, pp. 96-97).

Attila Gilányi: *Three-parameter families and generalized convex functions*
(Joint work with Mirosław Adamek, Kazimierz Nikodem and Zsolt Páles)

In 1937, E. F. Beckenbach introduced a generalization of the classical concept of convex functions. According to his definition, a set $\mathcal{B}(I)$ of real valued functions defined on the interval $I$ is said to be a two-parameter family on $I$, if, for any pairs of the set $I \times \mathbb{R}$ with distinct first coordinates, there exists a unique element of $\mathcal{B}(I)$ which lies on the points above. A function $f : I \to \mathbb{R}$ is called generalized convex if, for every two distinct points $(x_1, y_1)$ and $(x_2, y_2)$ of the graph of $f$, the (unique) element of $\mathcal{B}(I)$ determined by $(x_1, y_1)$ and $(x_2, y_2)$ lies above the graph of $f$ between $(x_1, y_1)$ and $(x_2, y_2)$. In this talk, we extend Beckenbach's concept for two-variable functions. Introducing three-parameter families, we define generalized convex functions of two variables and we investigate their basic properties.

Gabriella Hajdu: *Hosszú's equation over the Gaussian and Euler integers*
(Joint work with Lajos Hajdu)

We consider Hosszú's famous functional equation:

$$f(x) + f(y) = f(xy) + f(x + y - xy)$$

and completely describe the functions $f : T \to A$ satisfying this equation, where $T$ is the set of the Gaussian or Euler integers and $A$ is an arbitrary Abelian group.

Antal Járai: *On the measurable solutions of a functional equation*

The measurable solutions

$$f\colon \mathbb{R}^3 \setminus \{0\} \to \mathbb{C} \setminus \{0\}$$

and

$$(t,s) \mapsto G(t,s) \in \mathbb{C} \setminus \{0\}, \quad s \in \mathbb{R}^3, \quad t > |s| > 0$$

of the functional equation

$$f(x)f(y) = G\big(|x| + |y|, x + y\big), \quad x, y \in \mathbb{R}^3, \quad x \times y \neq 0$$

are considered and it is proved that they are continuous.

Zygfryd Kominek: *Some remarks on subquadratic functions*
(Joint work with Katarzyna Troczka)

In the talk $X$ is always a real linear space and $\mathbb{R}$ denotes the set of all reals. Every function $\varphi : X \to \mathbb{R}$ satisfying the following inequality

$$\varphi(x + y) + \varphi(x - y) \leq 2\varphi(x) + 2\varphi(y), \quad x, y \in X, \tag{1}$$

is called subquadratic. If the sign " $\leq$ " is replaced by " $=$ " in (1) then we say that $\varphi$ is quadratic function. There are plenty papers devoted to quadratic functions [1], [2], [3] (and references there). In this talk some properties of the solutions of (1) will be proved, particularly we will investigate nonpositive solutions of (1). Also interesting question of finding sufficient conditions on subquadratic function to be quadratic one will be considered.

References

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Károly Lajkó: *Sequenced problems for functional equations*
(Joint work with Zsolt Ádám, Gyula Maksa and Fruzsina Mészáros)

It is well-known that there are many possible methods to solve equations of the form

$$H\left(f\left(x+y\right),f\left(x-y\right),f\left(x\right),f\left(y\right),x,y\right)=0 \quad (x,y\in\mathbb{R}), \tag{1}$$

where $H:\mathbb{R}^6\to\mathbb{R}$ is a known function and $f:\mathbb{R}\to\mathbb{R}$ is the unknown function to be determined.

J. Aczél (e.g. in [1]) investigated the following special case of (1)

$$f\left(x\right)f\left(x+y\right)=\left[f\left(y\right)\right]^2\left[f\left(x-y\right)\right]^2a^{y+4} \quad (x,y\in\mathbb{R}), \tag{2}$$

where $a\in\mathbb{R}_+, a\neq 1$ is an arbitrary constant. He determined the identically zero and the nowhere zero solutions of (2) in the form

$$f\left(x\right)=0 \quad (x\in\mathbb{R}), \quad f\left(x\right)=a^{x+2} \quad (x\in\mathbb{R}), \quad f\left(x\right)=-a^{x+2} \quad (x\in\mathbb{R}).$$

But it is easy to see that functions

$$f\left(x\right)=\begin{cases} a^{x+2} & \text{for } x\in\mathbb{Q} \\ 0 & \text{for } x\in\mathbb{R}\setminus\mathbb{Q}\end{cases} \quad \text{or} \quad f\left(x\right)=\begin{cases} -a^{x+2} & \text{for } x\in\mathbb{Q} \\ 0 & \text{for } x\in\mathbb{R}\setminus\mathbb{Q}\end{cases}$$

are also solutions of (2).

Our principal aim in this talk is to obtain all solutions of the functional equation (2). For this purpose we shall create a sequence of problems for equations of type (1).

At the same time such sequenced problems are appropriate for the fostering of talented students on different level of mathematical education.


Reference

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László Losonczi: *Inequalities for two-variable means depending on a measure*

Let $I$ be an interval $f,g:I\to\mathbb{R}$ be given continuous functions such that $g(x)\neq 0$ for $x\in I$ and $h(x):=f(x)/g(x)$ $(x\in I)$ is strictly monotonic (thus invertible) on $I$. Let further $\mu$ be an increasing non-constant function on $[0,1]$ and

$$M_{f,g,\mu}(x,y) := h^{-1}\left(\frac{\int_0^1 f(tx+(1-t)y)d\mu(t)}{\int_0^1 g(tx+(1-t)y)d\mu(t)}\right) \quad (x,y\in I)$$

where the integrals are Riemann–Stieltjes ones. It is easy to see that $M_{f,g,\mu}$ is a two-variable mean on $I$.

By suitable choice of $\mu$ both the quasi-arithmetic means weighted with a weight function and the Cauchy (and also several other) mean can be obtained as special $M_{f,g,\mu}$ means.

We discuss the comparison problem

$$(1) \qquad M_{f,g,\mu}(x,y) \leq M_{F,G,\mu}(x,y) \quad (x,y\in I)$$

and give necessary conditions (which, in general, are not sufficient) and also sufficient conditions for (1) to hold.

GRAŻYNA ŁYDZIŃSKA: *On semicontinuity of some set-valued iteration semigroups*

Let $X$ be an arbitrary set, $A\colon X\to 2^{\mathbb{R}}$, $q := \sup A(X)$ and $F\colon (0,\infty)\times X \to 2^X$ be given by

$$(1) \qquad F(t,x) := A^{-1}\Big(A(x) + \min\big\{t, q-\inf A(x)\big\}\Big),$$

where

$$A^{-1}(V) := \{x\in X\colon\ A(x)\cap V \neq \emptyset\}$$

for every $V\subset\mathbb{R}$.

The formula (1) is a set-valued counterpart of a well-known form of iteration semigroups of single-valued functions on an interval.

We present a few theorems about lower semicontinuity of the multifunctions $F(t,\cdot)$ and $F(\cdot,x)$ in the case when $A$ is a single-valued function defined on a topological space.

GYULA MAKSA: *Quasi-sums in several variables*

(Joint work with E. Nizsalóczki)

In this talk we introduce the notions of quasi-sums in several variables and of the local quasi-sums in several variables. We present that the local quasi-sums are quasi-sums in all finite dimensions. Finally, we show how this result can be applied to find the continuous solutions of the functional equation

$$\begin{aligned} g(u_{11}+\cdots+u_{1N},\cdots,u_{M1}+\cdots+u_{MN}) \\ = f(g_1(u_{11},\ldots,u_{M1}),\ldots,g_N(u_{1N},\ldots,u_{MN})) \end{aligned}$$

that are strictly monotonic in each variable. This equation is an important particular case of a general aggregation equation.

JANUSZ MATKOWSKI: *On some generalized Gołąb-Schinzel functional equations*

Some results on functional equation

$$f(F(x+yf(x), y+xf(y))) = G(x, y, f(x), f(y)),$$

where $F$ and $G$ are given functions, will be presented.

FRUZSINA MÉSZÁROS: *Functional equations satisfied almost everywhere relating to characterization problems*
(Joint work with Károly Lajkó)

Functional equations are used in the characterization of joint distributions by means of conditional distributions. I look for the joint density function in special cases of the density functions for what I investigate the general measurable solution of functional equations satisfied almost everywhere.

PROBLEM 1. What is the general measurable solution of the functional equation

$$g_1\left(\frac{x - m_1 y - c_1}{\lambda_1 (y + a_1)}\right) f_Y(y) = g_2\left(\frac{y - m_2 x - c_2}{\lambda_2 (x + a_2)}\right) f_X(x),$$

satisfied for almost all $(x, y) \in D \subset \mathbb{R}^2$?

PROBLEM 2. Let $D_1, D_2$ be the following sets:

$$D_1 = \mathbb{R} \setminus \{-a_1, -a_2\}, D_2 = \mathbb{R} \setminus \{-b_1, -b_2\}.$$

What is the general measurable solution of the functional equation

$$g_1[(y + a_1)x + b_1 y + c_1] f_Y(y) = g_2[(x + b_2)y + a_2 x + c_2] f_X(x),$$

satisfied for almost all $(x, y) \in D = D_2 \times D_1$?

We present the general measurable solutions of these functional equations.

JANUSZ MORAWIEC: *Grincevičjus series and functional equations*
(Joint work with Rafał Kapica)

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $(\xi_n, \eta_n)_{n\in\mathbb{N}}$ be a sequence of independent identically distributed vectors of random variables. We consider the Grincevičjus series

$$\sum_{n=1}^{\infty} \eta_n \prod_{k=1}^{n-1} \xi_k$$

and its probability distribution function $F$ satisfying

$$F(x) = \int_{L>0} F(L(\omega)x + M(\omega))dP(\omega) + \int_{L<0} [1 - F(L(\omega)x + M(\omega))]dP(\omega),$$

where $L = \frac{1}{\xi_1}$ and $M = -\frac{\eta_1}{\xi_1}$.

Barbara Przebieracz: *Near iterability*

Inspired by Problem (3.1.12) posed by E. Jen in [T] we present various approaches to the concept of near-iterability. We deal with selfmappings of a real compact interval, characterize some classes of near-iterable functions which are generalizations of almost iterable ones. That will include:

almost iterable functions (W. Jarczyk [J]),

that is the continuous functions $f\colon I \to I$, for which there exists an iterable function $g\colon I \to I$ such that

$$\lim_{n\to\infty}(f^n(x) - g^n(x)) = 0 \tag{1}$$

for every $x \in I$ and the convergence is uniform on every interval with endpoints being two consecutive fixed points of function $f$;
functions satisfying some weaker conditions concerning the convergence (1), that is functions $f$ for which there exists an iterable function $g$ such that

(1) holds for every $x \in I$;

(1) holds for $x \in I \setminus M$,

where the set $M$ is "small" in a sense.

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Maciej Sablik: *On some functional equations arising in actuarial mathematics*

We present some functional equations that have naturally appeared in actuarial mathematics. Some of them have been treated by Gompertz, de Morgan, Chini, among others, and lead to a characterization of different analytical models of future lifetime. The author of the talk, together with T. Riedel and P. K. Sahoo, has also contributed to the area. Now, we discuss some new problems leading to functional equations and connected with assumming some (generally accepted in the actuarial calculus) hypotheses.

László Székelyhidi: *Functional equations on Sturm–Liouville hypergroups*

Sturm–Liouville hypergroups are closely related to Sturm–Liouville boundary value problems. Continuing our former investigations of functional equations on polynomial hypergroups here we present some new results concerning classical equations on Sturm–Liouville hypergroups. In particular, we describe the general form of exponentials, additive functions and moment functions on Sturm–Liouville hypergroups.

TOMASZ SZOSTOK: *A generalization of the sine function and geometrical properties of normed spaces*

Let $(X, \|\cdot\|)$ be a normed space. We define the function $s : X \times X \to \mathbb{R}$ by the formula

$$s(x,y) := \begin{cases} \inf\limits_{\lambda\in\mathbb{R}} \dfrac{\|x+\lambda y\|}{\|x\|} & x \neq 0\,, \\ 1 & x = 0\,. \end{cases}$$

It is worth noting that in an Euclidean space $s(x, y)$ is equal to the absolute value of the sine of the angle between $x$ and $y$. Using this function and vectors $x, y \in X$ satisfying some properties we define a new function $\varphi_{x,y}$ which is defined on $\mathbb{R}$ and plays the role of the sine function in the space $X$. Then we study the connections between the properties of $\varphi_{x,y}$ and properties of the space $X$.

## 2. Problems and Remarks

1. REMARK. Tropical mathematics has attracted the attention of several people nowadays. The basic structure is the min-plus semiring with the basic space $[0, +\infty]$ and with the two operations

$$x \oplus y = \min\{x, y\},$$
$$x \odot y = x + y\,.$$

It turns out that some classical functional equations have surprising solutions on this structure. Tropical exponentials, that is, solutions of the functional equation

$$m(x \oplus y) = m(x) \odot m(y)$$

are trivial: $m(x) = 0$ for each $x$, but tropical additive functions, that is, solutions of the functional equation

$$a(x \oplus y) = a(x) \oplus a(y)$$

are exactly the increasing functions which are bounded from above. The study of classical functional equations and inequalities may produce interesting results on this delicate structure.

L. SZÉKELYHIDI

2. REMARK (On the tropical logarithm). We consider the functional equation

$$f(x \odot y) = f(x) \oplus f(y), \tag{1}$$

that is,

$$f(x + y) = \min\{f(x), f(y)\}. \tag{2}$$

PROPOSITION. *A function* $f : [0,+\infty] \to [0,+\infty]$ *satisfies* (2) *for all* $x, y \in [0,+\infty]$ *if, and only if, there exist* $0 \le a \le b \le c \le +\infty$ *such that*

$$f(x) = \begin{cases} c\,, \text{ if } x = 0\,, \\ b\,, \text{ if } 0 < x < +\infty\,, \\ a\,, \text{ if } x = +\infty\,. \end{cases} \tag{3}$$

PROOF. Due to the symmetry of equation (2) with respect to the variables $x$ and $y$, one has to distinguish six cases in order to verify that every function of the form (3) satisfies (2).

Conversely, let us assume that $f$ is a solution of the functional equation (2). For $0 \le x < y \le +\infty$ we have

$$f(y) = f(x + (y - x)) = \min\{f(x), f(y - x)\} \le f(x). \tag{4}$$

Thus $f$ is decreasing. Moreover, for every $0 < x < +\infty$ we have

$$f(x) = f\left(\frac{x}{2} + \frac{x}{2}\right) = \min\left\{f\left(\frac{x}{2}\right), f\left(\frac{x}{2}\right)\right\} = f\left(\frac{x}{2}\right),$$

and thus

$$f(x) = f\left(\frac{x}{2^n}\right) \tag{5}$$

for all positive integer $n$. If $0 < x < y < +\infty$, then there exists a positive integer $n$ such that $\frac{y}{2^n} < x$, hence inequality (4) and equation (5) yield

$$f(x) \le f\left(\frac{y}{2^n}\right) = f(y).$$

Since the reversed inequality is established in (4), we have proved that $f$ is constant on the open interval $]0,+\infty[$. □

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