ON ENERGY FORMULAS FOR SYMMETRIC SEMIGROUPS

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Abstract. Let $\mathbb{P} := (P_t)_{t>0}$ be a strongly continuous contraction semigroup of symmetric operators on $L^2(m)$. Let β be a Bochner subordinator and let $\mathbb{P}^\beta$ be the subordinated semigroup of $\mathbb{P}$ by means of β, i.e. $P_t^\beta := \int_0^\infty P_s \beta_t(ds)$. We give in this paper an energy formula for the $\mathbb{P}^\beta$-potentials with finite energy in terms of the $\mathbb{P}$-exit laws and of β. We deduce an explicit energy formula for the α-potentials.

0. Introduction

Let $(E, \mathcal{E}, m)$ be a σ-finite measure space and let $\mathbb{P} := (P_t)_{t>0}$ be a strongly continuous contraction semigroup of m-symmetric operators on $L^2(m)$. Let A denote the generator of $\mathbb{P}$ with domain $D(A)$. The associated energy form is defined by $\mathbf{a}(f,g) :=< -Af, g >$ for $f, g \in D(A)$ and the energy norm $\mathbf{e}$ is defined on $D(A)$ by

$$\mathbf{e}(u) :=< -Au, u >^{1/2} \qquad (u \in D(A)) \tag{0.1}$$

Let $\mathcal{D}$ be the associated Dirichlet space, i.e. the completion of $D(A)$ in $L^2(m)$ with respect to any of the equivalent norms defined by the bilinear forms $\mathbf{a}_p(f,g) := \mathbf{a}(f,g) + p < f, g >$, $p > 0$. We denote by $\mathbf{a}$ (resp. $\mathbf{e}$) the extension of the energy form (resp. norm) on $\mathcal{D}$. A positive element u of $\mathcal{D}$ is called a $\mathbb{P}$-potential with finite energy if $e(u) < \infty$ and if $\mathbf{a}(u,f) \geq 0$ for each positive function $f \in \mathcal{D}$. We denote by $\mathcal{P}_f$ the set of such functions.

A classical problem in potential theory is the following: Find an "explicit" formula for the energy $\mathbf{e}$ on $\mathcal{P}_f$.

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Suppose for a moment that $\mathbb{P}$ admits a Green function G and that each potential $u \in \mathcal{P}_f$ admits an integral representation on the form $u = G\rho$ for some σ-finite positive measure on $(E, \mathcal{E})$ (cf. 2.3 below). In this case, the following energy formula is well known

$$\mathbf{e}^2(u) = \int\int G(x,y)\,d\rho(x)d\rho(y). \tag{0.2}$$

But there are many important situations for which a Green function does not exist (cf. Examples 1.2 and Remarks 2.2 below).

We present in this paper another integral representation type for the potentials of finite energy, valid in general settings and we deduce an energy formula. Our main tool is the notion of *exit law*. Note first that our approach is adapted from many papers ([1], [2], [6], [7] and [10–13]) which are devoted to this notion, but we are concerned with the general abstract case.

Let us return to the general case. A $\mathbb{P}$-exit law is a family $\varphi := (\varphi_t)_{t>0}$ of positive elements of $L^2(m)$ such that

$$P_s\varphi_t = \varphi_{s+t}, \qquad m-\text{a.e. } (s,t>0) \tag{0.3}$$

After some preliminaries in the first paragraph, we present in the second paragraph a general energy formula for $\mathbb{P}$ (proved in [7] if $\mathbb{P}$ is the transition function of a right process). Namely, for each $\mathbb{P}$-potential $u \in \mathcal{P}_f$, there exists a unique $\mathbb{P}$-exit law φ such that

$$P_t u = V\varphi_t, \quad m-\text{a.e.} \tag{0.4}$$

where $V := \int_0^\infty P_t dt$ is the potential operator of $\mathbb{P}$. We also deduce the energy formula

$$\mathbf{e}^2(u) = 2\int_0^\infty \|\varphi_t\|_2^2\,dt, \tag{0.5}$$

The third paragraph contains the main result of this paper, namely an energy formula for the subordinated semigroup: Let $\beta = (\beta_t)_{t>0}$ be a convolution semigroup on $[0,+\infty[$ and let $\mathbb{P}^\beta := (P_t^\beta)_{t>0}$ be the subordinated semigroup of $\mathbb{P}$ in the sense of Bochner by means of β, i.e.

$$P_t^\beta := \int_0^\infty P_s\,d\beta_t(s)\cdot \tag{0.6}$$

It is known that $\mathbb{P}^\beta$ is also a strongly continuous contraction semigroup of m-symmetric operators on $(E,\mathcal{E})$. If h is a $\mathbb{P}^\beta$-potential with finite energy, i.e. $\mathbf{e}_\beta(h) < \infty$, it is proved in this paper that there exists a unique $\mathbb{P}$-exit law $\varphi := (\varphi_t)_{t>0}$ such that

$$P_t h = V^\beta\varphi_t, \quad m-\text{a.e.} \tag{0.7}$$

where V^β is the potential operator of $\mathbb{P}^\beta$. We also obtain an energy formula for h on the form

$$\mathbf{e}_\beta^2(h) = \int_0^\infty \|\varphi_{s/2}\|_2^2\,\kappa(ds) \tag{0.8}$$

where $\kappa := \int_0^\infty \beta_t dt$.

If β is the Dirac subordinator, it can be easily seen that the energy formula (0.5) is a particular case of (0.8). If β is the subordinator "fractional power" of order $\alpha \in]0,1[$, then we have

$$\mathbf{e}_\alpha^2(h) = 2^\alpha \Gamma(\alpha) \int_0^\infty \|\varphi_s\|_2^2 \, s^{\alpha-1} \, ds \tag{0.9}$$

for each abstract Riesz potential h.

1. Preliminaries

Let $(E, \mathcal{E})$ be a measurable space and let m be a σ-finite positive measure on $(E, \mathcal{E})$. We denote by $< .,. >$ the inner product in $L^2(m)$ and by $\|.\|_2$ the associated norm. We say that a property holds m-a.e. if the set for which this property fails is m-negligible. If $\mathcal{T}$ is a set of functions defined on E, denote by $\mathcal{T}_+$ the set of positive elements of $\mathcal{T}$. Note that $< .,. >$ and $\|.\|_2$ may be extended to m-a.e. positive measurable functions defined on E in the usual way. Finally, ε_x will denote the Dirac measure at point $x \in E$.

1.1. Symmetric semigroups

For the following notions, we refer the reader to [15] or [16].
A strongly continuous m-symmetric contraction semigroup on E is a family $\mathbb{P} := (P_t)_{t>0}$ of linear operators on $L^2(m)$ such that

1. For each $t > 0$ and $f \in L^2(m)$ with $0 \le f \le 1$, we have $0 \le P_t f \le 1$ (*contraction property*).
2. For every $s, t > 0$: $P_s P_t = P_{s+t}$ (*semigroup property*).
3. For all $t > 0$ and $f, g \in L^2(m)$: $< P_t f, g >=< f, P_t g >$ (*symmetry property*).
4. $\lim_{t \to 0} \|P_t f - f\|_2 = 0$, for every $f \in L^2(m)$(*strong continuity*).

In this case, we say shortly that $\mathbb{P}$ is an m-symmetric semigroup on E.

Let $\mathbb{P}$ be an m-symmetric semigroup on E. The associated *generator* A is defined by

$$A(f) := \lim_{t \to 0} \frac{1}{t}(P_t f - f) \tag{1.1}$$

on its domain $D(A)$ which is the set of all functions $f \in L^2(m)$ for which this limit exists in $L^2(m)$. It is well known that $D(A)$ is dense in $L^2(m)$ and A is a closed operator on E. In the same way, the *potential operator* associated to $\mathbb{P}$ is defined by

$$V(f) := \int_0^\infty P_s f \, ds := \lim_{t \to +\infty} \int_0^t P_s f \, ds \tag{1.2}$$

on its domain $D(V)$ which is the set of all functions $f \in L^2(m)$ for which this limit exists in $L^2(m)$. Moreover, if $R(V)$ is the range of V then $R(V) \subset D(A)$ and (cf. [3], Chap. 11 for example).

$$A(Vf) = -f, \qquad (f \in D(V)) \tag{1.3}$$

1.2. EXAMPLES

For the following examples we refer the reader to ([5], XIII, 3).

(i) Let $\lambda^d(dx) = dx$ be the Lebesgue measure on $\mathbb{R}^d$, $d \geq 3$. For $t > 0$ and $x \in \mathbb{R}^d$, let $g_t(x) := \frac{1}{(2\pi t)^{d/2}} \exp(-\frac{|x|^2}{2t})$ be the Gaussian transition function on $\mathbb{R}^d$. For each $t > 0$ and $f \in L^2(m)$ define $B_t f := g_t * f$. Then $\mathbb{B} := (B_t)_{t>0}$ is a strongly continuous λ^d-symmetric semigroup on $L^2(\lambda^d)$, called *semigroup of the Brownian motion on* $\mathbb{R}^d$. For this example the associated generator is $A := \frac{1}{2}\Delta$ where Δ is the Laplacian operator. The associated potential operator is given by $Vf(x) = c_d \int |x-y|^{2-d} f(y)\,dy$ for some constant c_d.

(ii) Let μ be a convolution semigroup on $\mathbb{R}^d$, i.e., a family $\mu := (\mu_t)_{t>0}$ of sub-probability measures on $\mathbb{R}^d$ such that (cf. [3] Chap. 8 and Chap. 13)

1. $\mu_s * \mu_t = \mu_{s+t}$ for all $s, t > 0$.
2. $\lim_{t\to 0} \mu_t = \varepsilon_0$ vaguely.

We suppose that μ is symmetric, i.e.

3. $\mu_t(-\omega) = \mu_t(\omega)$ for each $t > 0$ and for each Borel subset ω of $\mathbb{R}^d$.

Let $P_t f := \mu_t * f$, then $\mathbb{P} := (P_t)_{t>0}$ is a λ^d-symmetric contraction semigroup on $\mathbb{R}^d$. If $\Upsilon := \int_0^\infty \mu_t\,dt$ and if $\Lambda := \lim_{t\to 0} \frac{1}{t}(\mu_t - \varepsilon_0)$ vaguely then $Vf = \Upsilon * f$ and $Af = \Lambda * f$. Moreover (1.3) is equivalent to $\Upsilon * \Lambda = -\varepsilon_0$ in this case. If we take $\mu_t := g_t.\lambda^d$, we obtain the Brownian semigroup.

(iii) For $x \in \mathbb{R}^d$, let $g(x) := g_1(x) = \frac{1}{(2\pi)^{d/2}} \exp(-\frac{|x|^2}{2})$ be the Gaussian function on $\mathbb{R}^d$ and let $m := g \cdot \lambda^d$ be the so called *Gaussian measure on* $\mathbb{R}^d$.
For $t > 0$, $f \in L^2(m)$ and $x \in \mathbb{R}^d$ let

$$U_t f(x) = \int f\left(x \exp\left(-\frac{t}{2}\right) + y\sqrt{1 - \exp(-t)}\right) m(dy)\cdot \tag{1.4}$$

(1.4) is the so called *Mehler* formula.

Then $\mathbb{U} := (U_t)_{t>0}$ is a strongly continuous m-symmetric semigroup on $\mathbb{R}^d$, called the semigroup of *Ornstein–Uhlenbeck* on $\mathbb{R}^d$. In this case, the associated generator is $A := \frac{1}{2}(\Delta - \langle .,\nabla \rangle)$ where ∇ is the gradient operator.

There is no explicit formula for the potential operator W of $\mathbb{U}$ but it can be expressed in terms of the *Hermite polynomials* of $\mathbb{R}^d$: Indeed, if H_α denotes the Hermite polynomial of multiindex $\alpha := (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d$ then

$$U_t H_\alpha = \exp\left(-\frac{t|\alpha|}{2}\right) H_\alpha; \quad (t \geq 0) \tag{1.5}$$

where $|\alpha| := \alpha_1 + ... + \alpha_d$. Hence, by integration of (1.5), we have $WH_\alpha = (\frac{\alpha}{2})H_\alpha$. In other words H_α is an eigen function of W for the eigen value $\alpha/2$.

2. Energy and exit laws

2.1. Energy

For the following notions and properties, we refer the reader to [6] and to the related references, in particular ([5], XIII, 4).

Let $\mathbb{P}$ be an m-symmetric semigroup on E with generator A defined on $D(A)$. The associated *energy form* $\mathbf{a}$ is defined by

$$\text{(2.1)} \qquad \mathbf{a}(f,g) :=< -Af, g > \qquad (f, g \in D(A)).$$

It is known that $\mathbf{a}$ is positive on $D(A)$, i.e. $\mathbf{a}(f,f) \geq 0$ for each $f \in D(A)$. Hence the associated *energy norm* $\mathbf{e}$ is defined by

$$\text{(2.2)} \qquad \mathbf{e}(f) := \mathbf{a}(f,f)^{1/2} =< -Af, f >^{1/2}; \qquad (f \in D(A)).$$

For $f, g \in D(A)$ and $p > 0$, let $a_p(f,g) :=< -Af, g > +p < f, g >$. The Dirichlet space $\mathcal{D}$ associated to $\mathbb{P}$, is defined as the completion of $D(A)$ in $L^2(m)$ with respect to any norm defined by a_p. We denote by $\mathbf{a}$ (resp. $\mathbf{e}$) the extension of the energy form (resp. norm) on $\mathcal{D}$. By the symmetry property, $\mathbf{e}$ is continuously extended and

$$\text{(2.3)} \qquad \mathbf{a}(f,g) \leq \mathbf{e}(f) \cdot \mathbf{e}(g); \qquad (f, g \in \mathcal{D}).$$

A (positive) element u of $\mathcal{D}$ is called a $\mathbb{P}$-*potential* if $\mathbf{a}(u,f) \geq 0$ for each positive function $f \in \mathcal{D}$. We denote by $\mathcal{P}$ the cone of the $\mathbb{P}$-potentials, i.e.

$$\mathcal{P} := \{u \in \mathcal{D}_+ : \mathbf{a}(u,f) \geq 0, \text{ for all } f \in \mathcal{D}_+\}$$

We denote by $\mathcal{P}_f$ the subset of $\mathbb{P}$-potentials with finite energy, i.e. $\mathcal{P}_f := \{u \in \mathcal{P}: \mathbf{e}(u) < \infty\}$. Note that

$$\text{(2.4)} \qquad D(A) \cap \mathcal{P} = D(A) \cap \mathcal{P}_f = \{u \in D(A) : u \geq 0 \text{ and } Au \leq 0 \ m\text{.a.e.}\}$$

For example, for each $f \in D_+(V)$, the function Vf is a $\mathbb{P}$-potential with finite energy. Moreover

$$\text{(2.5)} \qquad \mathbf{e}^2(Vf) =< f, Vf >= \int_0^\infty < f, P_t f > dt = 2\int_0^\infty \|P_t f\|_2^2 \, dt$$

by (1.2), (1.3), (2.2) and the symmetry property. The relation (2.4) will be generalized for each $u \in \mathcal{P}_f$.

2.2. Remarks

1. A *Green function* for $\mathbb{P}$ (if it exists) is a $(\mathcal{E}\otimes\mathcal{E}, \mathcal{B}([0,\infty])$-measurable function $G : E \times E \to [0,\infty]$ such that $Vf(x) = \int G(x,y)\, f(y)\, m(dy)$ for every $x \in E$ and $f \in D(V)$. Here $\mathcal{B}([0,\infty])$ stands for the Borel field of $[0,\infty]$.

If $\mathbb{P}$ admits a Green function G then, under some regularity hypothesis, each potential $u \in \mathcal{P}_f$ admits an integral representation on the form $u = G\rho := \int G(.,y)\, d\rho(y)$ for some σ-finite positive measure on $(E,\mathcal{E})$ (cf. [8] Chap. 5 and Chap. 6). In this case, the following energy formula is known (cf. [8] Chap. 11)

$$\mathrm{e}^2(u) = \int\int G(x,y)\, d\rho(x) d\rho(y). \tag{2.6}$$

2. If $\mathbb{P}$ is defined by a symmetric convolution semigroup μ, then (cf. [5], XIII, 4 for example).

$$\mathrm{e}^2(u) = \int |\hat{u}(x)|^2 \psi(x) dx \tag{2.7}$$

where ψ is the negative definite function defined by μ (i.e. $\hat{\mu}_t = \exp(-t\psi)$).

3. Let μ be a symmetric convolution semigroup on $\mathbb{R}^d$ and let $\mathbb{P}$ be the associated λ^d-symmetric semigroup. Then $\mathbb{P}$ admits a Green function if and only if $\Upsilon := \int_0^\infty \mu_t\, dt$ is absolutely continuous with respect λ^d. This is the case of the Brownian semigroup $\mathbb{B}$, where the associated Green function is $N(x,y) := |x-y|^{2-d}$ the so called *Newtonian Kernel.*

4. Note that, there is no Green function for the Ornstein-Uhlenbeck semigroup $\mathbb{U}$. Moreover $\mathbb{U}$ is not given by a convolution semigroup.

5. The aim of this paper is to find an energy formulae analogous to (2.6) or (2.7) but valid in the general case. Central to our development will be the notion of exit law.

2.3. Energy and capacity

The notion of energy is closely related to the notion of *capacity* in the following sense: A measurable subset F of E is said to be *capacitable* if

$$u_F := \inf\{u \in \mathcal{P}_f : u = 1 \text{ on } F\}$$

is a $\mathbb{P}$-potential with finite energy. In this case the *capacity* $\mathcal{C}(F)$ of F is given by $\mathcal{C}(F) := \mathrm{e}^2(u_F)$.

If there exists a Green function G and if $u_F = G\mu_F$ then $\mathcal{C}(F) = \mu_F(F)$. In fact, in the classical potential theory, the preceding relation is given as definition of capacity (cf. [8] for more details).

2.4. Exit laws

Let $\mathbb{P}$ be an m-symmetric semigroup. A family $\varphi = (\varphi_t)_{t>0} \subset L^2_+(m)$ is an *exit* *law* for $\mathbb{P}$ provided (cf. [7] for example) φ verifies the functional equation:

$$P_s \varphi_t = \varphi_{s+t}, \; m\text{-a.e.} \qquad (s,t>0) \tag{2.8}$$

Two exit laws φ and ψ are said *equivalent* if $\varphi_t = \psi_t . m$.a.e. for each $t > 0$.

Note that, for each $f \in L^2_+(m)$ the family $(P_t f)_{t>0}$ is a (so called *closed*) $\mathbb{P}$-exit law, by the semigroup property. Moreover the trivial relation $P_t(Vf) = V(P_t f); (t > 0)$ will be generalized in the following result for every $\mathbb{P}$-potential with finite energy. Let us remark also that: With the notation of Example 1.2.(i), it is easy to see that $B_t g_s = g_{s+t}$, hence $(g_t)_{t>0}$ is a $\mathbb{B}$-exit law. However $(g_t)_{t>0}$ is not closed.

Exit laws serve to represent potentials in a general setting (cf [6],[7],[10–13]). For the symmetric case, let us first recall the following result which is proved in ([7], Proposition (3.7)) if $\mathbb{P}$ is the transition function of a right process.

2.5. THEOREM

Let $\mathbb{P}$ be an m-symmetric semigroup with potential operator V. For each potential $u \in \mathcal{P}_f$, there exists a unique (up to equivalence) $\mathbb{P}$-exit law φ such that

$$P_t u = V\varphi_t, \; m-\text{a.e.} \qquad (t>0) \tag{2.9}$$

Moreover, we have

$$\mathrm{e}^2(u) = 2\int_0^\infty \|\varphi_t\|_2^2 \, dt\cdot \tag{2.10}$$

where $\varphi := (\varphi_t)_{t>0}$ is the associated $\mathbb{P}$-exit law.

2.6. REMARKS

1. The proof of 2.5 given in [7], can be adapted in the abstract case. Moreover, we can also adapt those of ([6], Proposition 3.10).

2. Let F be a capacitable subset of E, then by 2.5 we have $C(F) = 2\int_0^\infty \|\varphi_t\|_2^2 \, dt$ where φ is the $\mathbb{P}$-exit law associated to the potential u_F.

3. In what follows, we want to obtain similar Formulas for the subordinated semigroup, in the Bochner sense. In particular, Theorem 2.5 will be generalized and proved in the abstract framework.

3. Energy and subordination

3.1. BOCHNER SUBORDINATION

We consider $\mathbb{R}$ endowed with its Borel field, we denote by *supp* μ the support of the measure μ defined on $\mathbb{R}$ and by λ the Lebesgue measure on $[0, \infty[$.

A Bochner subordinator is a convolution semigroup $\beta = (\beta_t)_{t>0}$ of subprobability measures on $\mathbb{R}$ such that, for each $t > 0$, we have $\beta_t \neq \varepsilon_0$ and $supp\,\beta_t \subset [0, \infty[$ (cf. [3] Chap. 14).

According to ([3] Chap. 14), the associated potential $\kappa := \int_0^\infty \beta_s\,ds$ is a Borel measure on $[0, \infty[$. Moreover, let $\delta := \lim_{t\to 0} \frac{1}{t}(\beta_t - \varepsilon_0)$ be the associated Levy generator. Then (cf. Example 1.2.(ii))

$$\kappa * \delta = -\varepsilon_0 \tag{3.1}$$

Let $\mathbb{P}$ be an m-symmetric semigroup on E and let β be a Bochner subordinator. For every $t > 0$ and for every $f \in L^2(m)$, we may define

$$P_t^\beta f := \int_0^\infty P_s f \beta_t(ds) := \lim_{r\to\infty} \int_0^r P_t^\beta f\,dt = \int_0^\infty P_s f\,d\kappa(s) \tag{3.2}$$

According to [15] for example, $\mathbb{P}^\beta := (P_t^\beta)_{t>0}$ is a strongly continuous m-symmetric semigroup on E. It is said to be *subordinated* to $\mathbb{P}$ in the sense of Bochner by means of β (cf. for example [4],V,3 or [15] 5.3).

We index by "β" all entities associated to $\mathbb{P}^\beta$: In particular A^β is the associated generator, V^β is the potential kernel of $\mathbb{P}^\beta$, $\mathcal{P}^\beta$ is its set of potentials and $\mathbf{a}_\beta$ (resp. $\mathbf{e}_\beta$) the associated energy form (resp. norm). Note that, for $f \in D(V^\beta)$ we have

$$V^\beta f := \int_0^\infty P_t^\beta f\,dt = \int_0^\infty P_s f\,d\kappa(s)\cdot \tag{3.3}$$

3.2. Remarks

1. The most important example in Analysis is the so called *one-sided stable* subordinator of order $\alpha \in]0, 1[$, i.e. the unique convolution semigroup $\eta := (\eta_t)_{t>0}$ on $[0, \infty[$ such that for each $t > 0$, the Laplace Transform $\mathrm{L}(\eta_t)$ of η_t is given by $\mathrm{L}(\eta_t)(x) = \exp(-tx^\alpha)$, $x > 0$. In this case, we have $\kappa(dt) = 1_{]0,\infty[}(t)\Gamma(\alpha)t^{\alpha-1} \cdot dt$ (cf. [3] Chap. 14 for example). If A is the generator of an m-symmetric semigroup $\mathbb{P}$ and if A^η is the generator of $\mathbb{P}^\eta$ the subordinated of $\mathbb{P}$ by means of η then $A^\eta = -(-A)^\alpha$ the fractional power of order α of A (cf. [16]). In this case a $\mathbb{P}^\eta$-potential is called an α-potential (specifically a Riesz potential of order α if $\mathbb{P}$ is the Brownian semigroup on $\mathbb{R}^d$). Moreover the energy form will be denoted by $\mathbf{e}_\alpha$.

2. If $h \in \mathcal{P}_f^\beta$ is a $\mathbb{P}^\beta$-potential then by Theorem 2.5

$$\mathbf{e}_\beta^2(h) = 2\int_0^\infty \|\psi_t\|_2^2\,dt$$

where $\psi := (\psi_t)_{t>0}$ is the associated $\mathbb{P}^\beta$-exit law.

In the following, we want to express $\mathbf{e}_\beta(h)$ in terms of "the initial values", namely in terms of the $\mathbb{P}$-exit laws and the subordinator β.

3. For a $\mathbb{P}^\beta$-potential of the form $h := V^\beta f$ for some $f \in D_+(V^\beta)$, it can be seen that

$$\mathbf{e}_\beta^2(V^\beta f) = \int_0^\infty \|P_{s/2} f\|_2^2\,\kappa(ds)\cdot \tag{3.4}$$

The energy formula (3.4) will be generalized for every $\mathbb{P}^{\beta}$-potential with finite energy.

3.3. THEOREM

Let $\mathbb{P}$ be an m-symmetric semigroup, let β be a subordinator and let $\mathbb{P}^{\beta}$ be the subordinated of $\mathbb{P}$ by means of β. For each $\mathbb{P}^{\beta}$-potential $h \in \mathcal{P}_f^{\beta}$, there exists a unique (up to equivalence) $\mathbb{P}$-exit law φ such that

$$P_t h = V^{\beta}\varphi_t, \; m\text{.a.e.} \qquad (t > 0) \tag{3.5}$$

PROOF. Let $h \in \mathcal{P}_f^{\beta}$ and $t > 0$. Since $P_t(L^2(m)) \subset D(A)$ ([6], p. 292) and $D(A) \subset D(A^{\beta})$ ([15], p. 269) then $P_t h \in D(A^{\beta})$. We show now that $P_t h$ is a $\mathbb{P}^{\beta}$-potential. Indeed, by (2.4), it suffises to prove that $A^{\beta}(P_t h) \leq 0$. If $h \in D(A)$ then, by (2.4), $A^{\beta}h \leq 0$ since h is a $\mathbb{P}^{\beta}$-potential. Therefore, $A^{\beta}(P_t h) = P_t(A^{\beta}h) \leq 0$ since $\mathbb{P}$ is a positive operator. For the general case, it suffises to use the density of $D(A)$ in $\mathcal{D}$ and the continuity of the energy form $\mathbf{a}$.

In other words, we have proved that $\varphi_t := -A^{\beta}(P_t h)$ is well defined and $\varphi_t \in L^2_+(m)$. Moreover, since the Levy generator δ is a bounded measure on $[t, \infty[$ (cf. [10], Proposition 2) and $h \in L^2_+(m)$, we have

$$\varphi_t := -A^{\beta}(P_t h) = -\int_0^{\infty} P_{r+t}h \, d\delta(r) \tag{3.6}$$

by the well definition of A^{β}.

Now by the Theorem of Fubini and the semigroup property, we have m-a.e.

$$P_s\varphi_t := -\int_0^{\infty} P_s P_{r+t}h \, d\delta(r) = -\int_0^{\infty} P_{r+s+t}h \, d\delta(r) = \varphi_{s+t}$$

for all $s, t > 0$. Hence $\varphi := (\varphi_t)_{t>0}$ verifies (2.8).

Since κ is a Borel measure and δ is bounded on $[t, \infty[$, we may apply the Theorem of Fubini. Thus, by (2.8), (3.6) and (3.1), we have m-a.e.

$$\begin{aligned} V^{\beta}\varphi_t &= \textstyle\int_0^{\infty} P_s\varphi_t \, d\kappa(s) \\ &= \textstyle\int_0^{\infty} \varphi_{s+t} \, d\kappa(s) \\ &= -\textstyle\int_0^{\infty}\int_0^{\infty} P_{r+s+t}h \, d\delta(r)d\kappa(s) \\ &= -\textstyle\int_0^{\infty} P_{l+t}h \, d(\delta * \kappa)(l) \\ &= P_t h\cdot \end{aligned}$$

Finally, φ is a $\mathbb{P}$-exit law and (3.5) is verified. □

3.4. LEMMA

Let $\mathbb{P}$ be an m-symmetric semigroup, let β be a subordinator and $\mathbb{P}^{\beta}$ be the subordinated of $\mathbb{P}$ by means of β. For each $\mathbb{P}^{\beta}$-potential $h \in \mathcal{P}_f^{\beta}$

$$\lim_{t \to 0} \mathbf{e}_{\beta}(P_t h) = \mathbf{e}_{\beta}(h); \tag{3.7}$$

PROOF. Let $h \in \mathcal{P}_f^\beta$ and let $t > 0$, note first that $P_t h \in \mathcal{P}_f^\beta$ by Theorem 3.3. By the density of $D(A^\beta)$ in $\mathcal{D}^\beta$ and the continuity of $\mathbf{e}_\beta$, it suffises to prove (3.7) for $h \in D(A^\beta)$. In this case, we have $A^\beta P_t h = P_t A^\beta h$. Then by (2.2) and (2.4) applied to A^β instead of A, we have

$$\begin{aligned}\mathbf{e}_\beta^2(P_t h) - \mathbf{e}_\beta^2(h) &= < A^\beta h, h > - < A^\beta P_t h, P_t h > \\ &= < A^\beta h, h - P_t h > + < A^\beta h - A^\beta P_t h, P_t h > \\ &= < A^\beta h, h - P_t h > + < A^\beta h - P_t A^\beta h, P_t h >\end{aligned}$$

Then

$$|\mathbf{e}_\beta^2(h) - \mathbf{e}_\beta^2(P_t h)| \le \|A^\beta h\|_2 \|h - P_t h\|_2 + \|h\|_2 \|A^\beta h - P_t A^\beta h\|_2 \tag{3.8}$$

since $\|P_t h\|_2 \le \|h\|_2$ by the contraction property. The relation (3.7) is also a consequence of (3.8) and the strong continuity of $\mathbb{P}$. □

3.5. THEOREM

Let $\mathbb{P}$ be an m-symmetric semigroup, β be a subordinator with potential measure κ and let $\mathbb{P}^\beta$ be the subordinated of $\mathbb{P}$ by means of β. For each $\mathbb{P}^\beta$-potential $h \in \mathcal{P}^\beta$

$$\mathbf{e}_\beta^2(h) = \int_0^\infty \|\varphi_{s/2}\|_2^2 \,\kappa(ds) \tag{3.9}$$

where φ is the associated $\mathbb{P}$-exit law by Theorem 3.3.

PROOF: Let h be in $\mathcal{P}_f^\beta$ and let φ be the associated $\mathbb{P}$-exit law by Theorem 3.3 and let $t > 0$, then by (2.2), (3.5) and (1.3) we have

$$\mathbf{e}_\beta^2(P_t h) =< -A^\beta P_t h, P_t h >=< -A^\beta V^\beta \varphi_t, V^\beta \varphi_t >=< \varphi_t, V^\beta \varphi_t >$$

where V^β is the potential operator of $\mathbb{P}^\beta$. Moreover by (3.3) we obtain

$$< \varphi_t, V^\beta \varphi_t >=< \varphi_t, \int_0^\infty P_s \varphi_t \, d\kappa(s) >= \int_0^\infty < \varphi_t, P_s \varphi_t > \, d\kappa(s)$$

Finally by (2.8) we have

$$< \varphi_t, V^\beta \varphi_t >= \int_0^\infty \|\varphi_{t+(r/2)}\|_2^2 \, d\kappa(r). \qquad (t > 0)$$

Therefore

$$\mathbf{e}_\beta^2(P_t h) = \int_0^\infty \|\varphi_{t+(r/2)}\|_2^2 \, d\kappa(r). \qquad (t > 0) \tag{3.10}$$

Using Lemma 3.4, we obtain

$$\mathbf{e}_\beta^2(h) = \lim_{t \downarrow 0} \int_0^\infty \|\varphi_{t+(r/2)}\|_2^2 \, d\kappa(r). \tag{3.11}$$

On the other hand, since φ is a $\mathbb{P}$-exit law, it can be easily seen that $\|\varphi_{t+(r/2)}\|_2 \uparrow \|\varphi_{(r/2)}\|_2$ as $t \downarrow 0$. The energy formula (3.9) is a consequence of (3.11) and the monoton class Theorem.

3.6. REMARKS

1. Similar results are obtained in [1] by specific methods, if $\mathbb{P}$ is associated to a right process.

2. Let C^β be the β-capacity, i.e. the capacity defined by $\mathbb{P}^\beta$. For each β-capacitable subset F of E, let $h_F^\beta := \inf\{h \in \mathcal{P}_f^\beta : h = 1 \text{ on } F\}$ be the associated β-potentiel. By the proof of 3.3 and 3.5, we have

$$C^\beta(F) = \int_0^\infty \|A^\beta P_{s/2}\, h_F^\beta\|_2^2\, \kappa(ds)$$

3. If we take $\beta_t := \varepsilon_t$, then $\mathbb{P}^\beta = \mathbb{P}$ and $\kappa(dt) = dt$ the Lebesgue measure on $[0,\infty[$. Hence Theorem 2.5 is a particular case of Theorems 3.3 and 3.5. In particular, the energy formula (3.9) generalizes (2.10).

4. If κ is absolutely continuous with respect to the Lebesgue measure on $[0,\infty[$, i.e. $\kappa(dt) = k(t).dt$ for some Borel function $k : [0,\infty[\to [0,\infty[$, then (3.9) becomes

$$\mathbf{e}_\beta^2(h) = 2\int_0^\infty \|\varphi_s\|_2^2\, k(2s)\, ds \tag{3.12}$$

As application of (3.12), we obtain the following energy formula for the abstract Riesz potentials.

3.7. COROLLARY

Let $\mathbb{P}$ be an m-symmetric semigroup. Then, for each $\alpha \in]0,1[$ and for each α-potential h we have

$$\mathbf{e}_\alpha^2(h) = 2^\alpha\Gamma(\alpha)\int_0^\infty \|\varphi_s\|_2^2\; s^{\alpha-1}\, ds$$

where φ is the associated $\mathbb{P}$-exit law.

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