BOUNDEDNESS FOR MULTILINEAR MARCINKIEWICZ INTEGRAL OPERATORS ON HARDY AND HERZ–HARDY SPACES

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Abstract. The purpose of this paper is to establish the boundedness for some multilinear operators generated by Marcinkiewicz integral operators and Lipschitz functions on Hardy and Herz–Hardy spaces.

1. Introduction and Results

In this paper, we will consider a class of multilinear operators related to Marcinkiewicz integral operators, whose definitions are the following.

Let m be a positive integer and A be a function on R^n. Set

$$R_{m+1}(A;x,y) = A(x) - \sum_{|\alpha|\le m} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\alpha$$

and

$$Q_{m+1}(A;x,y) = R_m(A;x,y) - \sum_{|\alpha|=m} \frac{1}{\alpha!} D^\alpha A(x)(x-y)^\alpha.$$

Fix $\delta > 0$ and $0 < \gamma \le 1$. Suppose that S^{n-1} is the unit sphere of $R^n (n \ge 2)$ equipped with normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let Ω be homogeneous of degree zero and satisfy the following two conditions:

(i) $\Omega(x)$ is continuous on S^{n-1} and satisfies the Lip_γ condition on S^{n-1}, i.e.

$$|\Omega(x') - \Omega(y')| \le M|x'-y'|^\gamma, \quad x', y' \in S^{n-1};$$

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(ii) $\int_{S^{n-1}} \Omega(x')dx' = 0$.

We denote $\Gamma(x) = \{(y,t) \in R_+^{n+1} : |x-y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. The multilinear Marcinkiewicz integral operator is defined by

$$\mu_\Omega^A(f)(x) = \left[\int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t^3}\right]^{1/2},$$

where

$$F_t^A(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} \frac{R_{m+1}(A;x,y)}{|x-y|^m} f(y)dy.$$

The variant of μ_Ω^A is defined by

$$\tilde{\mu}_\Omega^A(f)(x) = \left[\int_0^\infty |\tilde{F}_t^A(f)(x)|^2 \frac{dt}{t^3}\right]^{1/2},$$

where

$$\tilde{F}_t^A(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} \frac{Q_{m+1}(A;x,y)}{|x-y|^m} f(y)dy.$$

We write

$$F_t(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-\delta-1}} f(y)dy.$$

We also define that

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

which are the Marcinkiewicz integral operator (see [16]).

Note that when $m = 0$ and $\delta = 0$, μ_Ω^A is just the commutator of Marcinkiewicz integral operators (see [9–11], [16]), while when $m > 0$, it is non-trivial generalizations of the commutators. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors when A has derivatives of order m in $BMO(R^n)$(see [2–5]). In [1], author obtain the boundedness of multilinear singular integral operators generated by singular integrals and Lipschitz functions on L^p $(p > 1)$ and some Hardy spaces. The main purpose of this paper is to discuss the boundedness properties of the multilinear Marcinkiewicz integral operators on Hardy and Herz–Hardy spaces. Let us first introduce some definitions (see [6], [7], [12–14]). Throughout this paper, $M(f)$ will denote the Hardy–Littlewood maximal function of f, Q will denote a cube of R^n with side parallel to the axes. Denote the Hardy spaces by $H^p(R^n)$. It is well known that $H^p(R^n)$ $(0 < p \le 1)$ has the atomic decomposition characterization(see[6]). The Lipschitz space $Lip_\beta(R^n)$ is the space of functions f such that

$$||f||_{Lip_\beta} = \sup_{\substack{x,h\in R^n \\ h\ne 0}} |f(x+h)-f(x)|/|h|^\beta < \infty,$$

where $\beta > 0$ (see [15]).

Let $B_k = \{x \in R^n : |x| \le 2^k\}, C_k = B_k \setminus B_{k-1}, k \in Z$.

DEFINITION 1. Let $0 < p,\ q < \infty$, $\alpha \in R$.

(1) The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha,p}(R^n) = \{f \in L_{loc}^q(R^n \setminus \{0\}) : ||f||_{\dot{K}_q^{\alpha,p}(R^n)} < \infty\},$$

where

$$||f||_{\dot{K}_q^{\alpha,p}} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}||f\chi_k||_{L^q}^p\right]^{1/p};$$

(2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,p}(R^n) = \{f \in L_{loc}^q(R^n) : ||f||_{K_q^{\alpha,p}(R^n)} < \infty\},$$

where

$$||f||_{K_q^{\alpha,p}} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p}||f\chi_k||_{L^q}^p + ||f\chi_{B_0}||_{L^q}^p\right]^{1/p}.$$

DEFINITION 2. Let $\alpha \in R$, $0 < p, q < \infty$.

(1) The homogeneous Herz type Hardy space is defined by

$$H\dot{K}_q^{\alpha,p}(R^n) = \{f \in S'(R^n) : G(f) \in \dot{K}_q^{\alpha,p}(R^n)\},$$

and

$$||f||_{H\dot{K}_q^{\alpha,p}} = ||G(f)||_{\dot{K}_q^{\alpha,p}};$$

(2) The nonhomogeneous Herz type Hardy space is defined by

$$HK_q^{\alpha,p}(R^n) = \{f \in S'(R^n) : G(f) \in K_q^{\alpha,p}(R^n)\},$$

and

$$||f||_{HK_q^{\alpha,p}} = ||G(f)||_{K_q^{\alpha,p}};$$

where $G(f)$ is the grand maximal function of f.

The Herz type Hardy spaces have the atomic decomposition characterization.

DEFINITION 3. Let $\alpha \in R$, $1 < q < \infty$. A function $a(x)$ on R^n is called a central (α, q)-atom (or a central (a, q)-atom of restrict type), if

1) Supp$a \subset B(0, r)$ for some $r > 0$ (or for some $r \geq 1$),
2) $||a||_{L^q} \leq |B(0,r)|^{-\alpha/n}$,
3) $\int_{R^n} a(x)x^\eta dx = 0$ for $|\eta| \leq [\alpha - n(1 - 1/q)]$.

LEMMA 1 (see [14]). *Let $0 < p < \infty$, $1 < q < \infty$ and $\alpha \geq n(1 - 1/q)$. A temperate distribution f belongs to $H\dot{K}_q^{\alpha,p}(R^n)$ (or $HK_q^{\alpha,p}(R^n)$) if and only if there exist central (α, q)-atoms (or central (α, q)-atoms of restrict type)a_j supported on $B_j = B(0, 2^j)$ and constants λ_j, $\sum_j |\lambda_j|^p < \infty$ such that $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ (or $f = \sum_{j=0}^{\infty} \lambda_j a_j$)in the $S'(R^n)$ sense, and*

$$||f||_{H\dot{K}_q^{\alpha,p}}(or\ ||f||_{HK_q^{\alpha,p}}) \sim \left(\sum_j |\lambda_j|^p\right)^{1/p}.$$

Now we can state our results as following.

THEOREM 1. *Let* $0 < \beta \leq 1$, $0 \leq \delta < n - \beta$, $\max(n/(n+\beta), n/(n+\gamma), n/(n+1/2)) < p \leq 1$ *and* $1/p - 1/q = (\delta+\beta)/n$. *If* $D^\alpha A \in Lip_\beta(R^n)$ *for* $|\alpha| = m$. *Then* μ_Ω^A *is bounded from* $H^p(R^n)$ *to* $L^q(R^n)$.

THEOREM 2. *Let* $0 < \beta < \min(1/2, \gamma)$, $0 \leq \delta < n - \beta$. *If* $D^\alpha A \in Lip_\beta(R^n)$ *for* $|\alpha| = m$. *Then* $\tilde{\mu}_\Omega^A$ *is bounded from* $H^{n/(n+\beta)}(R^n)$ *to* $L^{n/(n-\delta)}(R^n)$.

THEOREM 3. *Let* $0 < \beta < \min(1/2, \gamma)$, $0 < \delta < n - \beta$. *If* $D^\alpha A \in Lip_\beta(R^n)$ *for* $|\alpha| = m$. *Then* μ_Ω^A *is bounded from* $H^{n/(n+\beta)}(R^n)$ *to weak* $L^{n/(n-\delta)}(R^n)$.

THEOREM 4. *Let* $0 < \beta \leq 1$, $0 < \delta < n - \beta$, $0 < p < \infty$, $1 < q_1, q_2 < \infty$, $1/q_1 - 1/q_2 = (\delta+\beta)/n$ *and* $n(1-1/q_1) \leq \alpha < \min(n(1-1/q_1)+\beta, n(1-1/q_1)+\gamma, n(1-1/q_1)+1/2)$. *If* $D^\alpha A \in Lip_\beta(R^n)$ *for* $|\alpha| = m$. *Then* μ_Ω^A *is bounded from* $H\dot{K}_{q_1}^{\alpha,p}(R^n)$ *to* $\dot{K}_{q_2}^{\alpha,p}(R^n)$.

REMARK. Theorem 4 also hold for the nonhomogeneous Herz type Hardy space.

2. Some Lemmas

We begin with some preliminary lemmas.

LEMMA 2. *(see [4]). Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(A;x,y)| \leq C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}(x,y)$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

LEMMA 3. *Let* $0 < \beta \leq 1$, $1 < p < n/(\delta+\beta)$, $1/q = 1/p - (\delta+\beta)/n$ *and* $D^\alpha A \in Lip_\beta(R^n)$ *for* $|\alpha| = m$. *Then* μ_Ω^A *is bounded from* $L^p(R^n)$ *to* $L^q(R^n)$.

PROOF. By Minkowski inequality, we have

$$\begin{aligned}\mu_\Omega^A(f)(x) &\leq \int_{R^n} \frac{|\Omega(x-y)||R_{m+1}(A;x,y)|}{|x-y|^{m+n-1-\delta}} |f(y)| \left(\int_{|x-y|}^\infty \frac{dt}{t^3} \right)^{1/2} dy \\ &\leq C \int_{R^n} \frac{|R_{m+1}(A;x,y)|}{|x-y|^{m+n-\delta}} |f(y)| dy.\end{aligned}$$

□

Thus, the lemma follows from [1].

3. Proofs of Theorems

PROOF OF THEOREM 1. It suffices to show that there exists a constant $C > 0$ such that for every H^p-atom a,

$$||\mu_\Omega^A(a)||_{L^q} \le C.$$

Let a be a H^p-atom, that is that a supported on a cube $Q = Q(x_0, r)$, $||a||_{L^\infty} \le |Q|^{-1/p}$ and $\int a(x)x^\eta dx = 0$ for $|\eta| \le [n(1/p-1)]$. We write

$$\int_{R^n} [\mu_\Omega^A(a)(x)]^q dx = \left(\int_{|x-x_0|\le 2r} + \int_{|x-x_0|>2r}\right) [\mu_\Omega^A(a)(x)]^q dx = I + II.$$

For I, taking $1 < p_1 < n/(\delta+\beta)$ and q_1 such that $1/p_1 - 1/q_1 = (\delta+\beta)/n$, by Holder's inequality and the (L^{p_1}, L^{q_1})-boundedness of μ_Ω^A (see Lemma 3), we see that

$$I \le C||\mu_\Omega^A(a)||_{L^{q_1}}^q |2Q|^{1-q/q_1} \le C||a||_{L^{p_1}}^q |Q|^{1-q/q_1} \le C.$$

To obtain the estimate of II, we need to estimate $\mu_\Omega^A(a)(x)$ for $x \in (2Q)^c$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)_{\tilde{Q}} x^\alpha$. Then $R_m(A; x, y) = R_m(\tilde{A}; x, y)$ and $D^\alpha \tilde{A}(y) = D^\alpha A(y) - (D^\alpha A)_Q$. we have, by the vanishing moment of a,

$$\begin{aligned}
&|F_t^A(a)(x)| \\
&\le \int_{R^n} \left|\frac{\Omega(x-y)|}{|x-y|^{n+m-1-\delta}} - \frac{\Omega(x-x_0)}{|x-x_0|^{n+m-1-\delta}}\right| \chi_{\Gamma(x)}(y,t)|R_m(\tilde{A}; x, y)||a(y)|dy \\
&+ \int_{R^n} \frac{\chi_{\Gamma(x)}(y,t)|\Omega(x-x_0)|}{|x-x_0|^{n+m-1-\delta}} |R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x, x_0)||a(y)|dy \\
&+ \left|\int_{R^n} (\chi_{\Gamma(x)}(y,t) - \chi_{\Gamma(x)}(x_0,t)) \frac{\Omega(x-x_0)R_m(\tilde{A}; x, x_0)}{|x-x_0|^{n+m-1-\delta}} a(y)dy\right| \\
&+ \sum_{|\alpha|=m} \frac{1}{\alpha!} \left|\int_{\Gamma(x)} \frac{\Omega(x-y)(x-y)^\alpha D^\alpha A(y)}{|x-y|^{n+m-1-\delta}} a(y)dy\right| \\
&= II_1 + II_2 + II_3 + II_4.
\end{aligned}$$

For II_1, by Lemma 2 and the following inequality, for $b \in Lip_\beta(R^n)$,

$$|b(x) - b_Q| \le \frac{1}{|Q|}\int_Q ||b||_{Lip_\beta}|x-y|^\beta dy \le ||b||_{Lip_\beta}(|x-x_0| + r)^\beta,$$

we get

$$|R_m(\tilde{A}; x, y)| \le \sum_{|\alpha|=m} ||D^\alpha A||_{Lip_\beta}(|x-y| + r)^{m+\beta},$$

on the other hand, by the following inequality (see [16]):

$$\left|\frac{\Omega(x-y)}{|x-y|^{n+m-1-\delta}} - \frac{\Omega(x-x_0)}{|x-x_0|^{n+m-1-\delta}}\right| \le \left(\frac{r}{|x-x_0|^{n+m-\delta}} + \frac{r^\gamma}{|x-x_0|^{n+m+\gamma-1-\delta}}\right)$$

and note that $|x-y| \sim |x-x_0|$ for $y \in Q$ and $x \in R^n \setminus Q$, we obtain, similar to the proof of Lemma 3,

$$\left(\int_0^\infty |II_1|^2 dt/t^3\right)^{1/2}$$

$$\leq C\int_{R^n}\left(\frac{r}{|x-x_0|^{n+m+1-\delta}}+\frac{r^\gamma}{|x-x_0|^{n+m+\gamma-\delta}}\right)\sum_{|\alpha|=m}||D^\alpha A||_{Lip_\beta}|x-x_0|^{m+\beta}|a(y)|dy$$

$$\leq C\sum_{|\alpha|=m}||D^\alpha A||_{Lip\beta}\left(\frac{|Q|^{\beta/n+1-1/p}}{|x-x_0|^{n-\delta}}+\frac{|Q|^{\gamma/n+1-1/p}}{|x-x_0|^{n+\gamma-\delta-\beta}}\right);$$

For II_2, by the following equality (see [4]):

$$R_m(\tilde{A};x,y)-R_m(\tilde{A};x,x_0)=\sum_{|\eta|<m}\frac{1}{\eta!}R_{m-|\eta|}(D^\eta\tilde{A};x_0,y)(x-x_0)^\eta$$

we obtain

$$\left(\int_0^\infty |II_2|^2 dt/t^3\right)^{1/2} \leq C\sum_{|\alpha|=m}||D^\alpha A||_{Lip_\beta}\int\left(\sum_{|\eta|<m}\frac{|y-x_0|^{m+\beta-|\eta|}}{|x-x_0|^{n+m-|\eta|-\delta}}\right)|a(y)|dy$$

$$\leq C\sum_{|\alpha|=m}||D^\alpha A||_{Lip_\beta}\frac{|Q|^{\beta/n+1-1/p}}{|x-x_0|^{n-\delta}};$$

ForII_3, we have

$$\left(\int_0^\infty |II_3|^2 dt/t^3\right)^{1/2}$$

$$\leq C\int_{R^n}\frac{|R_m(\tilde{A};x,x_0)||a(y)|}{|x-x_0|^{n+m-1-\delta}}\left|\int_{R^n}\chi_{\Gamma(x)}(y,t)dt/t^3-\int\chi_{\Gamma(x)}(x_0,t)dt/t^3\right|^{1/2}dy$$

$$\leq C\int_{R^n}\frac{|R_m(\tilde{A};x,x_0)||a(y)||x_0-y|^{1/2}}{|x-x_0|^{n+m+1/2-\delta}}dy$$

$$\leq C\sum_{|\alpha|=m}||D^\alpha A||_{Lip_\beta}\frac{|Q|^{1+1/(2n)-1/p}}{|x-x_0|^{n+1/2-\delta-\beta}};$$

Similarly,

$$\left(\int_0^\infty |II_4|^2 dt/t^3\right)^{1/2}$$

$$\leq C\sum_{|\alpha|=m}||D^\alpha A||_{Lip\beta}\left(\frac{|Q|^{\beta/n+1-1/p}}{|x-x_0|^{n-\delta}}+\frac{|Q|^{1+\gamma/n-1/p}}{|x-x_0|^{n+\gamma-\delta-\beta}}+\frac{|Q|^{1+1/(2n)-1/p}}{|x-x_0|^{n+1/2-\delta-\beta}}\right).$$

Thus

$$II \le \sum_{k=1}^{\infty} \int_{2^{k+1}Q\setminus 2^kQ} [\mu_\Omega^A(a)(x)]^q dx$$
$$\le C\left(\sum_{|\alpha|=m} ||D^\alpha A||_{Lip_\beta}\right)^q \sum_{k=1}^{\infty}\Big[2^{kqn(1/p-(n+\beta)/n)} + 2^{kqn(1/p-(n+\gamma)/n)}$$
$$+ 2^{kqn(1/p-(n+1/2)/n)}\Big]$$
$$\le C\left(\sum_{|\alpha|=m} ||D^\alpha A||_{Lip_\beta}\right)^q,$$

which together with the estimate for I yields the desired result. This finishes the proof of Theorem 1. □

PROOF OF THEOREM 2. It suffices to show that there exists a constant $C>0$ such that for every $H^{n/(n+\beta)}$-atom a supported on $Q = Q(x_0, r)$, we have

$$||\tilde{\mu}_\Omega^A(a)||_{L^{n/(n-\delta)}} \le C.$$

We write

$$\int_{R^n} [\tilde{\mu}_\Omega^A(a)(x)]^{n/(n-\delta)} dx = \left[\int_{|x-x_0|\le 2r} + \int_{|x-x_0|>2r}\right] [\tilde{\mu}_\Omega^A(a)(x)]^{n/(n-\delta)} dx := J + JJ.$$

For J, by the following equality

$$Q_{m+1}(A;x,y) = R_{m+1}(A;x,y) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(x-y)^\alpha (D^\alpha A(x) - D^\alpha A(y)),$$

we have, similar to the proof of Lemma 3,

$$\tilde{\mu}_\Omega^A(a)(x) \le \mu_\Omega^A(a)(x) + C\sum_{|\alpha|=m} \int_{R^n} \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x-y|^{n-\delta}} |a(y)| dy,$$

thus, $\tilde{\mu}_\Omega^A$ is (L^p, L^q)-bounded by Lemma 3 and [8], where $1 < p < n/(\delta+\beta)$ and $1/q = 1/p - (\delta+\beta)/n$. We see that

$$J \le C||\tilde{\mu}_\Omega^A(a)||_{L^q}^{n/(n-\delta)} |2Q|^{1-n/((n-\delta)q)}$$
$$\le C||a||_{L^p}^{n/(n-\delta)} |Q|^{1-n/((n-\delta)q)}$$
$$\le C.$$

To obtain the estimate of JJ, we denote that $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)_{2Q} x^\alpha$. Then $Q_m(A;x,y) = Q_m(\tilde{A};x,y)$. We write, by the vanishing moment of a and $Q_{m+1}(A;x,y) = R_m(A;x,y) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(x-y)^\alpha D^\alpha A(x)$, for $x \in (2Q)^c$,

$$
\begin{aligned}
&\tilde{F}_t^A(a)(x)\\
&= \int_{\Gamma(x)} \frac{\Omega(x-y)R_m(\tilde{A};x,y)}{|x-y|^{n+m-1-\delta}} a(y)dy - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{\Gamma(x)} \frac{\Omega(x-y)D^\alpha \tilde{A}(x)(x-y)^\alpha}{|x-y|^{n+m-1-\delta}} a(y)dy\\
&= \int_{R^n} \left[\frac{\chi_{\Gamma(x)}(y,t)\Omega(x-y)R_m(\tilde{A};x,y)}{|x-y|^{n+m-1-\delta}} - \frac{\chi_{\Gamma(x)}(x_0,t)\Omega(x-x_0)R_m(\tilde{A};x,x_0)}{|x-x_0|^{n+m-1-\delta}} \right] a(y)dy\\
&\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \left[\frac{\chi_{\Gamma(x)}(y,t)\Omega(x-y)(x-y)^\alpha}{|x-y|^{n+m-1-\delta}} \right.\\
&\quad \left. - \frac{\chi_{\Gamma(x)}(x_0,t)\Omega(x-x_0)(x-x_0)^\alpha}{|x-x_0|^m} \right] D^\alpha \tilde{A}(x) a(y)dy,
\end{aligned}
$$

thus, similar to the proof of Theorem 1, we obtain, for $x \in (2Q)^c$

$$
\begin{aligned}
|\tilde{\mu}_\Omega^A(a)(x)| \le C|Q|^{-\beta/n} \sum_{|\alpha|=m} \Bigg[||D^\alpha A||_{Lip_\beta} \left(\frac{|Q|^{1/n}}{|x-x_0|^{n+1-\delta-\beta}} + \frac{|Q|^{1/(2n)}}{|x-x_0|^{n+1/2-\delta-\beta}} \right.\\
\left. + \frac{|Q|^{\gamma/n}}{|x-x_0|^{n+\gamma-\delta-\beta}} \right) + |D^\alpha \tilde{A}(x)| \left(\frac{|Q|^{1/n}}{|x-x_0|^{n+1-\delta}} \right.\\
\left. + \frac{|Q|^{1/(2n)}}{|x-x_0|^{n+1/2-\delta-\beta}} + \frac{|Q|^{\gamma/n}}{|x-x_0|^{n+\gamma-\delta}} \right) \Bigg],
\end{aligned}
$$

so that,

$$
\begin{aligned}
JJ \le C \left(\sum_{|\alpha|=m} ||D^\alpha A||_{Lip_\beta} \right)^{n/(n-\delta)} \sum_{k=1}^\infty \Big[2^{kn(\beta-1)/(n-\delta)} + 2^{kn(\beta-1/2)/(n-\delta)}\\
+ 2^{kn(\beta-\gamma)/(n-\delta)} \Big] \le C,
\end{aligned}
$$

which together with the estimate for J yields the desired result. This finishes the proof of Theorem 2. □

PROOF OF THEOREM 3. By the following equality

$$
R_{m+1}(A;x,y) = Q_{m+1}(A;x,y) + \sum_{|\alpha|=m} \frac{1}{\alpha!}(x-y)^\alpha (D^\alpha A(x) - D^\alpha A(y))
$$

and similar to the proof of Lemma 3, we get

$$
\mu_\Omega^A(f)(x) \le \tilde{\mu}_\Omega^A(f)(x) + C \sum_{|\alpha|=m} \int_{R^n} \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x-y|^{n-\delta}} |f(y)|dy,
$$

from Theorem 1 and 2 with [8], we obtain

$$\begin{aligned}
&|\{x \in R^n : \mu_\Omega^A(f)(x) > \lambda\}| \\
&\qquad \le |\{x \in R^n : \tilde{\mu}_\Omega^A(f)(x) > \lambda/2\}| \\
&\qquad + \left|\left\{x \in R^n : \sum_{|\alpha|=m} \int \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x-y|^{n-\delta}} |f(y)| dy > C\lambda \right\}\right| \\
&\qquad \le C(\lambda^{-1} ||f||_{H^{n/(n+\beta)}})^{n/(n-\delta)}.
\end{aligned}$$

This completes the proof of Theorem 3. □

PROOF OF THEOREM 4. Let $f \in H\dot{K}_{q_1}^{\alpha,p}(R^n)$ and $f(x) = \sum_{j=-\infty}^{\infty} \lambda_j a_j(x)$ be the atomic decomposition for f as in Lemma 1. We write

$$\begin{aligned}
||\mu_\Omega^A(f)||_{\dot{K}_q^{\alpha,p}}^p &\le \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-3} |\lambda_j| ||\mu_\Omega^A(a_j)\chi_k||_{L^{q_2}} \right)^p \\
&\quad + \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-2}^{\infty} |\lambda_j| ||\mu_\Omega^A(a_j)\chi_k||_{L^{q_2}} \right)^p \\
&= L_1 + L_2.
\end{aligned}$$

For L_2, by the (L^{q_1}, L^{q_2}) boundedness of μ_Ω^A(see Lemma 3), we have

$$\begin{aligned}
L_2 &\le C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-2}^{\infty} |\lambda_j| ||a_j||_{L^{q_1}} \right)^p \\
&\le \begin{cases} C\sum_{j=-\infty}^{\infty} |\lambda_j|^p \left(\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p}\right), & 0 < p \le 1 \\ C\sum_{j=-\infty}^{\infty} |\lambda_j|^p \left(\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p/2}\right) \left(\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p'/2}\right)^{p/p'}, & p > 1 \end{cases} \\
&\le C \sum_{j=-\infty}^{\infty} |\lambda_j|^p \\
&\le C||f||_{H\dot{K}_{q_1}^{\alpha,p}}^p.
\end{aligned}$$

For L_1, similar to the proof of Theorem 1, we have, for $x \in C_k$, $j \le k-3$,

$$\begin{aligned}
\mu_\Omega^A(a_j)(x) &\le C \left(\frac{|B_j|^{\beta/n}}{|x|^{n-\delta}} + \frac{|B_j|^{1/(2n)}}{|x|^{n+1/2-\delta-\beta}} + \frac{|B_j|^{\gamma/n}}{|x|^{n+\gamma-\delta-\beta}} \right) \int_{R^n} |a_j(y)| dy \\
&\le C \Big(2^{j(\beta+n(1-1/q_1)-\alpha)} |x|^{\delta-n} + 2^{j(1/2+n(1-1/q_1)-\alpha)} |x|^{\delta+\beta-n-1/2} \\
&\quad + 2^{j(\gamma+n(1-1/q_1)-\alpha)} |x|^{\delta+\beta-n-\gamma} \Big),
\end{aligned}$$

thus

$$||\mu_\Omega^A(a_j)\chi_k||_{L^{q_2}} \le C2^{-k\alpha}(2^{(j-k)(\beta+n(1-1/q_1)-\alpha)} + 2^{(j-k)(1/2+n(1-1/q_1)-\alpha)} + 2^{(k-j)(\gamma+n(1-1/q_1)-\alpha)}),$$

and

$$\begin{aligned} L_1 &\le C\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|(2^{(j-k)(\beta+n(1-1/q_1)-\alpha)} + 2^{(j-k)(1/2+n(1-1/q_1)-\alpha)} \right. \\ &\left. +2^{(j-k)(\gamma+n(1-1/q_1)-\alpha)}\right)^p \\ &\le \begin{cases} C\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+3}^{\infty}\big(2^{(j-k)(\beta+n(1-1/q_1)-\alpha)} \\ \quad +2^{(j-k)(1/2+n(1-1/q_1)-\alpha)} + 2^{(j-k)(\gamma+n(1-1/q_1)-\alpha)}\big)^p, & 0<p\le 1 \\ C\sum_{j=-\infty}^{\infty}|\lambda_j|^p\Big[\sum_{k=j+3}^{\infty}\big(2^{(j-k)p(\beta+n(1-1/q_1)-\alpha)/2} \\ \quad +2^{(j-k)p(1/2+n(1-1/q_1)-\alpha)/2} + 2^{(j-k)(\gamma+n(1-1/q_1)-\alpha)/2}\big)\Big], & p>1 \end{cases} \\ &\le C\sum_{j=-\infty}^{\infty}|\lambda_j|^p \\ &\le C||f||^p_{H\dot{K}^{\alpha,p}_{q_1}}. \end{aligned}$$

This finishes the proof of Theorem 4. □

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