MIXED STABILITY OF THE D'ALEMBERT FUNCTIONAL EQUATION

MACIEJ J. PRZYBYŁA

Abstract. In the present paper we will prove the theorem concerning the mixed stability of the d'Alembert functional equation, i.e. we will show that if $\varepsilon > 0$, $s \geq 1$, $\delta = [2^s + \sqrt{2^{2s} + 16\varepsilon + 8}]/4$, X is a real normed space and $f: X \to \mathbb{C}$ satisfies the inequality

$$|f(x+y) + f(x-y) - 2f(x)f(y)| \leq \varepsilon(\|x\|^s + \|y\|^s)$$

for all $x, y \in X$, then $|f(x)| \leq \delta\|x\|^s$ for all $x \in X$ such that $\|x\| \geq 1$, or $f(x+y) + f(x-y) = 2f(x)f(y)$ for all $x, y \in X$.

1. Introduction

In the paper [2] (see also [1]) P. Găvruţă has given an answer to a problem posed by Th. M. Rassias and J. Tabor concerning mixed stability of mappings. He has proved the following theorem

THEOREM 1. *Let* $\varepsilon > 0$, $s > 0$ *and* $\delta = [2^s + \sqrt{2^{2s} + 8\varepsilon}]/2$. *Let* B *be a normed algebra with multiplicative norm and* X *be a real normed space. If* $f: X \to B$ *satisfies the inequality*

$$|f(x+y) - f(x)f(y)| \leq \varepsilon(\|x\|^s + \|y\|^s)$$

for all $x, y \in X$, *then*

$$|f(x)| \leq \delta\|x\|^s \quad \textit{for all } x \in X \textit{ such that } \|x\| \geq 1,$$

or

$$f(x+y) = f(x)f(y)$$

Received: December 19, 2003. Revised: January 18, 2005.
(2000) Mathematics Subject Classification: 39B82.

for all $x, y \in X$.

In the present paper we will show the analogous theorem for the d'Alembert functional equation

$$f(x+y)+f(x-y)=2f(x)f(y).$$

2. Preliminaries

LEMMA 1. *Let* $\varepsilon > 0$, $s > 0$ *and let* X *be a real normed space. If* $f\colon X \to \mathbb{C}$ *satisfies the inequality*

$$|f(x+y)+f(x-y)-2f(x)f(y)| \le \varepsilon(\|x\|^s + \|y\|^s)$$

for all $x, y \in X$, *then either* $f(0)=0$ *or* $f(0)=1$.

PROOF. From the inequality for $x=y=0$ we get

$$f(0)[1-f(0)]=0.$$

Thus either $f(0)=0$ or $f(0)=1$. □

DEFINITION 1. Let G be an abelian group. Let us denote

$$A(f)(x,y)=f(x+y)+f(x-y)-2f(x)f(y)$$

for all $f\colon G \to \mathbb{C}$ and $x, y \in G$.

LEMMA 2. *Let* G *be an abelian group. Then for all* $x, u, v \in G$ *we have*

$$\begin{aligned} 2f(x)[A(f)(u,v)] &= A(f)(x+u,v)-A(f)(x,u+v)-A(f)(x,u-v) \\ &\quad +A(f)(x-u,v)+2f(v)A(f)(x,u). \end{aligned} \tag{1}$$

PROOF. Direct calculation. □

3. Mixed stability of the d'Alembert equation

THEOREM 2. *Let* $\varepsilon > 0$, $s \ge 1$ *and* $\delta = [2^s + \sqrt{2^{2s}+16\varepsilon+8}]/4$. *Let* X *be a real normed space. If* $f\colon X \to \mathbb{C}$ *satisfies the inequality*

$$|f(x+y)+f(x-y)-2f(x)f(y)| \le \varepsilon(\|x\|^s + \|y\|^s) \tag{2}$$

for all $x, y \in X$, *then*

$$|f(x)| \leq \delta\|x\|^s \quad \textit{for all } x \in X \textit{ such that } \|x\| \geq 1,$$

or

$$f(x+y) + f(x-y) = 2f(x)f(y)$$

for all $x, y \in X$.

REMARK 1. The method of the proof is similar to the method of the proof of P. Găvruţă from [2] and changes only in a few places.

PROOF. Let us assume that there exists $x_0 \in X$, $\|x_0\| \geq 1$ such that $|f(x_0)| > \delta\|x_0\|^s$. Hence there exists $\alpha > 0$ such that

$$|f(x_0)| > (\delta + \alpha)\|x_0\|^s.$$

From the inequality (2) we obtain

$$|f(2x_0) + f(0) - 2f^2(x_0)| \leq 2\varepsilon\|x_0\|^s.$$

By Lemma 1 we have $|f(0)| \leq 1$. Moreover, we get

$$\begin{aligned} |2f^2(x_0) - [f(2x_0) + f(0)]| &\geq |2f^2(x_0)| - |f(2x_0) + f(0)| \\ &\geq |2f^2(x_0)| - |f(2x_0)| - 1 \end{aligned}$$

and consequently

$$\begin{aligned} |f(2x_0)| &\geq |2f^2(x_0)| - |2f^2(x_0) - [f(2x_0) + f(0)]| - 1 \\ &> 2(\delta + \alpha)^2\|x_0\|^{2s} - 2\varepsilon\|x_0\|^s - 1 \\ &> [2(\delta + \alpha)^2 - 2\varepsilon - 1]\|x_0\|^s. \end{aligned}$$

From the definition of δ it follows that

$$2\delta^2 = 2^s\delta + 2\varepsilon + 1 \quad \text{and} \quad 2\delta > 2^s.$$

Thus we obtain

$$|f(2x_0)| > (\delta + 2\alpha)2^s\|x_0\|^s.$$

By mathematical induction we will show that for all $n \in \mathbb{N}$ we have

$$|f(2^n x_0)| > (\delta + 2^n\alpha)\|2^n x_0\|^s. \tag{3}$$

From the inequality (2) it follows that

$$|f(2^{n+1}x_0) + f(0) - 2f^2(2^n x_0)| \leq 2\varepsilon\|2^n x_0\|^s,$$

and from the properties of absolute value we deduce that

$$|f(2^{n+1}x_0)+f(0)-2f^2(2^nx_0)| \geq |2f^2(2^nx_0)| - |f(2^{n+1}x_0)| - 1$$

On account of previous inequalities and the inductive assumption we get

$$\begin{aligned} |f(2^{n+1}x_0)| &> |2f^2(2^nx_0)| - |f(2^{n+1}x_0)+f(0)-2f^2(2^nx_0)| - 1 \\ &> 2(\delta+2^n\alpha)^2\|2^nx_0\|^{2s} - 2\varepsilon\|2^nx_0\|^s - \|2^nx_0\|^s \\ &> [2(\delta+2^n\alpha)^2 - 2\varepsilon - 1]\|2^nx_0\|^s. \end{aligned}$$

And finally from the definition of δ

$$|f(2^{n+1}x_0)| > 2^s(\delta+2^{n+1}\alpha)\|2^nx_0\|^s,$$

which by the induction principle proves the inequality (3).
Let us denote $x_n = 2^nx_0$, then $\|x_n\| \geq 1$ for all $n \in \mathbb{N}$ and in view of the inequality (3) we get

$$\frac{1}{(\delta+2^n\alpha)} > \frac{\|x_n\|^s}{|f(x_n)|} > 0.$$

From the theorem of three sequences it follows that

$$\lim_{n\to\infty} \frac{\|x_n\|^s}{|f(x_n)|} = 0. \tag{4}$$

By Lemma 2 we have

$$\begin{aligned} 2f(x_n)[A(f)(u,v)] = {} & A(f)(x_n+u,v) - A(f)(x_n,u+v) - A(f)(x_n,u-v) \\ & + A(f)(x_n-u,v) + 2f(v)A(f)(x_n,u). \end{aligned}$$

Let us assume that $0^0 = 1$. Thus on account of the inequality (2) for all $u, v \in X$ we have

$$\begin{aligned} |2f(x_n)[A(f)(u,v)]| \leq {} & \varepsilon[\|x_n+u\|^s + \|v\|^s + \|x_n\|^s + \|u+v\|^s \\ & + \|x_n\|^s + \|u-v\|^s + \|x_n-u\|^s + \|v\|^s \\ & + |2f(v)|(\|x_n\|^s + \|u\|^s)] \\ \leq {} & 2\varepsilon[(\|x_n\| + \|u\|)^{\lceil s\rceil} + \|v\|^s + \|x_n\|^s + (\|u\| + \|v\|)^s \\ & + |f(v)|(\|x_n\|^s + \|u\|^s)] \\ \leq {} & 2\varepsilon\Bigg[\sum_{k=0}^{\lceil s\rceil}\binom{\lceil s\rceil}{k}\|x_n\|^{\lceil s\rceil-k}\|u\|^k + \|v\|^s + \|x_n\|^s \\ & + (\|u\| + \|v\|)^s + |f(v)|(\|x_n\|^s + \|u\|^s)\Bigg]. \end{aligned}$$

Because of the equality (4) it leads to

$$|A(f)(u,v)| \le 2\varepsilon \lim_{n\to\infty}\left(\frac{\sum_{k=0}^{\lceil s\rceil}\binom{\lceil s\rceil}{k}\|x_n\|^{\lceil s\rceil-k}\|u\|^k + \|x_n\|^s + |f(v)|\|x_n\|^s}{|f(x_n)|}\right.$$
$$\left. + \frac{\|v\|^s + (\|u\|+\|v\|)^s + |f(v)| + \|u\|^s}{|f(x_n)|}\right) = 0.$$

Thus we get that for all $u, v \in X$

$$A(f)(u,v) = f(u+v) + f(u-v) - 2f(u)f(v) = 0,$$

which completes the proof of the theorem. □

References

[1] Czerwik S.,: *Functional Equations and Inequalities in Several Variables*, World Scientific, New Jersey–London–Singapore–Hong Kong 2002.

[2] Găvrută P., *An answer to a question of Th. M. Rassias and J. Tabor on mixed stability of mappings*, Bul. Stiintific al Univ. Politehnica din Timisoara, **42** (1997), 1–6.

The paper was written when the author was a student in:

INSTITUTE OF MATHEMATICS
SILESIAN UNIVERSITY OF TECHNOLOGY
UL. KASZUBSKA 23
44-100 GLIWICE
POLAND

Currently the author is a Ph. D. student in:

SYSTEMS RESEARCH INSTITUTE
POLISH ACADEMY OF SCIENCES
UL. NEWELSKA 6
01-447 WARSZAWA
POLAND
e-mail: maciej_przybyla@bielsko.home.pl
www:http://maciek.przybyla.w.interia.pl