ON EQUILIBRIUM THEOREM

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Abstract. We prove that for given sets $A_0, ..., A_n \subset R^n$ such as $D_i \subset A_i$ for each $i = 0, ..., n$ exists a point $x \in D$ such as $d(x, A_0) = ... = d(x, A_n)$. This proof gives an algorithm of finding the point x.

The aim of this paper is to present the combinatorial and topological method of finding a point satisfying the thesis of theorem formulated above. It will be shown that using this method one can prove Sperner's Lemma and Equilibrium Theorem [3]. In [2] Kulpa used this lemma as a tool to generalise Equilibrium Theorem, therefore this theorem is a collolary of our theorem, too. Moreover, algorithm that allows to find the point mentioned in Sandwich Theorem [1] and Kuratowski–Steinhaus Theorem is obtained [4].

A subset $T \subset [0,1]^{n+1}$:

$$T:= \{t = (t_0, ..., t_n): \sum_{i=0}^{n} t_i = 1\}$$

is said to be the standard $n-$dimensional simplex.

Let $\{p_0, ..., p_n\}$ be an affinely independent set of $n+1$ points in R^n. Their convex hull:

$$\{x \in R^n: x = \sum_{i=0}^{n} t_i \cdot p_i, \ \sum_{i=0}^{n} t_i = 1\}$$

is called (n-dimensional) n-simplex with vertices $p_0, ..., p_n$ and is denoted $[p_0, ..., p_n]$.

The m-simplex spanned by any m+1 vertices $p_{i_o}, ..., p_{i_m}$ is called m-face of $[p_0, ..., p_n]$.

By $vert[p_0, ..., p_n]$ we call the set $\{p_0, ..., p_n\}$.

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Let $D: = [d_0, ..., d_n]$ be n-dimensional n-simplex.

A finite family $T(\epsilon)$ of simplexes contained in D is called a ϵ-triangulation of D if:

(i) the intersection of any two simplexes in $T(\epsilon)$ is empty or a common face of each.
(ii) if $\sigma \in T(\epsilon)$ then every face of σ is in $T(\epsilon)$.
(iii) $D = \bigcup\{\sigma: \sigma \in T(\epsilon)\}$.
(iv) for each $\sigma \in T(\epsilon)$ $diam[\sigma] < \epsilon$.

Every map $\phi: D \to \{0, ..., n\}$ is said to be the map colouring D and the set $A \subset D$ we call k-coloured if $\phi(A) = \{0, ..., k\}$.

Denote by $D_i: = [d_0, ..., d_{i-1}, d_{i+1}, ..., d_n]$ the i-th $(n-1)$-dimensional face of simplex D:

$$D_i: = \{x \in D: t_i = 0\}.$$

LEMMA 1. *For an arbitrary ϵ-triangulation of D and a map $\phi: D \to \{0, ..., n\}$ which for $i = 0, ..., n$ satisfies the condition:*

$$\phi(D_i) \neq i.$$

The number $\alpha(D)$ of all n-simplexes contained in D, for which the set of its vertices is n-coloured, is odd.

PROOF. We proceed the proof with the induction on n.

In case $n = 0$ it is clear that the lemma is true because $\phi(D) = \{0\}$.

Assuming that the lemma holds for an $(n-1)$-dimensional $(n-1)$-simplex the condition $\phi(D_i) \neq i$ implies that only $(n-1)$-coloured face of n-simplex D is D_n. Considering D_n to be $(n-1)$-simplex, by our inductive hypothesis the number $\alpha(D_n)$ of $(n-1)$-simplexes in $T(\epsilon)$ lying on D_n, whose vertices are $(n-1)$-coloured, is odd.

Let $\alpha(\sigma)$ denotes the number of $(n-1)$-faces of $\sigma \in T(\epsilon)$, of which set of vertices is $(n-1)$-coloured.

If the set of vertices of σ is n-coloured then $\alpha(\sigma) = 1$.

If the set of vertices of σ is $(n-1)$-coloured then $\alpha(\sigma) = 2$. Otherwise $\alpha(\sigma) = 0$.

Hence

$$\alpha(D) = \Sigma\alpha(\sigma), mod2$$

On the other hand, an $(n-1)$-face which vertices are $(n-1)$-coloured in $\Sigma\alpha(\sigma)$ is counted exactly once or twice, according it is subset of D_n or not. We have

$$\Sigma\alpha(\sigma) = \alpha(D_n), mod2$$

hence

$$\alpha(D_n) = \alpha(D), mod2.$$

But $\alpha(D_n)$ is odd. Thus $\alpha(D)$ is odd, too. □

THEOREM 1. *Let be given sets $A_0, ..., A_n \subset R^n$ such as $D_i \subset A_i$ for each $i = 0, ..., n$. Then there exists a point $x \in D$ such as*

$$d(x, A_0) = ... = d(x, A_n).$$

PROOF. Let us define a map $f: D \to R^n$, $f = (f_1, ..., f_n)$: $f_i(x) := d(x, A_i) - d(x, A_{i-1})$ for $i = 1, ..., n$ and the sets $H_i^- := f_i^{-1}((-\infty, 0]), H_i^+ := f_i^{-1}([0, \infty))$. Define a map $\phi: D \to \{0, ..., n\}$ by

$$\phi(x) := max\{j: x \in \bigcap_{i=0}^{j} F_i^+\}$$

where $F_0^+ = D$ and $F_i^+ = H_i^+ \setminus \bigcap_{l=i}^n D_l$ for each $i = 1, ..., n$.

Let n-simplex

$$D' := [a(d_0 - d_n) + d_n, ..., a(d_{n-1} - d_n) + d_n, d_n] = [d'_0, ..., d'_{n-1}, d'_n]$$

where $a > 1$ is fixed, be an extension of D.

Denote:

$$B_0 := [d'_0, d_n] \setminus [d_0, d_n],$$

$$B_i := [d'_0, ..., d'_i, d_n] \setminus [d_0, ..., d_i, d_n] \setminus \bigcup_{j=0}^{i-1} B_j \text{ for } i = 1, ..., n-1.$$

Define the extension of the map ϕ :

$$\phi'(x) := \begin{cases} \phi(x) & \text{for } x \in D, \\ 0 & \text{for } x \in B_0, \\ \vdots & \\ n-1 & \text{for } x \in B_{n-1}. \end{cases}$$

Let us prove that for every $\epsilon > 0$ exists n-simplex $\sigma_\epsilon \subset D$ such as the set of its vertices is n-coloured.

Let us take $\epsilon > 0$ and the ϵ-triangulation of n-simplex D' denoted by $T'(\epsilon)$ hence that

(v) for all $\sigma \in T'(\epsilon)$ if $\sigma \cap D' \setminus D \neq \emptyset$ then $\sigma \cap Int[D] = \emptyset$.

The map ϕ' has following properties: $\phi'(D'_i) \neq i$ and $\phi'(d_i) = i$ for each $i = 0, ..., n$, hence the only $(n-1)$-coloured $(n-1)$-face is D'_n.

From lemma 1 the number of $(n-1)$-faces of $\sigma \in T'(\epsilon)$ which are included in D'_n hence that their vertices are $(n-1)$-coloured is odd. Moreover ϕ' is defined in the way that there exists exactly one $(n-1)$-simplex that lies in D'_n so that its vertices are $(n-1)$-coloured. Besides the point d'_0 is one of its vertices. This simplex is the $(n-1)$-face of exactly one n-simplex which is denoted by σ_0.

Define induction procedure:

If $\phi'(vert\sigma_k) = \{0, ..., n-1\}$ then take "unused" $(n-1)$-face of σ_k so that its vertices are $(n-1)$-coloured. For this face we have exactly one n-simplex different from σ_k which contains our face. Let us call it σ_{k+1}.

Otherwise we have $\phi'(vert\sigma_k) = \{0, ..., n\}$ end.

The procedure must stop because the number of simplexes is finite. The last n-simplex we call σ_ϵ.

The map ϕ' was constructed in the way that $\phi'(vert\sigma_\epsilon) = \{0, ..., n\}$ and the condition (v) implies $\sigma_\epsilon \subset D$.

Observe that the condition $x_{i-1}, x_i \in \sigma_\epsilon$, $\phi(x_{i-1}) = i-1$ and $\phi(x_i) = i$ implies $x_{i-1} \in H_i^-$, $x_i \in H_i^+$ for $i = 1, ..., n$. From Bolzano Theorem we have such $c_i \in [x_{i-1}, x_i]$ that $f(c_i) = 0$ for $i = 1, ..., n$. Without the loss of generality we can take $\epsilon = \frac{1}{n'}$. From $diam[S_\epsilon] < \epsilon$ we obtain $lim_{n' \to \infty} c_i^{n'} \to c$ and from the continuity of f_i we get $f_i(c) = 0$ for $i = 1, ..., n$. Now it is clear that

$$d(x, A_0) = ... = d(x, A_n).$$

□

COROLLARY 1 (Sperner). *Let* $\{A_0, ..., A_n\} \subset R^n$ *be closed covering of* D. *If* $D_i \subset A_i$ *for each* $i = 0, ..., n$, *then* $A_0 \cap ... \cap A_n \neq \emptyset$.

PROOF. From the theorem 1 there exists a point $x \in D$ that $d(x, A_0) = ... = d(x, A_n)$. Since $\{A_0, ..., A_n\}$ covers D, there exists index j such that $x \in A_j$, it implies $d(x, A_0) = ... = d(x, A_n) = 0$ but A_i is closed for $i = 0, ..., n$ hence we have

$$A_0 \cap ... \cap A_n \neq \emptyset.$$

□

THEOREM 2 (Equilibrium Theorem). *Let* $f\colon D \to [0, \infty)^{n+1}$, $f = (f_0, ..., f_n)$ *be a continuous map such that* $f_i(D_i) = \{0\}$ *for* $i = 0, ..., n$. *Then for each* $t \in T$, *there exists a point* $x \in D$ *such that*

$$f(x) = \mid f(x) \mid \cdot t.$$

PROOF. Let us define sets:

$$A_i := \{x \in D : f_i(x) \leq \mid f(x) \mid \cdot t_i\}.$$

Observe that the family $\{A_0, ..., A_n\}$ covers simplex D, because if not, there exists $a \in D \setminus (A_0 \cup ... \cup A_n)$ and we have $f_i(a) > \mid f(a) \mid \cdot t_i$ for each $i = 0, ..., n$, and

$$\mid f(a) \mid = \sum_{i=0}^{n} f_i(a) > \sum_{i=0}^{n} \mid f(a) \mid \cdot t_i = \mid f(a) \mid$$

contradiction.

The condition $f_i(D_i) = \{0\}$ implies $D_i \subset A_i$ for $i = 0, ..., n$. From the theorem 1 there exists $x \in D$ such that $d(x, A_0) = ... = d(x, A_n)$ but we know $x \in A_j$ hence $d(x, A_0) = ... = d(x, A_n) = 0$. Moreover from continuity of f every set A_i is a closed subset of compact space D that is why $x \in (A_0 \cap ... \cap A_n)$ and therefore:

$$\mid f(a) \mid = \sum_{i=0}^{n} f_i(a) \leq \sum_{i=0}^{n} \mid f(a) \mid \cdot t_i = \mid f(a) \mid$$

yields:

$$f(x) = \mid f(x) \mid \cdot t.$$

The proofs of corollaries formulated below the reader will find in [2], [3] .

Let $\mu(A)$ means the n-dimensional Lebesgue measure of the set $A \subset R^n$. For any point $x \in D$ let us denote

$$D_i(x) := conv\{d_0, ..., d_{i-1}, x, d_{i+1}, ..., d_n\}.$$

□

COROLLARY 2 (Sandwich Theorem). *Let $A \subset D$ be a measurable set. Then for any point $t \in T$ there exists a point $x \in D$ such that for each $i = 0, ..., n$*

$$\mu[A \cap D_i(x)] = t_i \cdot \mu(A).$$

For a given set $A \subset R^n$ and a point $x \in R^n$ let

$$A - x := \{a - x : a \in A\}$$

means a translation of the set A.

Assume $0 \in IntD$. Let for each $i = 0, ..., n$ M_i be the cone consisting of the union of all rays joining 0 to the points of D_i.

COROLLARY 3 (Kuratowski–Steinhaus Theorem). *Let $A \subset R^n$ be a bounded Lebesgue measurable set. Then for each $t \in T$ there exists $x \in R^n$ such that for each $i = 0, ..., n$*

$$\mu[(A - x) \cap M_i] = \mu(A) \cdot t_i.$$

References

[1] Borsuk K., *An application of the theorem on antipodes to the measure theory,* Bull. Acad. Polon. Sci., **1** (1953), 87–90.

[2] Kulpa W., *Convexity and The Brouwer Fixed Point Theorem,* Topology Proc., **22** (1997), 211–235.

[3] Kulpa W. *Sandwich Type Theorems,* Acta Univ. Carol. Math. Phys. **35** (2), 45–50.

[4] Kuratowski K. and Steinhaus H., *Une application géométrique du théoréme de Brouwer sur les points invariants,* Bull. Acad. Polon. Sci., **1** (1953), 83–86.

[5] Sperner E., *Neuer Beweis für die Invarianz der Dimensionzahl und des Gebieties,* Abh. Math. Sem. Ham. Univ., **6** (1928), 265–272.

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