## Report of Meeting

### The Fifth Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities February 2–5, 2005, Będlewo, Poland

The Fifth Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities was held from February 2 through February 5, 2005, at the Mathematical Research and Conference Center of Polish Academy of Sciences, Będlewo, Poland.

24 participants came from the Silesian University of Katowice (Poland) and the University of Debrecen (Hungary) at 12 from each of both cities.

Professor Roman Ger opened the Seminar and welcomed the participants to Będlewo.

The scientific talks presented at the Seminar focused on the following topics: equations in a single and several variables, iteration theory, equations on algebraic structures, difference equations, Hyers-Ulam stability, applications of functional equations. Interesting discussions were generated by the talks.

There was a very profitable Problem Session.

The social program consisted of an excursion to the city of Poznań where the participants visited the Cathedral of St. Peter and St. Paul, then they visited "Muzeum Bambrów Poznańskich" and took part in a festive dinner.

The closing address was given by Professor Zsolt Páles. His invitation to hold the Sixth Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities in February 2006 in Hungary was gratefully accepted.

Summaries of the talks in alphabetic order of the authors follow in section 1, problems and remarks in approximate chronological order in section 2, and the list of participants in the final section.

## 1. Abstracts of talks

Roman Badora: *On the Hahn-Banach theorem for amenable groups*

We generalize the classical Hahn-Banach theorem to the class of amenable groups. As a consequence we obtain that every amenable group $G$ has the following property:

(*) for every subadditive functional $p$ on $G$ there exists an additive functional $a$ on $G$ such that $a \leq p$.

Next, we prove that the Hyers theorem on stability of the Cauchy functional equation holds on groups having the property (*). We also present the version of the Hahn-Banach theorem for groups satisfying (*).

Karol Baron: *Continuity of Carathéodory's solutions of the translation equation* (Joint work with Witold Jarczyk)

Inspired by a (private) question of M.C. Zdun we are interested in finding conditions under which any Carathéodory solution $F : (0, \infty) \times X \to X$ of the translation equation

$$F(s+t, x) = F(t, F(s, x)) \tag{1}$$

is continuous without assuming the compactness of the metric space $X$. The result that we obtain concerns, in fact, functions satisfying

$$F(s+t, x) = h(s, x, g(t, f(s, x))). \tag{2}$$

In the compact case this was studied in [1]. Considering (2) allows us to treat uniformly (1) and some other equations of iteration theory. As in [1] we base our results on [2] by K.–G. Grosse–Erdmann, a procedure which (as it seems) is different from that one elaborated by A. Járai (cf. [3]). Applying the result concerning (2) to (1) we obtain the following answer to the question of M.C. Zdun which both extends his result of [4] to locally compact separable metric spaces and provide the topological counterpart.

*Let $X$ be a locally compact separable metric space and $F : (0, \infty) \times X \to X$ be a solution of* (1) *such that $F(t, \cdot)$ is continuous for every $t \in (0, \infty)$. Suppose $t_0 \in (0, \infty)$.*

(A) *If there exists a Lebesgue measurable set $M \subset (0, t_0)$ of positive Lebesgue measure such that $F(\cdot, x)|_M$ is Lebesgue measurable for every $x \in X$ then $F|_{(t_0, \infty) \times X}$ is continuous.*

(B) *If there exists a set $M \subset (0, t_0)$ of second category with the property of Baire such that $F(\cdot, x)|_M$ is Baire measurable for every $x \in X$ then $F|_{(t_0, \infty) \times X}$ is continuous.*

### References

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MIHÁLY BESSENYEI: *Characterizations of convexity via Hadamard's inequality* (Joint work with Zsolt Páles)

The classical Hermite–Hadamard inequality is not merely the consequence of convexity but, under some weak regularity assumptions, also *characterizes* it. Our goal is to verify analogous results in the case of generalized convexity induced by two dimensional Chebyshev systems.

ZOLTÁN BOROS: *Approximate Jensen-convexity of the Takagi function*

The Takagi function

$$T(x) = \sum_{n=0}^{\infty} \frac{\operatorname{dist}(2^n x, \mathbb{Z})}{2^n} \qquad (x \in \mathbb{R})$$

plays a specific role in the theory of approximately convex functions ([1], [2]). Motivated by this experience, in 2003 (during the 41$^{\mathrm{st}}$ International Symposium on Functional Equations [3]), Zsolt Páles formulated the conjecture that the inequality

$$T\left(\frac{x+y}{2}\right) \leq \frac{T(x)+T(y)}{2} + \frac{1}{2}|x-y| \tag{1}$$

holds for every $x, y \in \mathbb{R}$. We present a proof for this conjecture.

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PÁL BURAI: *Extension theorem for a functional equation*

In this presentation we prove an extension theorem for the following functional equation

$$\varphi^{-1}\left(\frac{x\varphi(x)+y\varphi(y)}{x+y}\right) + \psi^{-1}\left(\frac{x\psi(x)+y\psi(y)}{x+y}\right) = x+y,$$

where $\varphi$, $\psi$ are continuous and strictly monotone on the same interval.

Zoltán Daróczy: *Remarks and problems on functional equations*

**1.** Let $(G,+)$ be an abelian group. We consider the functional equations

$$f(x+2f(y)) = f(x) + y + f(y) \qquad (x, y \in G)$$

and

$$f(x+2f(y)) + f(y+2f(x)) = 2f(x) + 2f(y) + x + y \qquad (x, y \in G)$$

for the unknown function $f\colon G \to G$.

**2.** Let $I \subset \mathbb{R}$ be a non-void open interval and let $\mathrm{CM}(I)$ denote the class of continuous and strictly monotone functions defined on the interval $I$. A function $M : I \times I \to I$ is called a *symmetrized weighted quasi-arithmetic mean on* $I$ if there exist $0 < \alpha < 1$ and $\phi \in \mathrm{CM}(I)$ such that

$$M(x,y) = \frac{1}{2}\left(\phi^{-1}\left(\alpha\phi(x) + (1-\alpha)\phi(y)\right) + \phi^{-1}\left((1-\alpha)\phi(x) + \alpha\phi(y)\right)\right)$$

for all $x, y \in I$. In the case $\alpha = \frac{1}{2}$, $M$ is a quasi-arithmetic mean. We investigated the following problems.

(i) Which symmetrized weighted quasi-arithmetic means are quasi-arithmetic means?

(ii) What is the necessary and sufficient condition for the equality of two symmetrized weighted quasi-arithmetic means?

Włodzimierz Fechner: *On functions with the Cauchy difference bounded by a functional*

Let $(X,+)$ be an abelian group. We consider the functional inequality

$$f(x+y) - f(x) - f(y) \geq \phi(x,y), \quad x, y \in X, \tag{1}$$

where $\phi\colon X \times X \to \mathbb{R}$ and $f\colon X \to \mathbb{R}$ are unknown mappings.

It is easy to check that if $\phi\colon X \times X \to \mathbb{R}$ is biadditive and symmetric, $A\colon X \to \mathbb{R}$ is subadditive and a function $f\colon X \to \mathbb{R}$ is defined by the formula $f(x) := \frac{1}{2}\phi(x,x) - A(x)$ for $x \in X$, then (1) holds. The following two theorems provide conditions under which the converse implication is valid.

Theorem 1. *Assume that $f\colon X \to \mathbb{R}$ and $\phi\colon X \times X \to \mathbb{R}$ satisfy* (1),

$$\phi(x,-y) \geq -\phi(x,y), \quad x, y \in X, \tag{2}$$

$$\begin{cases} \limsup_{n\to+\infty} \frac{1}{4^n}\phi(2^n x, 2^n x) < +\infty, & x \in X, \\ \liminf_{n\to+\infty} \frac{1}{4^n}\phi(2^n x, 2^n y) \geq \phi(x,y), & x, y \in X \end{cases} \tag{3}$$

*and*

$$\phi(-x,-y) = \phi(x,y), \quad x, y \in X. \tag{4}$$

*Then there exists a subadditive function* $A\colon X \to \mathbb{R}$ *such that*

$$f(x) = \frac{1}{2}\phi(x,x) - A(x), \quad x \in X.$$

*Moreover,* $\phi$ *is biadditive and symmetric.*

THEOREM 2. *Assume* $X$ *to be uniquely* 2*-divisible. If* $f\colon X \to \mathbb{R}$, $\phi\colon X \times X \to \mathbb{R}$ *satisfy* (1), (2),

$$\phi(2x,2x) \le 4\phi(x,x), \quad x \in X \tag{5}$$

*and*

$$f(x) + f(-x) \ge 0, \quad x \in X, \tag{6}$$

*then there exists an additive function* $a\colon X \to \mathbb{R}$ *such that*

$$f(x) = \frac{1}{2}\phi(x,x) + a(x), \quad x \in X.$$

*Moreover,* $\phi$ *is biadditive and symmetric.*

ROMAN GER: *Ring homomorphism equation revisited*
(Joint work with Ludwig Reich)

We deal with the functional equation

$$af(xy) + bf(x)f(y) + cf(x+y) + df(x) + kf(y) = 0 \tag{$*$}$$

yielding a joint generalization of equations that has been studied under different assumptions by J. Dhombres [2] (the case $k = d$), H. Alzer, S. Ruscheweyh & L. Salinas [1], R. Ger [3], [4] and C. Hammer (the case $a = c = 1$, $b = d = k = -1$). We are looking for vanishing at 0 solutions $f$ of equation $(*)$ mapping a given unitary ring into another one possessing no zero divisors. Our main aim is to find suitable conditions under which a function $f$ satisfying $(*)$ yields a homomomorphism between the rings spoken of. On the other hand we try to keep the assumptions upon the rings in question as weak as possible.

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Attila Gilányi: *On asymptotically monomial functions*

During the 42nd International Symposium on Functional Equations (2004, Opava, Czech Republic) János Aczél posed the general problem of solving conditional functional equations. Motivated by his announcement and related to the results presented in the papers [1], [2], [3], we prove a conditional stability theorem for monomial functional equations. As a consequence of this result, we characterize monomial functions using asymptotic conditions.

References

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Zoltán Kaiser: *An example of a stable functional equation when the Hyers method does not work*
(Joint work with Zsolt Páles)

We show that the functional equation

$$g\left(\frac{x+y}{2}\right) = \sqrt[4]{g(x)g(y)}$$

is stable in the classical sense on arbitrary $\mathbb{Q}$-algebraically open convex sets, but the Hyers method does not work.

Zygfryd Kominek: *Nonstability results in the theory of convex functions*
(Joint work with Jacek Mrowiec)

We show that the inequality defining convex functions (convex in the sense of Wright) is not stable in infinitely-dimensional spaces. The inequality defining Jensen-convex functions is not stable too, even if its domain is a real open interval.

Károly Lajkó: *Remarks to a paper by S. Narumi*
(Joint work with Fruzsina Mészáros)

The functional equation

$$\Psi_1\left(\frac{x-m_1y-c_1}{\lambda_1(y+a_1)}\right) f_Y(y) = \Psi_2\left(\frac{y-m_2x-c_2}{\lambda_2(x+a_2)}\right) f_X(x) \tag{1}$$

for unknown functions $\Psi_1$, $\Psi_2$, $f_Y$, $f_X$ is connected to the characterization of joint distributions by means of conditional distributions.

Here we present the general measurable positive valued solution of (1), where $m_1, m_2, c_1, c_2, a_1, a_2 \in \mathbb{R}$; $\lambda_1, \lambda_2 \in \mathbb{R}_+$ are arbitrary constants such that

$$K_1 = m_1a_1 - c_1 - a_2 \geq 0, \quad K_2 = m_2a_2 - c_2 - a_1 \geq 0,$$

and (1) is satisfied for almost all

$$(x,y) \in D = \{(x,y) \mid y + a_1 > 0\,,\ x + a_2 > 0\}.$$

ZITA MAKÓ: *On the Lipschitz perturbation of monotonic functions*
(Joint work with Zsolt Páles)

Let $d : I^2 \to I$ be a semimetric. A real valued function $f$ defined on a real interval $I$ is called $d$-Lipschitz if it satisfies

$$|\ell(x) - \ell(y)| \le d(x,y)$$

for $x, y \in I$.

The main result states that a function $p : I \to \mathbb{R}$ can be written in the form $p = q + \ell$ where $q$ is increasing and $\ell$ is $d$-Lipschitz if and only if the inequality

$$\sum_{i=1}^{n} \Big(p(s_i) - p(t_i) - d(t_i, s_i)\Big)^+ \le \sum_{j=1}^{m} \Big(p(v_j) - p(u_j) + d(u_j, v_j)\Big)$$

is fulfilled for all real numbers $t_1 < s_1, \ldots, t_n < s_n, u_1 < v_1, \ldots, u_m < v_m$ in $I$ satisfying the condition

$$\sum_{i=1}^{n} 1_{]t_i, s_i]} = \sum_{j=1}^{m} 1_{]u_j, v_j]}.$$

GYULA MAKSA: *On a functional equation arising in probability theory*

In this talk we discuss the following problem. Find all density functions $f$ satisfying the following properties.

($a$) $f(u) = 0$ for all $u < 0$.

($b$) There exist $0 \le n \in \mathbb{Z}$ and $-1 < \beta \in \mathbb{R}$ such that the function $p$ defined on $\mathbb{R}^2$ by $p(u,v) = 0$ if $u < 0$ or $v < 0$ and

$$p(u,v) = \int_0^{+\infty} f(u)\,(F(u) - F(s+u))^n f(s+u) f(s+u+v) F(s+u+v)^\beta ds$$

for all $u \ge 0$ and $v \ge 0$, where $F(u) = \int_u^{+\infty} f \quad (u \ge 0)$ is the joint density function of some two independent random variables.

In this talk we present the solution of this problem under the following additional suppositions.

($c$) $f|_{[0,+\infty[}$ is continuous.

($d$) $f(u) > 0$ for all $u \ge 0$.

JANUSZ MATKOWSKI: *On a generalization of convex functions*

Given a real function $\alpha$ defined on a convex subset $U$ of a linear space and $t \in (0;1)$, we define: convexity, $t$-convexity, Jensen convexity, affininity, $t$-affinity and Jensen affinity of a function $f : U \to \mathbb{R}$ with respect to $\alpha$ (wrt $\alpha$). The counterparts of Berstein–Doetsch theorem, Sierpiński theorem and the “sandwich theorem” are proved. Natural generalizations of Jensen and Cauchy functional equations are considered.

Janusz Morawiec: *On Probability Distribution Solutions of a Functional Equation*
(Joint work with Ludwig Reich)

Given $0 < \beta < \alpha < 1$ and $p, q \in (0,1)$ such that $p + q = 1$ we consider the functional equation

$$\varphi(x) = p\varphi\left(\frac{x-\beta}{1-\beta}\right) + q\varphi\left(\min\left\{\frac{x}{\alpha}, \frac{x(\alpha-\beta)+\beta(1-\alpha)}{\alpha(1-\beta)}\right\}\right)$$

and its solutions $\varphi\colon \mathbb{R} \to \mathbb{R}$ in the following two classes of functions

$$\mathcal{I} = \{\varphi\colon \mathbb{R} \to \mathbb{R} \mid \varphi \text{ is increasing}, \varphi|_{(-\infty,0]} = 0, \varphi|_{[1,+\infty)} = 1\},$$

$$\mathcal{C} = \{\varphi\colon \mathbb{R} \to \mathbb{R} \mid \varphi \text{ is continuous}, \varphi|_{(-\infty,0]} = 0, \varphi|_{[1,+\infty)} = 1\}.$$

We prove that the above equation has in the class $\mathcal{C}$ at most one solution and that for some parameters $\alpha, \beta, p, q$ such a solution exists and for some parameters does not exist. We also determine all solutions of that equation in the class $\mathcal{I}$.

Ágota Orosz: *Difference equations on hypergroups I.*

The theory of linear homogeneous difference equations with nonconstant coefficients cannot offer a wide range of solutions for different practical problems. Here we present a new method depending on polynomial hypergroups based on the idea to reduce the original equation to another one with constant coefficients on a reasonable polynomial hypergroup. The main result shows that these homogeneous linear equations with constant coefficients on polynomial hypergroups can be treated in a similar way like in the classical theory on difference equations on the nonnegative integers.

Zsolt Páles: *Bernstein–Doetsch theorem revisited*

Given a nonempty open interval $I \subset \mathbb{R}$, the set of upper diagonal points of $I$ is denoted by $\Delta(I)$, i.e.,

$$\Delta(I) := \{(x,y) \in I^2 : x \le y\}.$$

We consider the general functional inequality

$$f\big(M(x,y)\big) \le \sum_{i=0}^{n} \lambda_i(x,y) f\big(M_i(x,y)\big) + \varepsilon(x,y), \quad (x,y) \in \Delta(I), \tag{1}$$

where $\lambda_0, \dots, \lambda_n : \Delta(I) \to \mathbb{R}$ and $\varepsilon : \Delta(I) \to \mathbb{R}$ are given nonnegative continuous functions and $M, M_0, \dots, M_n : \Delta(I) \to \mathbb{R}$ are given continuous means satisfying certain further conditions.

Assume that there exists a two-dimensional Chebyshev system $\omega_1, \omega_2 : I \to \mathbb{R}$ on $I$ such that, for $j = 1, 2$,

$$\omega_j\big(M(x,y)\big) = \sum_{i=0}^{n} \lambda_i(x,y)\omega_j\big(M_i(x,y)\big), \qquad (x,y) \in \Delta(I)$$

holds. Then we prove that, for all $(x,y)\in\Delta(I)$, there exists a continuous function $\varepsilon^*_{x,y}:[x,y]\to\mathbb{R}$ such that any locally upper bounded solution $f:I\to\mathbb{R}$ of the above functional inequality 1 satisfies

$$f(u)\le\frac{\begin{vmatrix}\omega_1(u)&\omega_1(y)\\ \omega_2(u)&\omega_2(y)\end{vmatrix}}{\begin{vmatrix}\omega_1(x)&\omega_1(y)\\ \omega_2(x)&\omega_2(y)\end{vmatrix}}f(x)+\frac{\begin{vmatrix}\omega_1(x)&\omega_1(u)\\ \omega_2(x)&\omega_2(u)\end{vmatrix}}{\begin{vmatrix}\omega_1(x)&\omega_1(y)\\ \omega_2(x)&\omega_2(y)\end{vmatrix}}f(y)+\varepsilon^*_{x,y}(u).$$

BARBARA PRZEBIERACZ: *Near embeddability of continuous selfmappings of a compact interval into iteration semigroup.*

We present various approaches to the concept of near-iterability. We will define, characterize and compare some classes of near-iterable functions in a sense. That will include among others:

- almost iterable functions (W. Jarczyk):
  *A continuous function $g\colon I\to I$ is called almost iterable if and only if there exists an iterable function $f\colon I\to I$ such that*

  $$\lim_{n\to\infty}(f^n(x)-g^n(x))=0,\qquad x\in I,$$

  *and the convergence is uniform on every component of the set $[a_g,b_g]\setminus Per(g,1)$.*
- quasi-iterable functions:
  *Continuous function $g\colon I\to I$ is called quasi-iterable if and only if it satisfies condition:*
  *for every $\varepsilon>0$ there exists an iterable function $f\colon I\to I$ and a natural $n_0$ such that*

  $$|f^n(x)-g^n(x)|<\varepsilon,\qquad n\ge n_0,\ x\in I.$$
- functions satisfying condition:
  *For every $\varepsilon>0$ there exists an iterable function $f\colon I\to I$ such that*

  $$|f^n(x)-g^n(x)|<\varepsilon,\qquad n\in\mathbb{N},\ x\in I.$$

MACIEJ SABLIK: *On a functional equation of Alsina*
(Joint work with Bruce Ebanks)

We deal with the functional equation

$$(A)\qquad 2F(y)-2F(x)=(y-x)\left[f\left(\frac{x+y}{2}\right)+\frac{f(x)+f(y)}{2}\right]$$

which has appeared in C. Alsina's talk at the 42nd ISFE in Opava, Czech Republic, June 2004, and then in the paper [1]. In [1] the equation (A) is solved for functions mapping $(0,\infty)$ into $\mathbb{R}$ under the assumption of differentiability of both $F$ and $f$. In the talk we present two ways of solving (A) with no regularity assumptions. The first one is based upon a lemma from [2], and yields the general solution in the class of functions mapping $\mathbb{R}$ into $\mathbb{R}$, while the second one gives a direct solution in the case where $F$ and $f$ are mapping an integral domain in which each element is uniquely divisible by 2 into itself. In any case the solutions are of the form

$$f(x) = ax + b,\ F(x) = \frac{a}{2}x^2 + bx + c$$

for some constants $a, b, c$.

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JUSTYNA SIKORSKA: *On a functional equation based on a result of Gaspard Monge* (Joint work with Claudi Alsina and Maciej Sablik)

Motivated by a geometrical result of Gaspard Monge we study the functional equation

$$\left|\frac{1}{2}(y-x)f\left(\frac{x+y}{2}\right) - \frac{1}{2}(f(y)-f(x))\frac{x+y}{2}\right| = \int_x^y f(t)dt + \frac{1}{2}xf(x) - \frac{1}{2}yf(y)$$

and we show that under some regularity assumptions only affine functions are the general solution of the above equation.

DARIUSZ SOKOŁOWSKI: *On connections between linear functional equations and real roots of their characteristic equations*

Inspired by R.O. Davies and A.J. Ostaszewski [J. Math. Anal. Appl. 247 (2000), 608–626] we investigate how the existence of solutions $\varphi$ having a constant sign in a vicinity of infinity of the equation

$$\varphi(x) = \int_S \varphi(x + M(s))\sigma(ds) \tag{1}$$

depends on real roots $\lambda$ of its characteristic equation

$$\int_S e^{\lambda M(s)}\sigma(ds) = 1. \tag{2}$$

Here $(S, \Sigma, \sigma)$ is a measure space with a finite measure $\sigma$ and $M: S \to \mathbb{R}$ is a $\Sigma$-measurable bounded function. By a *solution* of (1) we mean a Borel measurable real function $\varphi$ defined on an infinite real interval $I$, Lebesgue integrable on every finite interval contained in $I$ and such that for every $x \in I - \sup\{|M(s)| :\ s \in S\}$ the integral $\int_S \varphi(x + M(s))\sigma(ds)$ exists and (1) holds.

If $\lambda$ is a root of (2), then the function $x \mapsto e^{\lambda x}, x \in \mathbb{R}$, is a positive solution of (1). We have also the following converse statement.

THEOREM 3. *If* (1) *has a solution with a constant sign, then* (2) *has a real root.*

LÁSZLÓ SZÉKELYHIDI: *Difference equations on hypergroups II.*

Spectral analysis and spectral synthesis results on polynomial hypergroups are presented. As an application we exhibit results for linear homogeneous difference equations on polynomial hypergroups.

TOMASZ SZOSTOK: *On some functional equations connected with Hadamard inequality*
(Joint work with Barbara Koclęga–Kulpa)

It is well known that every convex function $f : \mathbb{R} \to \mathbb{R}$ satisfies the inequalities

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y f(t)dt \leq \frac{f(x)+f(y)}{2}. \tag{1}$$

It means that the expression $\frac{1}{y-x}\int_x^y f(t)dt$ lies in the interval $\left[f\left(\frac{x+y}{2}\right), \frac{f(x)+f(y)}{2}\right]$. However for the function $f(x) = x^2$ we have much more precise information. Namely it satisfies the equation

$$\frac{f(x)+f(y)}{2} - \frac{1}{y-x}\int_x^y f(t)dt = 2\left(\frac{1}{y-x}\int_x^y f(t)dt - f\left(\frac{x+y}{2}\right)\right). \tag{2}$$

Motivated by this fact we consider the generalized version of (2)

$$(y-x)\left[\frac{f(x)+f(y)}{2} + kf\left(\frac{x+y}{2}\right)\right] = (k+1)[F(y)-F(x)] \tag{3}$$

in a rather general situation i.e. for functions acting on an integral domain.

## 2. Problems and Remarks

1. PROBLEM. (A sharpening of the Hermite–Hadamard inequality)

Motivated by Professor Sikorska's talk, consider the following functional inequality which is strongly related to the Hermite–Hadamard inequality:

$$\frac{2}{y-x}\int_x^y f(t)dt \leq \frac{f(x)+f(y)}{2} + f\left(\frac{x+y}{2}\right), \qquad x, y \in I,\ x < y. \tag{1}$$

We show that this inequality holds for all convex functions defined on $I$. Indeed, applying the Hermite–Hadamard inequality on the intervals $[x, (x+y)/2]$ and $[(x+y)/2, y]$, we get that, for all $x < y$ in $I$,

$$\frac{2}{y-x}\int_x^{\frac{x+y}{2}} f(t)dt \leq \frac{f(x)+f\left(\frac{x+y}{2}\right)}{2}$$

and

$$\frac{2}{y-x}\int_{\frac{x+y}{2}}^{y} f(t)dt \leq \frac{f\left(\frac{x+y}{2}\right)+f(y)}{2}.$$

After adding up these inequalities, we get (1), which is an obvious sharpening of the Hermite–Hadamard inequality.

It is more or less known that, for a continuous function $f : I \to \mathbb{R}$, the validity of the right as well as of the left hand side inequalities in the Hermite–Hadamard inequality is equivalent to the convexity convexity of $f$. The problem is to prove or to disprove that, assuming the continuity of $f$, (1) holds for all $x < y$ in $I$ if and only if $f$ is convex over $I$.

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2. Problem. (The stability of the Hermite–Hadamard inequality)

If $f : I \to \mathbb{R}$ is $\varepsilon$-convex, i.e., if it satisfies the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon, \qquad x, y \in I,\, x < y,\, t \in [0,1],$$

By the Hyers–Ulam stability theorem of convexity, $f$ is $\varepsilon$-convex if and only if it is of the form $f = g+h$, where $g$ is convex on $I$ and $h$ is bounded on $I$ with $\|h\| \leq \varepsilon/2$.

It is very easy to show that $\varepsilon$-convex functions also satisfy the inequalites

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y f(t)dt + \varepsilon \qquad \text{and} \qquad \frac{1}{y-x}\int_x^y f(t)dt \leq \frac{f(x)+f(y)}{2} + \varepsilon$$

for all $x < y$ in $I$. The problem is about the converse of this statement, more precisely, it is an open question if the validity of either of the left hand side or of the right hand side inequality (for all $x < y$ in $I$) and the continuity of $f$ does or does not imply the $c\varepsilon$-convexity of $f$ for some constant $c > 0$ (which is independent of $f$).

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3. Problem. Find sufficient (or, rather, necessary and sufficient) conditions for the difference equation

$$f(n+2) + B_n f(n+1) + C_n f(n) = 0 \quad n = 0, 1, 2, \ldots \tag{1}$$

so that there exists a polynomial hypergroup $(K, *)$ for which (1) is equivalent to

$$f(n*1) - \lambda f(n) = 0 \quad n = 0, 1, 2, \ldots \tag{2}$$

with some complex number $\lambda$. The same problems arises concerning the more general difference equation

$$\sum_{k=0}^{2N} A_{k,n} f(n+k) = 0 \quad n = 0, 1, 2, \ldots \tag{3}$$

with $A_{N,n} \equiv 1$ where (2) is replaced by

$$Q(T)f(n) = 0 \quad n = 0, 1, 2, \ldots \tag{4}$$

with some complex polynomial $Q$ of degree $N$ and with the notation $Tf(n) = f(n * 1)$.

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4. PROBLEM. Let $I \subset \mathbb{R}$ be an open interval and let $f : I \to \mathbb{R}$ be a function. For $a \in (0, \infty)$ we define a function $\varphi_a(x) : I \cap (I - a) \to \mathbb{R}$ by the formula $\varphi_a(x) := f(x + a) - f(x)$. It was shown in [1] that if $\varphi_a$ is monotonic for every $a \in \mathbb{R}$ and $f$ is continuous, then $\varphi_a$ must be either increasing for every $a > 0$ or decreasing for every $a > 0$. A natural question arises whether an analogue result is true without the continuity assumption.

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