EXISTENCE OF POSITIVE SOLUTIONS OF SOME INTEGRAL EQUATIONS

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Abstract. We study the existence of positive solutions of the integral equation

$$x(t) = \int_0^1 k(t,s) f(s, x(s), x'(s), \dots, x^{(n-1)}(s))\, ds, \qquad n \geq 2$$

in both $C^{n-1}[0,1]$ and $W^{n-1,p}[0,1]$ spaces, where $p \geq 1$. The Krasnosielskii fixed point theorem on cone is used.

1. Introduction

In analyzing nonlinear phenomena many mathematical models give rise to problems for which only nonnegative solutions make sense. This paper deals with existence of positive solutions of the integral equations of the form

$$x(t) = \int_0^1 k(t,s) f(s, x(s), x'(s), \dots, x^{(n-1)}(s))\, ds, \qquad n \geq 2. \tag{1.1}$$

Throughout this article k is nonnegative. The literature on positive solutions is for the most part devoted to (1.1) when f is not dependent on derivatives of the function x (see [1]–[5]). Existence in this paper will be established using Krasnosielskii's fixed point theorem in a cone, which we state here for the convenience of the reader.

THEOREM 1.1. (K. Deimling [4], D. Guo [5]). *Let $E = (E, \|\cdot\|)$ be a Banach space and let $K \subset E$ be a cone in E. Assume Ω_1 and Ω_2 are bounded open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$ and let $A\colon K \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow K$ be continuous and completely continuous. In addition suppose either*

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$$\|Au\| \le \|u\| \quad \textit{for } u \in K \cap \partial\Omega_1 \qquad \textit{and} \qquad \|Au\| \ge \|u\| \quad \textit{for } u \in K \cap \partial\Omega_2$$

or

$$\|Au\| \ge \|u\| \quad \textit{for } u \in K \cap \partial\Omega_1 \qquad \textit{and} \qquad \|Au\| \le \|u\| \quad \textit{for } u \in K \cap \partial\Omega_2$$

hold. Then A *has a fixed point in* $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2. Main results

In this section we present some results for the integral equation (1.1).

THEOREM 2.1. *Suppose the following conditions are satisfied:*

(2.1) $k\colon [0,1] \times [0,1] \longrightarrow [0,\infty)$, $\frac{\partial^l k}{\partial t^l}$ $(l = 0,1,\dots,n-2)$ *exist and are continuous on* $[0,1] \times [0,1]$,

(2.2) *there exists* $\frac{\partial^{n-1} k(t,s)}{\partial t^{n-1}}$ *for all* $t \in [0,1]$ *and a.e.* $s \in [0,1]$,

(2.3) *there exist* $k^* \in C[0,1]$, $\overline{k}_i \in L^1[0,1]$ *and* $M > 0$ *such that*

(a) $k^*(t) > 0$ *for a.e.* $t \in [0,1]$,

(b) $\overline{k}_i(s) \ge 0$ *and* $\int_0^1 \overline{k}_i(s)\,ds > 0$ *for* $i = 0,1,\dots,n-1$ *and a.e.* $s \in [0,1]$,

(c) $Mk^*(t)\overline{k}_i(s) \le \left|\frac{\partial^i k}{\partial t^i}(t,s)\right| \le \overline{k}_i(s)$ *for* $i = 0,1,\dots,n-1$; $t \in [0,1]$ *and a.e.* $s \in [0,1]$,

(2.4) *the map*

$$t \longrightarrow \frac{\partial^{n-1}}{\partial t^{n-1}} k(t,s)$$

is continuous from $[0,1]$ *to* $L^1[0,1]$,

(2.5) *there exists a function* $d \in C[0,1]$ *with* $d(t) > 0$ *for a.e.* $t \in [0,1]$ *such that*

$$\begin{aligned} k(t,s) - d(t)\left[\left|\frac{\partial k}{\partial t}(t,s)\right| + \dots + \left|\frac{\partial^{n-1} k}{\partial t^{n-1}}(t,s)\right|\right] \\ \ge d(t)\left[k(t,s) + \left|\frac{\partial k}{\partial t}(t,s)\right| + \dots + \left|\frac{\partial^{n-1}}{\partial t^{n-1}} k(t,s)\right|\right] \end{aligned}$$

for all $t \in [0,1]$ *and a.e.* $s \in [0,1]$,

(2.6) $f\colon [0,1] \times [0,\infty) \times (-\infty,\infty)^{n-1} \longrightarrow [0,\infty)$ *is continuous and there exists a function* $\psi(u)$ *such that*

$$f(t, v_0, v_1, \dots, v_{n-1}) \le \psi(v_0 + |v_1| + \dots + |v_{n-1}|)$$

on $[0,1] \times [0,\infty) \times (-\infty,\infty)^{n-1}$, *where* $\psi\colon [0,\infty) \longrightarrow [0,\infty)$ *is continuous, nondecreasing and* $\psi(u) > 0$ *for* $u > 0$, (2.7) *there exists* $r > 0$ *with*

$$\frac{r}{\psi(r)} \geq \sum_{i=0}^{n-1} \sup_{t\in[0,1]} \int_0^1 \left|\frac{\partial^i k(t,s)}{\partial t^i}\right| ds,$$

(2.8) $f(t, v_0, v_1, \ldots, v_{n-1}) \geq g(v_0)$ *for*

$$(t, v_0, v_1, \ldots, v_{n-1}) \in [0,1] \times [0,\infty) \times (-\infty,\infty)^{n-1}$$

with $g\colon [0,\infty) \longrightarrow [0,\infty)$ *continuous and nondecreasing and* $g(u) > 0$ *for* $u > 0$,

(2.9) *there exist* $R > 0$ *and* $t_0 \in [0,1]$ *such that* $R > r$, $k^*(t_0) > 0$, $d(t_0) > 0$ *and*

$$R \leq \int_0^1 \left(k(t_0,s) + \left|\frac{\partial k}{\partial t}(t_0,s)\right| + \ldots + \left|\frac{\partial^{n-1}}{\partial t^{n-1}} k(t_0,s)\right|\right) d(t_0) g(RMd(s)k^*(s))\, ds.$$

Then (1.1) *has a positive solution* $x \in C^{n-1}_{[0,1]}$ *with* $x(t) > 0$ *for a.e.* $t \in [0,1]$.

PROOF. Let

$$\|u\|_{n-1} = \sup_{t\in[0,1]} \left[\,|u(t)| + |u'(t)| + \ldots + |u^{(n-1)}(t)|\,\right], \quad E = (C^{n-1}[0,1], \|\cdot\|_{n-1})$$

and

$$K = \{u \in C^{n-1}[0,1] :\ u(t) - d(t)[\,|u'(t)| + \ldots + |u^{(n-1)}(t)|\,] \geq Md(t)k^*(t)\|u\|_{n-1} \quad \text{for } t \in [0,1]\}.$$

Clearly K is a cone of E. Let

$$\Omega_1 = \{u \in C^{n-1}[0,1] :\ \|u\|_{n-1} < r\}, \qquad \Omega_2 = \{u \in C^{n-1}[0,1] :\ \|u\|_{n-1} < R\}$$

and $f^*(s, u(s)) = f(s, u(s), u'(s), \ldots, u^{(n-1)}(s))$. Now, let $A\colon K \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow C^{n-1}[0,1]$ be defined by

$$Ax(t) = \int_0^1 k(t,s) f^*(s, x(s))\, ds. \tag{2.10}$$

First we show $A\colon K \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow K$. If $x \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ and $t \in [0,1]$, then relations (2.1), (2.5) imply

$$Ax(t) - d(t)\left[\,|(Ax)'(t)| + \ldots + |(Ax)^{(n-1)}(t)|\,\right]$$
$$= \int_0^1 k(t,s) f^*(s,x(s))\, ds - d(t)\left[\left|\int_0^1 \frac{\partial k(t,s)}{\partial t} f^*(s,x(s))\, ds\right|\right.$$

$$+\dots+\left|\int_0^1 \frac{\partial^{n-1}}{\partial t^{n-1}}k(t,s)f^*(s,x(s))\,ds\right|$$

$$\geq d(t)\int_0^1 \left[k(t,s)+\left|\frac{\partial k}{\partial t}(t,s)\right|+\dots+\left|\frac{\partial^{n-1}}{\partial t^{n-1}}k(t,s)\right|\right]f^*(s,x(s))\,ds$$

and this together with (2.3) yields

$$\|Ax\|_{n-1}\geq Ax(t)-d(t)\big[\left|(Ax)'(t)\right|+\dots+\left|(Ax)^{(n-1)}(t)\right|\big]$$

$$\geq d(t)\left(\sum_{i=0}^{n-1}Mk^*(t)\int_0^1 \overline{k}_i(s)f^*(s,x(s))\,ds\right). \tag{2.11}$$

On the other hand (2.3) implies

$$\|Ax\|_{n-1}\leq\sum_{i=0}^{n-1}\int_0^1 \overline{k}_i(s)f^*(s,x(s))\,ds. \tag{2.12}$$

Taking into account (2.11)–(2.12) we conclude that

$$Ax(t)-d(t)\big[\left|(Ax)'(t)\right|+\dots+\left|(Ax)^{(n-1)}(t)\right|\big]\geq Md(t)k^*(t)\|Ax\|_{n-1}.$$

Consequently $Ax\in K$ so $A\colon K\cap(\overline{\Omega}_2\setminus\Omega_1)\longrightarrow K$. We now show

$$\|Ax\|_{n-1}\leq\|x\|_{n-1}\qquad\text{for } x\in K\cap\partial\Omega_1. \tag{2.13}$$

To see this let $x\in K\cap\partial\Omega_1$. Then $\|x\|_{n-1}=r$ and $x(t)\geq Md(t)k^*(t)r$ for $t\in[0,1]$. Also for $t\in[0,1]$ we have

$$\sum_{i=0}^{n-1}\left|(Ax)^{(i)}(t)\right|\leq\sum_{i=0}^{n-1}\int_0^1\left|\frac{\partial^i k(t,s)}{\partial t^i}\right|f^*(s,x(s))\,ds.$$

This together with (2.6)–(2.7) yields

$$\|Ax\|_{n-1}\leq\psi(\|x\|_{n-1})\left(\sum_{i=0}^{n-1}\sup_{t\in[0,1]}\int_0^1\left|\frac{\partial^i k}{\partial t^i}(t,s)\right|ds\right)\leq r=\|x\|_{n-1},$$

so (2.13) holds. Next we show

$$\|Ax\|_{n-1}\geq\|x\|_{n-1}\qquad\text{for } x\in K\cap\partial\Omega_2. \tag{2.14}$$

To see it let $x\in K\cap\partial\Omega_2$. Then we get $\|x\|_{n-1}=R$ and $x(t)\geq RMd(t)k^*(t)$ for $t\in[0,1]$. Now with t_0 as in (2.9) we have

$$\|Ax\|_{n-1} \geq Ax(t_0) - d(t_0)\big[\,|(Ax)'(t_0)| + \ldots + |(Ax)^{(n-1)}(t_0)|\,\big]$$
$$\geq d(t_0) \int_0^1 \left[k(t_0,s) + \left|\frac{\partial k}{\partial t}(t_0,s)\right| + \ldots + \left|\frac{\partial^{n-1}}{\partial t^{n-1}} k(t_0,s)\right| \right] f^*(s,x(s))\,ds$$
$$\geq d(t_0) \int_0^1 \left[k(t_0,s) + \left|\frac{\partial k}{\partial t}(t_0,s)\right| + \ldots + \left|\frac{\partial^{n-1}}{\partial t^{n-1}} k(t_0,s)\right| \right] g(x(s))\,ds.$$

This together with (2.8)–(2.9) yields

$$\|Ax\|_{n-1} \geq d(t_0) \int_0^1 g(RMd(s)k^*(s)) \left[k(t_0,s) + \left|\frac{\partial k}{\partial t}(t_0,s)\right| + \ldots + \left|\frac{\partial^{n-1}}{\partial t^{n-1}} k(t_0,s)\right| \right] ds \geq R = \|x\|_{n-1}.$$

Hence we obtain (2.14). By (2.3)–(2.4) and the Arzela–Ascoli theorem we conclude that $A\colon K \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow K$ is continuous and compact. Theorem 1.1 implies A has a fixed point $x \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, i.e. $r \leq \|x\|_{n-1} \leq R$ and $x(t) \geq Md(t)k^*(t)r$ for $t \in [0,1]$. This completes the proof of Theorem 2.1. □

REMARK 2.2. To illustrate the applicability of Theorem 2.1 we consider the following boundary value problems:

$$\begin{cases} x''(t) + f(t,x(t),x'(t)) = 0 \\ x(0) = x(1) = 0, \end{cases} \tag{2.15}$$

$$\begin{cases} x'''(t) = f(t,x(t),x'(t),x''(t)) \\ x(0) = x(1) = x'(1) = 0, \end{cases} \tag{2.16}$$

$$\begin{cases} x^{(4)}(t) = f(t,x(t),x'(t),x''(t),x'''(t)) \\ x(0) = x'(0) = x(1) = x'(1) = 0, \end{cases} \tag{2.17}$$

$$\begin{cases} x^{(4)}(t) = f(t,x(t),x'(t),x''(t),x'''(t)) \\ x(0) = x''(0) = x(1) = x''(1) = 0. \end{cases} \tag{2.18}$$

Of course the problems (2.15)–(2.18) are equivalent to the problem of determining the fixed point of the operators T_i of the form:

$$T_1(x)(t) = \int_0^1 G_1(t,s) f(s,x(s),x'(s))\,ds, \tag{2.19}$$

$$T_2(x)(t) = \int_0^1 G_2(t,s) f(s,x(s),x'(s),x''(s))\,ds, \tag{2.20}$$

$$T_3(x)(t) = \int_0^1 G_3(t,s) f(s,x(s),x'(s),x''(s),x'''(s))\,ds, \tag{2.21}$$

$$T_4(x)(t) = \int_0^1 G_4(t,s) f(s, x(s), x'(s), x''(s), x'''(s))\, ds, \tag{2.22}$$

where the Green functions G_i are defined as follows

$$G_1(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1 \\ s(1-t), & 0 \le s \le t \le 1, \end{cases} \tag{2.23}$$

$$G_2(t,s) = \begin{cases} \dfrac{t(1-s)(2s-st-t)}{2}, & 0 \le t \le s \le 1 \\ \dfrac{s^2(1-t)^2}{2}, & 0 \le s \le t \le 1, \end{cases} \tag{2.24}$$

$$G_3(t,s) = \begin{cases} \dfrac{(s-1)^2 t^2(-2st-t+3s)}{6}, & 0 \le t \le s \le 1 \\ \dfrac{s^2[(-2s+3)t^3+3t^2(s-2)+3t-s]}{6}, & 0 \le s \le t \le 1, \end{cases} \tag{2.25}$$

$$G_4(t,s) = \begin{cases} \dfrac{(s-1)[t^3+ts(s-2)]}{6}, & 0 \le t \le s \le 1 \\ \dfrac{t^3 s - 3st^2 + t(s^3+2s) - s^3}{6}, & 0 \le s \le t \le 1. \end{cases} \tag{2.26}$$

REMARK 2.3. There are many functions $k(t,s)$ that satisfy condition (2.5). It is not difficult to check that the function

$$d(t) = \begin{cases} \min\left[\dfrac{t}{t+2}, \dfrac{1-t}{3-t}\right], & \text{if } k(t,s) = G_1(t,s) \\ \min\left[\left(\dfrac{1-t}{3-t}\right)^2, \dfrac{t^2(1-t)}{2t+12}\right], & \text{if } k(t,s) = G_2(t,s) \\ \min\left[\dfrac{t^3(1-t)}{75}, \dfrac{(1-t)^3 t}{75}\right], & \text{if } k(t,s) = G_3(t,s) \\ \min\left[\dfrac{t(1-t)^2}{20}, \dfrac{t^2(1-t)}{20}\right], & \text{if } k(t,s) = G_4(t,s) \end{cases}$$

(with $t \in [0,1]$) fulfills condition (2.5). It is easy to see that the functions $G_j(t,s)$ do not satisfy (2.3)(c) for $j = 1,2,3,4$.

We will give later on a theorem on the existence of positive solutions of the problems (2.16)–(2.18).

EXAMPLE 2.4. Consider the problem

$$\begin{cases} x''(t) + (x(t) + |x'(t)|)^2 = 0 \\ x(0) = x'(0), \quad x(1) = -x'(1). \end{cases} \tag{2.27}$$

The problem (2.27) is equivalent to the problem of determining the fixed point of the operator T_5 of the form

$$T_5(x)(t) = \int_0^1 G_5(t,s)(x(s) + |x'(s)|)^2\,ds,$$

where the Green function G_5 is defined as follows

$$G_5(t,s) = \begin{cases} \dfrac{(2-t)(1+s)}{3}, & 0 \le s \le t \le 1 \\ \dfrac{(2-s)(1+t)}{3}, & 0 \le t \le s \le 1. \end{cases}$$

Fix $t_0 = \frac{1}{2}$, $d(t) = \frac{1}{4}$, $M = \frac{1}{4}$, $k^*(t) = 1$, $\overline{k}_0(s) = \overline{k}_1(s) = \frac{4}{3}$ and $\psi(u) = g(u) = u^2$ for $t \in [0,1]$ and $u \in [0,\infty)$. Clearly

$$g(RMd(s)k^*(s)) = R^2M^2d^2(s)k^{*2}(s) = \frac{1}{256}R^2$$

and

$$d\left(\frac{1}{2}\right)\int_0^1 \left(G_5\left(\frac{1}{2},s\right) + \left|\frac{\partial G_5}{\partial t}\left(\frac{1}{2},s\right)\right|\right) g(RMd(s)k^*(s))\,ds$$
$$\ge \frac{R^2}{1024}\int_0^1 \left[G_5\left(\frac{1}{2},s\right) + \left|\frac{\partial G_5}{\partial t}\left(\frac{1}{2},s\right)\right|\right] ds \ge R$$

for sufficiently large R. Next we claim (2.7) holds for $r = \frac{1}{2}$. To see this notice that

$$\sup_{t\in[0,1]} \int_0^1 G_5(t,s)\,ds + \sup_{t\in[0,1]} \int_0^1 \left|\frac{\partial G_5}{\partial t}(t,s)\right| ds = \frac{9}{8} \le \frac{r}{\psi(r)} = 2.$$

So (2.7) holds. Thus all conditions of Theorem 2.1 are satisfied and the problem (2.27) has a positive solution $x \in C^2[0,1]$ with $x(t) > 0$ for $t \in (0,1)$.

It is possible to obtain another existence results for (1.1) if we change some conditions on the nonlinearity f and some of the conditions on the kernel k.

In the sequel, we will assume that $\frac{\partial^{n-1}k}{\partial t^{n-1}}(t,s)$ is continuous with respect to (t,s) on triangles $0 \le t < s \le 1$ and $0 \le s < t \le 1$. By $\frac{\partial^{n-1}k}{\partial t^{n-1}}(s+0,s)$ ($\frac{\partial^{n-1}k}{\partial t^{n-1}}(s-0,s)$) we will denote the right–hand (the left–hand) side derivative of order $n-1$ of k at (s,s).

THEOREM 2.5. *Let conditions* (2.1), (2.6)–(2.7) *be satisfied. Moreover, we assume that*

(2.28) $\frac{\partial^{n-1}}{\partial t^{n-1}}k$ *is continuous with respect to* (t,s) *on triangles* $0 \le t < s \le 1$ *and* $0 \le s < t \le 1$,

(2.29) *there exist constants* $d_0 \ge 1$, $M_0 > 0$ *and* $m \in \left(0, \frac{1}{2}\right)$ *such that*

$$\text{(a)} \quad \begin{cases} d_0k(t,s) - \left(\left|\dfrac{\partial k}{\partial t}(t,s)\right| + \ldots + \left|\dfrac{\partial^{n-1}k}{\partial t^{n-1}}(t,s)\right|\right) \\ \qquad \geq k(s,s) + \left|\dfrac{\partial k}{\partial t}(s,s)\right| + \ldots + \left|\dfrac{\partial^{n-1}}{\partial t^{n-1}}k(s-0,s)\right| \\ \qquad\qquad \textit{for } t \in [m, 1-m],\ s \in [0,1] \textit{ and } t < s, \end{cases}$$

$$\text{(b)} \quad \begin{cases} d_0k(t,s) - \left(\left|\dfrac{\partial k}{\partial t}(t,s)\right| + \ldots + \left|\dfrac{\partial^{n-1}k}{\partial t^{n-1}}(t,s)\right|\right) \\ \qquad \geq k(s,s) + \left|\dfrac{\partial k}{\partial t}(s,s)\right| + \ldots + \left|\dfrac{\partial^{n-1}}{\partial t^{n-1}}k(s+0,s)\right| \\ \qquad\qquad \textit{for } t \in [m, 1-m],\ s \in [0,1] \textit{ and } s < t, \end{cases}$$

$$\text{(c)} \quad \begin{cases} k(s,s) + \left(\left|\dfrac{\partial k}{\partial t}(s,s)\right| + \ldots + \left|\dfrac{\partial^{n-1}k}{\partial t^{n-1}}(s-0,s)\right|\right) \\ \qquad \geq M_0\left(k(t,s) + \left|\dfrac{\partial k}{\partial t}(t,s)\right| + \ldots + \left|\dfrac{\partial^{n-1}}{\partial t^{n-1}}k(t,s)\right|\right) \\ \qquad\qquad \textit{for } t \in [0,1],\ s \in [0,1] \textit{ and } t < s, \end{cases}$$

$$\text{(d)} \quad \begin{cases} k(s,s) + \left(\left|\dfrac{\partial k}{\partial t}(s,s)\right| + \ldots + \left|\dfrac{\partial^{n-1}k}{\partial t^{n-1}}(s+0,s)\right|\right) \\ \qquad \geq M_0\left(k(t,s) + \left|\dfrac{\partial k}{\partial t}(t,s)\right| + \ldots + \left|\dfrac{\partial^{n-1}}{\partial t^{n-1}}k(t,s)\right|\right) \\ \qquad\qquad \textit{for } t \in [0,1],\ s \in [0,1] \textit{ and } s < t, \end{cases}$$

(2.30) *there exist* $\tau \in C[m, 1-m]$ *and* $g \in C[0,\infty)$ *such that* $\tau > 0$ *on* $[m, 1-m]$, $g\colon [0,\infty) \longrightarrow [0,\infty)$, g *is nondecreasing,* $g(n) > 0$ *for* $n > 0$ *and*

$$f(t, v_0, v_1, \ldots, v_{n-1}) \geq \tau(t)g(v_0) \qquad \textit{for } t \in [m, 1-m]$$

and $(v_0, v_1, \ldots, v_{n-1}) \in [0,\infty) \times (-\infty,\infty)^{n-1}$,

(2.31) *there exist numbers* $R > 0$ *and* $t_0 \in [m, 1-m]$ *with* $R > r$ *and*

$$d_0R \leq \int_m^{1-m} \left[d_0k(t_0,s) - \left(\left|\frac{\partial k}{\partial t}(t_0,s)\right| + \ldots + \left|\frac{\partial^{n-1}}{\partial t^{n-1}}k(t_0,s)\right|\right)\right] g\left(\frac{RM_0}{d_0}\right)\tau(s)\,ds.$$

Then (1.1) *has a positive solution* $x \in C^{n-1}[0,1]$ *such that* $\min_{t\in[m,1-m]} d_0x(t) \geq M_0r$.

PROOF. Let

$$K = \Big\{u \in C^{n-1}[0,1] : \ u(t) \geq 0, \ \min_{t\in[m,1-m]} \big[d_0 u(t) - \big(|u'(t)| + \ldots + |u^{(n-1)}(t)|\big) \geq M_0 \|u\|_{n-1}\Big\}.$$

Clearly K is a cone of E. Let Ω_1, Ω_2, $f^*(s,x(s))$, and A be defined as in the proof of Theorem 2.1. First we show $A: K\cap(\overline{\Omega_2}\setminus\Omega_1) \longrightarrow K$. Let $x \in K\cap(\overline{\Omega_2}\setminus\Omega_1)$, $s \in [0,1]$ and $t \in [m, 1-m]$. Then relations (2.1), (2.29) imply

$$\begin{aligned}
&d_0 Ax(t) - \big[|(Ax)'(t)| + \ldots + |(Ax)^{(n-1)}(t)|\big] \\
&\geq \int_0^1 \Big[d_0 k(t,s) - \Big(\Big|\frac{\partial k}{\partial t}(t,s)\Big| + \ldots + \Big|\frac{\partial^{n-1} k}{\partial t^{n-1}}(t,s)\Big|\Big)\Big] f^*(s,x(s))\,ds \\
&\geq \int_0^t \Big[k(s,s) + \Big|\frac{\partial k}{\partial t}(s,s)\Big| + \ldots + \Big|\frac{\partial^{n-1} k}{\partial t^{n-1}}(s+0,s)\Big|\Big] f^*(s,x(s))\,ds \\
&\quad + \int_t^1 \Big[k(s,s) + \Big|\frac{\partial k}{\partial t}(s,s)\Big| + \ldots + \Big|\frac{\partial^{n-1} k}{\partial t^{n-1}}(s-0,s)\Big|\Big] f^*(s,x(s))\,ds.
\end{aligned}$$

On the other hand, by (2.29) we get for $s, \bar{t} \in [0,1]$

$$\begin{aligned}
&\int_0^t \Big[k(s,s) + \Big|\frac{\partial k}{\partial t}(s,s)\Big| + \ldots + \Big|\frac{\partial^{n-1} k}{\partial t^{n-1}}(s+0,s)\Big|\Big] f^*(s,x(s))\,ds \\
&\quad + \int_t^1 \Big[k(s,s) + \Big|\frac{\partial k}{\partial t}(s,s)\Big| + \ldots + \Big|\frac{\partial^{n-1} k}{\partial t^{n-1}}(s-0,s)\Big|\Big] f^*(s,x(s))\,ds \\
&\geq M_0 \int_0^t \Big[k(\bar{t},s) + \Big|\frac{\partial k}{\partial t}(\bar{t},s)\Big| + \ldots + \Big|\frac{\partial^{n-1} k}{\partial t^{n-1}}(\bar{t},s)\Big|\Big] f^*(s,x(s))\,ds \\
&\quad + M_0 \int_t^1 \Big[k(\bar{t},s) + \Big|\frac{\partial k}{\partial t}(\bar{t},s)\Big| + \ldots + \Big|\frac{\partial^{n-1} k}{\partial t^{n-1}}(\bar{t},s)\Big|\Big] f^*(s,x(s))\,ds \\
&\geq M_0\big[|Ax(\bar{t})| + |(Ax)'(\bar{t})| + \ldots + |(Ax)^{(n-1)}(\bar{t})|\big].
\end{aligned}$$

Thus

$$\min_{t\in[m,1-m]} \big[d_0 Ax(t) - \big(|(Ax)'(t)| + \ldots + |(Ax)^{(n-1)}(t)|\big] \geq M_0 \|Ax\|_{n-1}$$

and in consequence $A(K) \subset K$. We now show

(2.32) $$\|Ax\|_{n-1} \leq \|x\|_{n-1} \qquad \text{for } x \in K \cap \partial\Omega_1\,.$$

To see this let $x \in K \cap \partial\Omega_1$. Then $\|x\|_{n-1} = r$ and $d_0 x(t) \geq M_0 r$ for $t \in [m, 1-m]$. Also for $t \in [0,1]$ we have

$$\sum_{i=0}^{n-1} |(Ax)^{(i)}(t)| \le \sum_{i=0}^{n-1} \int_0^1 \left|\frac{\partial^i k(t,s)}{\partial t^i}\right| f^*(s,x(s))\,ds$$
$$\le \left(\sum_{i=0}^{n-1} \sup_{t\in[0,1]} \int_0^1 \left|\frac{\partial^i k}{\partial t^i}(t,s)\right| ds\right) \psi(\|x\|_{n-1}) \le r = \|x\|_{n-1}\,.$$

Thus (2.32) holds. Next we show

$$\|Ax\|_{n-1} \ge \|x\|_{n-1} \qquad \text{for } x \in K \cap \partial\Omega_2\,. \tag{2.33}$$

To see it let $x \in K \cap \partial\Omega_2$. Then we get $\|x\|_{n-1} = R$ and $d_0 x(t) \ge M_0 R$ for $t \in [m, 1-m]$. By (2.30)–(2.31) we have

$$\begin{aligned} d_0\|Ax\|_{n-1} &\ge d_0 Ax(t_0) - \left[\,|(Ax)'(t_0)| + \ldots + |(Ax)^{(n-1)}(t_0)|\,\right] \\ &\ge \int_m^{1-m} \left[d_0 k(t_0,s) - \left(\left|\frac{\partial k}{\partial t}(t_0,s)\right| + \ldots + \left|\frac{\partial^{n-1} k}{\partial t^{n-1}}(t_0,s)\right|\right)\right] f^*(s,x(s))\,ds \\ &\ge \int_m^{1-m} \left[d_0 k(t_0,s) - \left(\left|\frac{\partial k}{\partial t}(t_0,s)\right| + \ldots + \left|\frac{\partial^{n-1} k}{\partial t^{n-1}}(t_0,s)\right|\right)\right] g\left(\frac{RM_0}{d_0}\right) \tau(s)\,ds \\ &\ge d_0 R = d_0\|x\|_{n-1}\,. \end{aligned}$$

Thus (2.33) holds. The standard arguments show that the operator A is continuous and compact. Theorem 1.1 implies A has a fixed point $x \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ i.e. $r \le \|x\|_{n-1} \le R$ and $d_0 x(t) \ge M_0 r$ for $t \in [m, 1-m]$. This completes the proof of Theorem 2.5. □

COROLLARY 2.6. *Let assumptions* (2.1), (2.6), (2.28)–(2.31) *be satisfied. Moreover, let there exists* $r > 0$ *with* $\frac{r}{\psi(r)} \ge a$, *where*

$$a = \sum_{i=0}^{n-1} \sup_{t,s\in[0,1]} \left|\frac{\partial^i k(t,s)}{\partial t^i}\right|.$$

Then (1.1) *has a positive solution* $x \in C^{n-1}[0,1]$ *with* $\min_{t\in[m,1-m]} d_0 x(t) \ge M_0 r$.

REMARK 2.7. It is not difficult to verify that the following constants

$$M_0 = \begin{cases} \frac{1}{2}, & \text{if } k(t,s) = G_1(t,s) \\ \frac{1}{6}, & \text{if } k(t,s) = G_2(t,s) \\ \frac{1}{16}, & \text{if } k(t,s) = G_3(t,s) \\ \frac{3}{16}, & \text{if } k(t,s) = G_4(t,s) \end{cases} \quad \text{and} \quad d_0 = \begin{cases} 12, & \text{if } k(t,s) = G_1(t,s) \\ 144, & \text{if } k(t,s) = G_2(t,s) \\ 20000, & \text{if } k(t,s) = G_3(t,s) \\ 1280, & \text{if } k(t,s) = G_4(t,s) \end{cases}$$

satisfy condition (2.29) with $m = \frac{1}{4}$.

REMARK 2.8. Let $i \in \{1,2,3,4\}$ and let functions G_i be defined by relations (2.15)–(2.18). It is not difficult to check that:

$$\sup_{t,s\in[0,1]} G_1(t,s) + \sup_{t,s\in[0,1]} \left|\frac{\partial G_1}{\partial t}(t,s)\right| = \frac{5}{4},$$

$$\sup_{t,s\in[0,1]} G_2(t,s) + \sup_{t,s\in[0,1]} \left|\frac{\partial G_2}{\partial t}(t,s)\right| + \sup_{t,s\in[0,1]} \left|\frac{\partial^2 G_2(t,s)}{\partial t^2}\right| = \frac{1}{4}(5\sqrt{5}-6),$$

$$\sup_{t,s\in[0,1]} G_3(t,s) + \sup_{t,s\in[0,1]} \left|\frac{\partial G_3}{\partial t}(t,s)\right| + \sup_{t,s\in[0,1]} \left|\frac{\partial^2 G_3}{\partial t^2}(t,s)\right| + \sup_{t,s\in[0,1]} \left|\frac{\partial^3 G_3}{\partial t^3}(t,s)\right|$$
$$= \frac{39\sqrt{13}+48}{162} + \frac{1}{192},$$

$$\sup_{t,s\in[0,1]} G_4(t,s) + \sup_{t,s\in[0,1]} \left|\frac{\partial G_4}{\partial t}(t,s)\right| + \sup_{t,s\in[0,1]} \left|\frac{\partial^2 G_4}{\partial t^2}(t,s)\right| + \sup_{t,s\in[0,1]} \left|\frac{\partial^3 G_4}{\partial t^3}(t,s)\right|$$
$$= \frac{61}{48} + \frac{\sqrt{3}}{27}.$$

REMARK 2.9. Consider the following boundary value problem

$$\begin{cases} x^{(4)}(t) = \frac{1}{8}t(x(t)^\alpha + |x'(t)|^\beta + |x''(t)|^\gamma + |x'''(t)|^\delta) \\ x(0) = x'(0) = x(1) = x'(1) = 0, \end{cases} \tag{2.34}$$

where $t \in [0,1]$, $\alpha > 1$ and $\alpha, \beta, \gamma, \delta > 0$.

Let n be a natural number such that $n \geq \max(\alpha, \beta, \gamma, \delta)$. Then

$$\begin{aligned} &\frac{1}{8}t(v_0^\alpha + |v_1|^\beta + |v_2|^\gamma + |v_3|^\delta) \\ &\quad \leq \frac{1}{8}(4 + v_0^n + |v_1|^n + |v_2|^n + |v_3|^n) \\ &\quad \leq \frac{1}{8}[4 + (v_0 + |v_1| + |v_2| + |v_3|)^n] \qquad \text{for } v_0 \in [0,\infty). \end{aligned}$$

We put

$$\psi(u) = \frac{1}{8}(4+u^n), \quad g(v_0) = \frac{1}{8}v_0^\alpha, \quad \tau(t) = t, \quad m = \frac{1}{4}, \quad r = 1, \quad d_0 = 20000,$$
$$t_0 = \frac{1}{2}, \quad M_0 = \frac{1}{16} \quad \text{and} \quad k(t,s) = G_3(t,s)$$

(where G_3 is defined by (2.25)). Then

$$\frac{r}{\psi(r)} = \frac{8r}{4+r^n} = \frac{8}{5} > \frac{39\sqrt{13}+48}{162} + \frac{1}{192} = a$$

and

$$\int_m^{1-m} \left[d_0 G_3(t_0, s) - \left|\frac{\partial G_3}{\partial t}(t_0, s)\right| - \left|\frac{\partial^2 G_3}{\partial t^2}(t_0, s)\right| - \left|\frac{\partial^3 G_3}{\partial t^3}(t_0, s)\right| \right] g\left(\frac{RM_0}{d_0}\right) \tau(s)\, ds$$

$$= \frac{1}{8} \int_{\frac{1}{4}}^{\frac{3}{4}} \left[20000 G_3\left(\frac{1}{2}, s\right) - \left|\frac{\partial G_3}{\partial t}\left(\frac{1}{2}, s\right)\right| - \left|\frac{\partial^2 G_3}{\partial t^2}\left(\frac{1}{2}, s\right)\right| \right.$$

$$\left. - \left|\frac{\partial^3 G_3}{\partial t^3}\left(\frac{1}{2}, s\right)\right| \right] \frac{sR^\alpha\, ds}{320000^\alpha} \geq d_0 R = 20000R$$

for sufficiently large R.

It is easy to check that the function

$$f(t, v_0, v_1, v_2, v_3) = \frac{1}{8} t(v_0^\alpha + |v_1|^\beta + |v_2|^\gamma + |v_3|^\delta)$$

fulfills all assumptions of Theorem 2.5. So the problem (2.34) has a positive solution $x \in C^4[0,1]$ with $x(t) > 0$ for $t \in \left(\frac{1}{4}, \frac{3}{4}\right)$.

Notice the function

$$\tilde{f}(t, v_0, v_1, v_2, v_3) = \frac{1}{8} t(v_0 + |v_1| + |v_2| + |v_3|)$$

has not property (2.31). To see this let x be a solution of the problem

$$\text{(2.34)}' \qquad \begin{cases} x^{(4)}(t) = \frac{1}{8} t(x(t) + |x'(t)| + |x''(t)| + |x'''(t)|) \\ x(0) = x'(0) = x(1) = x'(1) = 0, \quad t \in [0,1]. \end{cases}$$

Then

$$x(t) = \frac{1}{8} \int_0^1 G_3(t,s)(x(s) + |x'(s)| + |x''(s)| + |x'''(s)|) s\, ds.$$

Hence

$$\|x\|_3 \leq \frac{1}{8} a \|x\|_3 .$$

This together with $\frac{1}{8}a < 1$ yields $x(t) \equiv 0$ for $t \in [0,1]$ and in consequence the problem (2.34)′ has not positive solutions. So $\tilde{f}$ does not satisfy (2.31).

It is not difficult to verify that the function

$$f(t, v_0, v_1, v_2, v_3) = \frac{1}{8} t(v_0^\alpha + |v_1|^\beta + |v_2|^\gamma + |v_3|^\delta)$$

satisfies all assumptions of Theorem 2.5 with

$$k(t,s) = G_4(t,s), \quad M_0 = \frac{3}{16}, \quad d_0 = 1280, \quad m = \frac{1}{4},$$

$$g(v_0) = \frac{1}{8} v_0^\alpha, \quad \Psi(u) = \frac{1}{8}(4 + u^n), \quad r = 1; \quad \alpha, \beta, \gamma, \delta > 0 \text{ and } \alpha > 1.$$

So the problem

$$\begin{cases} x^{(4)}(t) = \frac{1}{8}t(x(t)^\alpha + |x'(t)|^\beta + |x''(t)|^\gamma + |x'''(t)|^\delta) \\ x(0) = x''(0) = x(1) = x''(1) = 0, \quad x \in [0,1] \end{cases}$$

has a positive solution $x \in C^4[0,1]$ such that $x(t) > 0$ for $t \in \left(\frac{1}{4}, \frac{3}{4}\right)$.

Proceeding analogously to problem (2.34) we can prove that the problems

$$\begin{cases} x'''(t) = \frac{1}{8}t(x(t)^\alpha + |x'(t)|^\beta + |x''(t)|^\gamma) \\ x(0) = x(1) = x'(1) = 0, \quad t \in [0,1], \ \alpha, \beta, \gamma, \delta > 0, \ \alpha > 1 \end{cases}$$

and

$$\begin{cases} x''(t) + \frac{1}{8}t(x'(t) + |x'(t)|^\beta) = 0 \\ x(0) = x(1) = 0, \quad t \in [0,1]; \ \alpha, \beta, \gamma, \delta > 0, \ \alpha > 1 \end{cases}$$

have positive solutions x such that $x(t) > 0$ for $t \in \left(\frac{1}{4}, \frac{3}{4}\right)$.

Before formulating a next theorem we will introduce some notation. For $p \geq 1$, $L^p[0,1]$ is the Banach space of functions x such that $|x|^p$ is Lebesgue integrable on $[0,1]$ with the norm

$$\|x\|_p^* = \left(\int_0^1 |x(t)|^p \, dt \right)^{\frac{1}{p}}.$$

The symbol $W^{n-1,p}[0,1]$ $(n \geq 2)$ denotes the set of all functions x with $x^{(n-2)}$ absolutely continuous and $x^{(n-1)} \in L^p[0,1]$. For $x \in W^{n-1,p}[0,1]$ we introduce the following norm

$$\|x\|_{n-1,p} = \sup_{t \in [0,1]} \left[\sum_{i=0}^{n-2} |x^{(j)}(t)| \right] + \|x^{(n-1)}\|_p^*.$$

The space $(W^{n-1,p}[0,1], \|\cdot\|_{n-1,p})$ is the Banach space. We adopt the following convention $y(t+\tau) = 0$ if $t + \tau \notin [0,1]$ and $y \in L^p[0,1]$.

A function

$$f\colon [0,1] \times [0,\infty) \times (-\infty,\infty)^{n-1} \longrightarrow [0,\infty)$$

is a Carathéodory function provided: If $f = f(t,z)$, then

(i) the map $z \longrightarrow f(t,z)$ is continuous for almost all $t \in [0,1]$,

(ii) the map $t \longrightarrow f(t,z)$ is measurable for all z in $[0,\infty) \times (-\infty,\infty)^{n-1}$.

If f is a Carathéodory function, by a solution to (1.1) we will mean a function x which has an absolutely continuous $(n-2)st$ derivative such that x satisfies the integral equation (1.1) almost everywhere in $[0,1]$.

THEOREM 2.10. *Assume that conditions* (2.1)–(2.2), (2.5) *are satisfied and* p, q *are such that* $p, q \geq 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Suppose the following conditions are satisfied:*

(2.35) *there exist* $k^* \in C[0,1]$, $\overline{k}_i \in L^p[0,1]$ *and* $M > 0$ *such that*

(a) $k^*(t) > 0$ *for a.e.* $t \in [0,1]$,

(b) $\overline{k}_i(s) \geq 0$ *and* $\int_0^1 \overline{k}_i(s)\,ds > 0$ *for* $i = 0, 1, \dots, n-1$ *and a.e.* $s \in [0,1]$,

(c) $Mk^*(t)\overline{k}_i(s) \leq \left|\frac{\partial^i k(t,s)}{\partial t^i}\right| \leq \overline{k}_i(s)$ *for* $i = 0, 1, \dots, n-1$; $t \in [0,1]$ *and a.e.* $s \in [0,1]$,

(d) *the map* $(t,s) \longrightarrow \frac{\partial^{n-1}}{\partial t^{n-1}} k(t,s)$ *is measurable,*

(2.36) $f\colon [0,1] \times [0,\infty) \times (-\infty,\infty)^{n-1} \longrightarrow [0,\infty)$ *is a Carathéodory function and there exist nonnegative functions* $p_j \in L^q[0,1]$ *and a constant* $p_n > 0$ *with*

$$f(t, v_0, v_1, \dots, v_{n-1}) \leq \sum_{i=0}^{n-2} p_i(t)|v_i| + p_{n-1}(t) + p_n |v_{n-1}|^{\frac{p}{q}}$$

for $j = 0, 1, \dots, n-1$ *and a.e.* $t \in [0,1]$,

(2.37) $f(t, v_0, v_1, \dots, v_{n-1}) \leq \Psi(v_0 + |v_1| + \dots + |v_{n-1}|)$ *for a.e.* $t \in [0,1]$ *and* $(v_0, v_1, \dots, v_{n-1}) \in [0,\infty) \times (-\infty,\infty)^{n-1}$ *with* $\Psi\colon [0,\infty) \longrightarrow [0,\infty)$ *continuous and nondecreasing and* $\Psi(u) > 0$ *for* $u > 0$,

(2.38) *there exists* $\varphi \in C[0,1]$ *with*

$$\left\|\Psi(x + |x'| + \dots + |x^{(n-1)}|)\right\|_q^* \leq \varphi(\|x\|_{n-1,p}) \qquad \text{for all } x \in W^{n-1,p}[0,1],$$

(2.39) $f(t, v_0, v_1, \dots, v_{n-1}) \geq g(v_0)$ *for a.e.* $t \in [0,1]$ *and all* $(v_0, v_1, \dots, v_{n-1}) \in [0,\infty) \times (-\infty,\infty)^{n-1}$ *with* $g\colon [0,\infty) \longrightarrow [0,\infty)$ *continuous and nondecreasing and* $g(u) > 0$ *for* $u > 0$,

(2.40) *there exists* $r > 0$ *with*

$$\frac{r}{\varphi(r)} \geq (b + \|\overline{k}_{n-1}\|_p^*),$$

where

$$b = \sum_{i=0}^{n-2} \sup_{t\in[0,1]} \left\| \frac{\partial^i k}{\partial t^i}(t,\cdot) \right\|_p^*,$$

(2.41) *there exist* $R > 0$ *and* $t_0 \in [0,1]$ *such that* $R > r$, $k^*(t_0) > 0$, $d(t_0) > 0$ *and*

$$R \leq \int_0^1 d(t_0) g(RMk^*(s)d(s)) \left[k(t_0,s) + \left|\frac{\partial k}{\partial t}(t_0,s)\right| + \dots + \left|\frac{\partial^{n-1}}{\partial t^{n-1}} k(t_0,s)\right|\right] ds.$$

Then (1.1) *has a solution* $x \in W^{n-1,p}[0,1]$ *with* $x(t) > 0$ *for a.e.* $t \in [0,1]$.

PROOF. Let $a(t) = Md(t)k^*(t)$ and let

$$K = \{u \in W^{n-1,p}[0,1]:\ u(t) - d(t)\big[|u'(t)| + \ldots + |u^{(n-1)}(t)|\big] \geq a(t)\|u\|_{n-1,p} \text{ for a.e. } t \in [0,1]\}.$$

Clearly K is a cone of $W^{n-1,p}[0,1]$. Let

$$\Omega_1 = \{u \in W^{n-1,p}[0,1]:\ \|u\|_{n-1,p} < r\},$$
$$\Omega_2 = \{u \in W^{n-1,p}[0,1]:\ \|u\|_{n-1,p} < R\}$$

and let

$$f^*(s,u(s)) = f\big(s,u(s),u'(s),\ldots,u^{(n-1)}(s)\big).$$

Let

$$A\colon K \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow W^{n-1,p}[0,1]$$

be defined by

$$Ax(t) = \int_0^1 k(t,s) f^*(s,x(s))\,ds.$$

Then

$$\big|(Ax)^{(n-1)}(t)\big| \leq \int_0^1 \overline{k}_n(s) f^*(s,x(s))\,ds \tag{2.42}$$

and

$$\begin{aligned}&\big|Ax(t) + \big|(Ax)'(t)\big| + \ldots + \big|(Ax)^{(n-2)}(t)\big|\\ &\quad\leq \sum_{i=0}^{n-2}\int_0^1 \left|\frac{\partial^i k}{\partial t^i}(t,s)\right| f^*(s,x(s))\,ds \leq \sum_{i=0}^{n-2}\int_0^1 \overline{k}_i(s) f^*(s,x(s))\,ds.\end{aligned} \tag{2.43}$$

From relations (2.42)–(2.43), (2.37)–(2.38) and Hölder's inequality it follows

$$\begin{aligned}\|Ax\|_{n-1,p} &\leq \sum_{i=0}^{n-1}\int_0^1 \overline{k}_i(s) f^*(s,x(s))\,ds \leq \sum_{i=0}^{n-1} \|\overline{k}_i\|_p^*\, \|f^*(s,x(s))\|_q^*\\ &\leq \sum_{i=0}^{n-1} \varphi(\|x\|_{n-1,p})\|\overline{k}_i\|_p^*.\end{aligned} \tag{2.44}$$

Note that A is well defined and A is a bounded operator. Now we will prove $A\colon K \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow K$. If $x \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ and $t \in [0,1]$, then (2.35), (2.5), (2.44) imply

$$
\begin{aligned}
&Ax(t) - d(t)\big[\,|(Ax)'(t)| + \ldots + |(Ax)^{(n-1)}(t)|\,\big] \\
&\quad \geq d(t)\int\limits_0^1 \left[k(t,s) + \left|\frac{\partial k}{\partial t}(t,s)\right| + \ldots + \left|\frac{\partial^{n-1}}{\partial t^{n-1}}k(t,s)\right|\right] f^*(s,x(s))\,ds \\
&\quad \geq d(t)Mk^*(t)\Big(\sum_{i=0}^{n-1}\int\limits_0^1 \overline{k}_i(s)f^*(s,x(s))\,ds\Big) \\
&\quad \geq Md(t)k^*(t)\|Ax\|_{n-1,p} \geq a(t)\|Ax\|_{n-1,p}\,.
\end{aligned}
$$

Thus $Ax \in K$ and $A\colon K \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow K$. Now we will prove that A is a continuous operator. It is enought to show that the Niemytzki operator $H\colon W^{n-1,p}[0,1] \longrightarrow L^q[0,1]$ defined by

$$Hx(t) = f\big(t, x(t), x'(t), \ldots, x^{(n-1)}(t)\big)$$

is continuous. The proof of the continuity of H is similar to the proof of Theorem 1.2 in [6]. Let $\{\overline{x}_\nu\}$ be a sequence of elements of $W^{n-1,p}[0,1]$ converging to $\overline{x}$ in $W^{n-1,p}[0,1]$. Then there exists a subsequence $\{\overline{x}_{\nu_\lambda}^{(n-1)}(t)\}$ such that

$$\lim_{\lambda\to\infty} \overline{x}_{\nu_\lambda}^{(n-1)}(t) = \overline{x}^{(n-1)}(t) \qquad \text{for a.e. } t \in [0,1].$$

Moreover, there exists a function $g \in L^p[0,1]$ satisfying the following condition

$$\big|\overline{x}_{\nu_\lambda}^{(n-1)}(t)\big| \leq g(t) \qquad \text{for a.e. } t \in [0,1]$$

([6], Lemma 2.1). Hence by (2.36) we conclude that there exists a function $h \in L^1[0,1]$ such that

$$\big|f\big(t,\overline{x}(t),\overline{x}'(t),\ldots,\overline{x}^{(n-1)}(t)\big) - f\big(t,\overline{x}_{\nu_\lambda}(t),\overline{x}'_{\nu_\lambda}(t),\ldots,\overline{x}_{\nu_\lambda}^{(n-1)}(t)\big)\big| \leq h(t)$$

for a.e. $t \in [0,1]$.

From the Lebesgue dominated convergence theorem it follows that the Niemytzki operator H is continuous at the point $\overline{x}$. We next show that A is completely continuous. Let Ω be a bounded set in $(W^{n-1,p}[0,1],\ \|\cdot\|_{n-1,p})$. Then, by virtue of (2.44) we have $A(\Omega)$ is bounded in $(W^{n-1,p}[0,1],\ \|\cdot\|_{n-1,p})$. We need to prove that $A(\Omega)$ is relatively compact. We will use the Arzela-Ascoli and the Riesz theorems. In fact, let $y_\nu \in A(\Omega)$ i.e.

$$y_\nu = A(x_\nu), \qquad x_\nu \in \Omega.$$

Since $A(\Omega)$ is bounded in $(W^{n-1,p}[0,1],\ \|\cdot\|_{n-1,p})$ there exist subsequences $\{x_{\nu_\mu}^{(j)}\}$ and $\{y_{\nu_\mu}^{(j)}\}$ of sequences $\{x_\nu^{(j)}\}$ and $\{y_\nu^{(j)}\}$ uniformly convergent to $x^{(j)}$ and $y^{(j)}$ respectively for $j = 0,1,\ldots,n-2$. Without loss of generality we can assume that the sequences $\{x_\nu^{(j)}\}$ and $\{y_\nu^{(j)}\}$ are uniformly convergent to $x^{(j)}$ and $y^{(j)}$, respectively. We will prove that there exists a subsequence $\{y_{\nu_\lambda}^{(n-1)}\}$ of the sequence $\{y_\nu^{(n-1)}\}$ such that

$$\lim_{\lambda\to\infty} \left\|y_{\nu_\lambda}^{(n-1)} - \overline{y}\right\|_p^* = 0, \qquad \text{where } \overline{y} \in L^p[0,1].$$

In fact, for fixed $\tau > 0$, we have by the Hölder inequality and the Fubini theorem that

$$\int_0^1 \left|(Ax_\nu)^{(n-1)}(t+\tau) - (Ax_\nu)^{(n-1)}(t)\right|^p dt$$

$$\leq \int_0^1 \left(\int_0^1 \left|\frac{\partial^{n-1}}{\partial t^{n-1}}k(t+\tau,s) - \frac{\partial^{n-1}}{\partial t^{n-1}}k(t,s)\right|^p ds\right) dt \int_0^1 \left(\int_0^1 |f^*(s,x_\nu(s))|^q\, ds\right)^{\frac{p}{q}} dt$$

$$\leq (\varphi(\|x_\nu\|_{n-1,p}))^p \int_0^1 \left(\int_0^1 \left|\frac{\partial^{n-1}}{\partial t^{n-1}}k(t+\tau,s) - \frac{\partial^{n-1}}{\partial t^{n-1}}k(t,s)\right|^p dt\right) ds.$$

Now using the fact that translates of L^p functions are continuous in norm we see that

$$\int_0^1 \left|(Ax)^{(n-1)}(t+\tau) - (Ax)^{(n-1)}(t)\right|^p dt \to 0$$

as $\tau \to 0$ uniformly. From the Riesz compactness criteria it follows that there exists a subsequence $\{y_{\nu_\lambda}^{(n-1)}\}$ of the sequence $\{y_\nu^{(n-1)}\}$ convergent in $L^p[0,1]$ to a function $\overline{y} \in L^p[0,1]$. It is easy to notice that $(y^{(0)})^{(n-1)}(t) = \overline{y}(t)$ for a.e. $t \in [0,1]$. So $A(\Omega)$ is relatively compact, i.e. A is completely continuous. Next we show that

$$\|Ax\|_{n-1,p} \leq \|x\|_{n-1,p} \qquad \text{for } x \in K \cap \partial\Omega_1\,. \tag{2.45}$$

Let $x \in K \cap \partial\Omega_1$, so $\|x\|_{n-1,p} = r$ and $x(t) \geq a(t)r$ for a.e. $t \in [0,1]$. The relation (2.37)–(2.40), (2.42)–(2.44) yield

$$\sum_{j=0}^{n-2} \left|(Ax)^{(j)}(t)\right| \leq b\varphi(\|x\|_{n-1,p}) \tag{2.46}$$

and

$$\sum_{j=0}^{n-2} \left|(Ax)^{(j)}(t)\right| + \left\|(Ax)^{(n-1)}\right\|_p^* \leq \varphi(\|x\|_{n-1,p})(b + \|\overline{k}_{n-1}\|_p^*) \leq r. \tag{2.47}$$

Now, taking into account the relations (2.46)–(2.47) and (2.40) we get

$$\|Ax\|_{n-1,p} \leq \|x\|_{n-1,p}\,.$$

So (2.45) holds. We finally show that

$$\|Ax\|_{n-1,p} \geq \|x\|_{n-1,p} \qquad \text{for } x \in K \cap \partial\Omega_2\,. \tag{2.48}$$

To see this let $x \in K \cap \partial\Omega_2$, so $\|x\|_{n-1,p} = R$ and $x(t) \geq a(t)R$ for a.e. $t \in [0,1]$. Thus for a.e. $t \in [0,1]$ we have

$$\|Ax\|_{n-1,p} \geq Ax(t_0) - d(t_0)\left[\,|(Ax)'(t_0)| + \ldots + |(Ax)^{(n-1)}(t_0)|\,\right]$$
$$\geq d(t_0) \int_0^1 \left[k(t_0,s) + \left|\frac{\partial k}{\partial t}(t_0,s)\right| + \ldots + \left|\frac{\partial^{n-1} k}{\partial t^{n-1}}(t_0,s)\right|\right] g(x(s))\, ds.$$

This together with (2.41) yields

$$\|Ax\|_{n-1,p} \geq R = \|x\|_{n-1,p}\,.$$

Thus (2.48) holds. Now Theorem 1.1 implies A has a fixed point $x \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ i.e. $x(t) \geq a(t)r$ for a.e. $t \in [0,1]$. This proves Theorem 2.10. □

References

[1] R.P. Agarwal, S.R. Grace, D. O'Regan, *Existence of positive solutions to semipositone Fredholm integral equation*, Funkc. Equac. **45** (2002), 223–235.

[2] R.P. Agarwal, D. O'Regan, *Infinite Interval Problems for Differential, Difference and Integral Equations*, Kluwer, Dordrecht – Boston – London 2001.

[3] R.P. Agarwal, D. O'Regan, P.J.Y. Wang, *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer, Dordrecht – Boston – London 1999.

[4] K. Deimling, *Nonlinear Functionals Analysis*, Springer, New York 1985.

[5] D. Guo, V. Laksmikannthan, *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego 1988.

[6] A. Galewski, *On a certain generalization of the Krasnosielskii theorem*, Journ. of Applied Anal. **1** (2003), 139–147.

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