ON DIVISIBILITY OF THE NUMBERS $H_n(1)$, $H_n(2)$ AND $H_n(3)$

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Abstract. We will deal with numbers given by the relation

$$H_n(k) = \frac{(k+1)^n - \binom{n}{2} k^2 - nk - 1}{k^3},$$

where k is equal to 1, 2 or 3. These numbers arise from a generalization Bernoulli's inequality. In this paper some results about divisibility and primality of the numbers $H_n(1)$, $H_n(2)$ and $H_n(3)$ are found. For example any positive integer $n > 1$ does not divide $H_n(2)$ and $n \equiv 2 \bmod 4$ is the necessary condition for divisibility $H_n(1)$ and $H_n(3)$ by $n > 2$. In addition certain properties of their divisibility are used for finding primes among these numbers.

1. Introduction

Some properties of different types of numbers arising from terms in Bernoulli's inequality $(1+x)^n \geqslant 1 + nx$ were dealt in our previous papers [1], [2] and [3].

In [1] the numbers b_n (denoted by $\mathcal{J}_n$ there) given by the relation

$$b_n = 2^n - n - 1 \,, n \in \mathbb{N}$$

were studied with respect to their divisibility and primality.

In [2] we dealt with a generalization of these numbers, concretely the numbers in the form

$$b_n(k) = \frac{(k+1)^n - nk - 1}{k^2} \,,$$

Received: 8.10.2002. Revised: 4.03.2003.

(2000) Mathematics Subject Classification: Primary 11A51, 11A07, 11Y11.

Key words and phrases: Divisibility, congruence, primality.

where k was any positive integer and n any nonnegative integer. The main results concerning divisibility of these numbers by 2 and 3 for arbitrary k were derived. Some of them were used for testing of primality of the numbers $b_n(k)$ by computer.

In paper [3] some new results were shown about divisibility of the numbers $b_n(k)$. Specially we found a congruence for the numbers $b_n(al+b)$ under $\pmod a$ (Theorem 1 in [3]). Further we proved that any positive integer $n > 2$ does not divide $b_n(2)$ and $b_n(4)$. But for arbitrary positive integer $k > 1$ there exists infinite number of integers n which divide $M_n(k) = \frac{(k+1)^n - 1}{k}$. $M_n(k)$ are a natural generalization of Mersenne numbers $2^n - 1$ for any positive integer k.

But it seems to be interesting to investigate a similar type of numbers close to the terms of the generalization of Bernoulli's inequality in the form $(1+x)^n \geqslant 1 + nx + \binom{n}{2} x^2$. In fact, these numbers $H_n(k)$ are given by the following relation

$$H_n(k) = \frac{(k+1)^n - \binom{n}{2} k^2 - nk - 1}{k^3},$$

where k is any positive integer and n is any nonnegative integer. In this paper we deal with the numbers $H_n(k)$ only for $k = 1, 2, 3$. For example in the case $k = 1$ we get the relation to the previous types of the numbers

$$H_n(1) = b_n(1) - \binom{n}{2} = 2^n - \sum_{i=0}^{2} \binom{n}{i} = 2^n - 1 - \frac{n(n+1)}{2}.$$

Some results about their divisibility and primality are found. Specially any positive integer $n > 1$ does not divide $H_n(2)$ and n congruent to $2 \pmod 4$ is the necessary condition for divisibility $H_n(1)$ and $H_n(3)$ by $n > 2$.

In addition certain properties of their divisibility are used for finding primes among these numbers.

2. The main results

THEOREM 1. *If a positive integer $n > 2$ divides $H_n(1)$ or $H_n(3)$ then $n \equiv 2 \pmod 4$.*

THEOREM 2. *Let $n > 1$. Then*

$$n \nmid H_n(2).$$

3. Some lemmas and preliminary results

LEMMA 1. *Let p be any prime and i, l be any nonnegative integers. Then*

$$(1) \qquad p^{i+1} \mid (lp+1)^{p^i} - 1 ,$$

$$(2) \qquad \frac{(lp+1)^{p^i}-1}{p^{i+1}} \equiv \begin{cases} l \pmod p , & p \neq 2 \\ 0 \pmod p , & p = 2 , \end{cases}$$

for $i \geqslant 1$

$$(3) \quad \frac{(lp+1)^{p^i}-1}{p^{i+1}} \equiv \begin{cases} l - \frac{l^2}{2}p \pmod{p^2} , & p \geqslant 5 , \\ l - l^2 + \frac{4}{3}l^3 - 2l^4 \equiv l + l^2 \pmod{p^2} , & p = 2 \\ l - 3\frac{l^2}{2} + 3l^3 \equiv l - 6l^2 + 3l^3 \pmod{p^2} , & p = 3 . \end{cases}$$

PROOF. We use the binomial theorem

$$(lp+1)^{p^i} = 1 + \binom{p^i}{1}(lp)^1 + \binom{p^i}{2}(lp)^2 + \binom{p^i}{3}(lp)^3 + \ldots + (lp)^{p^i} ,$$

therefore

$$\frac{(lp+1)^{p^i}-1}{p^{i+1}} = l + l^2\frac{p^i-1}{2}p + l^3\frac{(p^i-1)(p^i-2)}{2\cdot 3}p^2 + \ldots + l^{p^i}p^{p^i-i-1}$$

and all assertions are clear after simplification. □

LEMMA 2. *Let m be any nonnegative integer. Then*

$$\frac{4^m-1}{3} \equiv m \pmod 3, \qquad \frac{4^m-1}{3^2} \equiv \frac{3m^2-m}{6} \pmod 3 .$$

PROOF. For $m = 0$ the assertion is obvious and for $m \geqslant 1$ we use the binomial theorem

$$\frac{4^m-1}{3} = \frac{(3+1)^m-1}{3} = 3^{m-1} + \binom{m}{1}3^{m-2} + \binom{m}{2}3^{m-3} + \cdots + \binom{m}{3}3^2 + \binom{m}{2}3 + \binom{m}{1} .$$

Hence,

$$\frac{4^m-1}{3} \equiv \binom{m}{1} \pmod 3 , \qquad \frac{4^m-1}{3^2} \equiv \binom{m}{2} + \frac{m}{3} \pmod 3 .$$

□

LEMMA 3. *Let m be any nonnegative integer. Then*

$$\frac{4^m-1}{9} - 2\left(\frac{4^m-1}{3}\right)^2 + \left(\frac{4^m-1}{3}\right)^3 + \frac{m}{6} \equiv m \pmod 3 .$$

PROOF. After simplification the assertion is a clear consequence of Lemma 2 and the congruence $m^3 \equiv m \pmod 3$. □

LEMMA 4. *Let i be any positive integer. Then* $3^i \nmid H_{3^i}(3)$.

PROOF. As

$$H_n(3) = \frac{4^n - \binom{n}{2} 3^2 - 1}{3^3} = \frac{4^n - \binom{3n}{2} - 1}{3^3} ,$$

then

$$3^i \nmid H_{3^i}(3) = \frac{4^{3^i} - 1 - \binom{3^{i+1}}{2}}{3^3} \iff 3^{i+3} \nmid 4^{3^i} - 1 - \frac{3^{i+1}(3^{i+1}-1)}{2} .$$

Using the congruence

$$4^{3^i} - 1 - 7 \cdot 3^{i+1} \equiv 0 \pmod{3^{i+3}} ,$$

which follows from (3), we obtain

$$4^{3^i} - 1 - \frac{3^{i+1}(3^{i+1}-1)}{2} \equiv \frac{15}{2} 3^{i+1} - \frac{1}{2}(3^{i+1})^2 \equiv 3^{i+2}\frac{5-3^i}{2} \equiv$$
$$3^{i+2} + 3^{i+3}\frac{1-3^{i-1}}{2} \equiv 3^{i+2} \pmod{3^{i+3}} .$$

Hence $H_{3^i}(3) \equiv 3^{i-1} \pmod{3^i}$ and the assertion holds. □

LEMMA 5. *Let $n = m3^i$, where m, i are any positive integers, $m \not\equiv 0 \pmod 3$. Then $n \nmid H_n(3)$.*

PROOF.

$$3^3 \cdot H_{m3^i}(3) = 4^{m3^i} - 1 - m3^{i+1}\frac{m3^{i+1}-1}{2} = (3s+1)^{3^i} - 1 - \frac{m^2}{2}\left(3^{i+1}\right)^2 + \frac{m}{2}3^{i+1},$$

where we denote $s = \frac{4^m-1}{3}$ (it is clear that s must be an integer). Thus using (3) and Lemma 2

$$3^3 \cdot H_{m3^i}(3) \equiv s \cdot 3^{i+1} - 6s^2 \cdot 3^{i+1} + 3s^3 \cdot 3^{i+1} - \frac{m^2}{2}\left(3^{i+1}\right)^2 + \frac{m}{2}3^{i+1} \equiv$$

$$3^{i+2}\left(\frac{4^m-1}{3^2} - 2\left(\frac{4^m-1}{3}\right)^2 + \left(\frac{4^m-1}{3}\right)^3 + \frac{m}{6}\right) \equiv m \cdot 3^{i+2} \pmod{3^{i+3}}$$

and $H_{m3^i}(3) \equiv m3^{i-1} \pmod{3^i}$. □

LEMMA 6. *Let n be any positive integer, $2 \nmid n$ and $3 \nmid n$. Then $n \nmid H_n(3)$.*

PROOF. Suppose conversely that $n \mid H_n(3)$ for some positive integer n which is not divisible by 2 and 3. Such number n can be written as $n = p_1^{a_1} \cdot p_2^{a_2} \cdot \dots \cdot p_s^{a_s}$, where all a_i are positive integers and for primes p_i the relation $5 \leqslant p_1 < p_2 < \dots < p_s$ holds. It is easy to see that $p_1 \mid \binom{3n}{2}$ and we show that $4^n \not\equiv 1 \pmod{p_1}$. If m is the order of the cyclic group generated by 4 under multiplication $\pmod{p_1}$ then the congruence $4^n \equiv 1 \pmod{p_1}$ may be true iff $m \mid n$. As the congruence $4^1 \equiv 1 \pmod{p_1}$ does not hold the number m has to be greater than 1. Therefore $2 \leqslant m < p_1 < p_2 < \dots < p_s$. But by Lagrange Theorem $m \mid p_1 - 1$ because $p_1 - 1$ is the order of the group of the numbers relatively prime to p_1 under multiplication $\pmod{p_1}$. It means that m does not divide n, which is a contradiction. □

4. The proofs of the main theorems

PROOF OF THEOREM 1.

(i) First consider the numbers $H_n(1)$.

Let n be any odd positive integer, then $n \mid \binom{n+1}{2}$ and $n \nmid 2^n - 1$ (see the proof in [1] or in [4], [5], [6] with a proof due to A. Schinzel). Hence the fact that n has to be *even* follows easily from the relation $H_n(1) = 2^n - 1 - \binom{n+1}{2}$. Suppose conversely that $n = 2m$ and m is an even positive integer. Then

$$H_{2m}(1) = 2^{2m} - 1 - \binom{2m+1}{2} = 4^m - 1 - m(2m+1)$$

is an odd number, thus $2m \nmid H_{2m}(1)$. It means that if $2m \mid H_{2m}(1)$ then m must be odd. Hence if $n \mid H_n(1)$ then $n \equiv 2 \pmod 4$.

(ii) The proof of the assertion for the numbers $H_n(3)$.

For an odd integer n the proof of the assertion is clear by Lemma 5 and Lemma 6.

Let $n = 2m$, where m is an even integer. Then in an analogous way as in (i) we can write

$$H_{2m}(3) = \frac{1}{27}\left(4^{2m} - 1 - \frac{3 \cdot 2m(6m-1)}{2}\right) = \frac{1}{27}(4^{2m} - 1 - 3m(2 \cdot 3m - 1)).$$

It means that if $2m \mid H_{2m}(3)$, then m is odd because $2m \nmid H_{2m}(3)$. The proof is complete. □

Proof of Theorem 2.

Let n be a number such that $n \equiv 0 \pmod 3$. Then n does not divide the number $H_n(2) = \frac{3^n-1-2n^2}{8}$ because 3 divides n and does not divide $3^n - 1$. Now let us assume that $n \not\equiv 0 \pmod 3$ is odd. The number n can be written as $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_s^{a_s}$, where a_i, $i = 1, 2, \ldots, s$, are positive integers and the primes $3 < p_1 < p_2 < \ldots < p_s$. Suppose there exists a number n such that $n \mid H_n(2)$. Thus $p_1 \mid H_n(2)$. As $n \mid 3^n - 1$, then $p_1 \mid 3^n - 1$, too. It means that $3^n \equiv 1 \pmod{p_1}$, but we will show that this congruence does not hold. The group of numbers relatively prime to p_1 under multiplication $\pmod{p_1}$ has the order $p_1 - 1$. By Lagrange Theorem $m \mid p_1 - 1$, where m is the order of the cyclic subgroup generated by number 3 under multiplication $\pmod{p_1}$. Thus the last congruence can be true if and only if $m \mid n$. But m has to be greater than 1, because the congruence $3^1 \equiv 1 \pmod{p_1}$ does not hold. It means that $2 \leqslant m < p_1 < p_2 < \ldots < p_s$ and m cannot divide n, which is a contradiction.

If n is even it can be written in the form $n = 2^{a_0} p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s}$, where a_i, $i = 0, 1, \cdots, s$, are positive integers, s is a nonnegative integer and the primes $3 < p_1 < p_2 < \ldots < p_s$. It is easy to see that $8 \mid 3^n - 1$. But this relation is true for $a_0 \geqslant 2$. Then $n \mid H_n(2)$ only if $2^{a_0+3} \mid 3^n - 1$ because $n \mid \frac{2n^2}{8}$. We can write

$$\begin{aligned}3^n - 1 &= (3^{2^{a_0}})^{p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s}} - 1 = \\ &= (3^{2^{a_0}} - 1)\left((3^{2^{a_0}})^{p_1^{a_1} \cdots p_s^{a_s} - 1} + (3^{2^{a_0}})^{p_1^{a_1} \cdots p_s^{a_s} - 2} + \cdots + 1\right).\end{aligned}$$

The term in the parentheses is odd (the sum of an odd number of odd numbers) and the factor $3^{2^{a_0}} - 1$ is not divisible by 2^{a_0+3} with respect to (3). But it means that n does not divide $H_n(2)$ for any even $n \equiv 0 \pmod{4}$.

Finally, suppose that $n \equiv 2 \pmod{4}$. Then $n = 2l$, where $l = p_1^{a_1} \cdots p_s^{a_s}$, primes $3 < p_1 < p_2 < \cdots < p_s$ and a_i, $i = 1, 2, \cdots, s$, are positive integers. We can write $H_{2l}(2) = \frac{3^{2l}-1-2(2l)^2}{2^3} = \frac{3^{2l}-1}{2^3} - l^2$. It is possible to show that p_1 does not divide $3^{2l} - 1$. Suppose conversely that $3^{2l} \equiv 1 \pmod{p_1}$ or $9^l - 1 \equiv 0 \pmod{p_1}$. But we can prove that this congruence is not true in the same way as the proof was done for $3^l - 1$. The proof of Theorem 2 is finished. □

5. Further results about divisibility of the numbers $H_n(1)$ and $H_n(3)$

Lemma 7. *Let i, k be positive integers such that $k \mid (p+1)^{p^i} - 1$. Then $p^{i+1}k^{j+1} \mid (p+1)^{p^i k^j} - 1$ holds for all positive integers j.*

Proof. For a fixed k we will prove the assertion by induction on j. The assertion for $j = 1$ is a consequence of the fact that

$$
\begin{aligned}
&(p+1)^{p^i k} - 1 = \left((p+1)^{p^i}\right)^k - 1 = \\
&= \left((p+1)^{p^i} - 1\right)\left(\left((p+1)^{p^i}\right)^{k-1} + \cdots + (p+1)^{p^i} + 1\right) = \\
&= \left((p+1)^{p^i} - 1\right)\left(\left(\left((p+1)^{p^i}\right)^{k-1} - 1\right) + \cdots + \left(\left((p+1)^{p^i} - 1\right) + k\right)\right.
\end{aligned}
$$

and using Lemma 1 the proof is finished since the term in the parentheses is divisible by k.

Suppose that the assertion holds for a positive integer j and we will show that it holds for $j+1$, too. We can write

$$
\begin{aligned}
&(p+1)^{p^i k^{j+1}} - 1 = \left((p+1)^{p^i k^j}\right)^k - 1 = \\
&= \left((p+1)^{p^i k^j} - 1\right)\left(\left((p+1)^{p^i k^j}\right)^{k-1} + \cdots + (p+1)^{p^i k^j} + 1\right) = \\
&= \left((p+1)^{p^i k^j} - 1\right)\left(\left(\left((p+1)^{p^i k^j}\right)^{k-1} - 1\right) + \cdots + \left(\left((p+1)^{p^i k^j} - 1\right) + k\right)\right.
\end{aligned}
$$

and it is easy to see that this number is divisible by $p^{i+1}k^{j+1}$. □

Lemma 8. *Let $p \leqslant 100000$ be a prime and i a positive integer. Then the relation $p \mid 4^{3^i} - 1$, where $i \geqslant t$, holds only for the primes p (and starting with the value t) in the following table:*

p	t	p	t	p	t
3	1	487	5	52489	8
7	1	1459	5	71119	4
19	2	2593	4	80191	6
73	2	17497	7	87211	3
163	4	39367	7	97687	6

PROOF. The assertion can be proved by using Lagrange theorem about the order of the cyclic subgroup. For example if $p = 17$ we get the condition $4^{3^i} = 2^{2\cdot 3^i} \equiv 1 \pmod{17}$ and the smallest number e satisfying the condition $2^e \equiv 1 \pmod{17}$ is $e = 8$. But as $8 \nmid 2\cdot 3^i$ for any i then $17 \nmid 4^{3^i} - 1$ for any positive integer i. Further if $p = 19$ we get the condition $4^{3^i} = 2^{2\cdot 3^i} \equiv 1 \pmod{19}$ and the smallest number e satisfying the condition $2^e \equiv 1 \pmod{19}$ is $e = 18$. And as $18 \mid 2\cdot 3^i$ for $i \geqslant t$, $t = 2$, then $19 \mid 4^{3^i} - 1$ for any positive integer $i \geqslant 2$. As the proof can be done in the same way for any prime p it is possible to use computer for it. □

THEOREM 3. *Let i, k be any positive integers such that $k \mid 4^{3^i} - 1$ and j be any positive integer. If $n = 2\cdot 3^i k^j$ then*

$$n \mid H_n(1) .$$

PROOF. Since

$$H_{2\cdot 3^i k^j}(1) = 4^{3^i k^j} - 1 - 3^i k^j (2\cdot 3^i k^j + 1) ,$$

divisibility by 2 is clear and divisibility by $3^i k^j$ follows from Lemma 7 for $p = 3$. □

THEOREM 4. *Let i be any nonnegative integer. If $n = 2\cdot 5^i$ then*

$$n \mid H_n(3) .$$

PROOF. We can write

$$H_{2\cdot 5^i}(3) = \frac{16^{5^i} - 1 - 3\cdot 5^i(2\cdot 3\cdot 5^i - 1)}{27}$$

and we get the assertion using Lemma 1 and clear divisibility by 2. □

6. Remark on primality of the numbers $H_n(1)$, $H_n(2)$ and $H_n(3)$

The following theorems are the basis for our computer testing of primality of the numbers $H_n(1)$, $H_n(2)$ and $H_n(3)$.

THEOREM 5. *Let $n \geqslant 2$ be any positive integer. Then*

$$\begin{aligned} 2 \mid H_n(1) &\Longleftrightarrow n \equiv 1, 2 \pmod 4 , \\ 3 \mid H_n(1) &\Longleftrightarrow n \equiv 0, 1, 2 \pmod 6 , \\ 5 \mid H_n(1) &\Longleftrightarrow n \equiv 0, 1, 2, 4, 13 \pmod{20} , \\ 7 \mid H_n(1) &\Longleftrightarrow n \equiv 0, 1, 2, 6, 11, 19 \pmod{21} , \\ 11 \mid H_n(1) &\Longleftrightarrow n \equiv 0, 1, 2, 7, 10, 31, 47, 52, 104 \pmod{110} . \end{aligned}$$

PROOF. All cases can be proved in a similar way. Therefore we take only the case of divisibility by 3. Suppose $n \equiv 0 \pmod 6$, thus $n = 6m$, where m is a positive integer. Then

$$H_{6m}(1) = 2^{6m} - 1 - 3m(6m+1) = 64^m - 1 - 3m(6m+1)$$

and $64^m - 1$ is divisible by 3 for all positive integers m, which is obvious. Similarly we can prove the cases $n \equiv 1, 2 \pmod 6$.

Now suppose $n \equiv 3 \pmod 6$, thus $n = 6m+3$, where m is a nonnegative integer. Then 3 does not divide

$$H_{6m+3}(1) = 2^{6m+3} - 1 - (3m+2)(6m+3)$$

as $3 \mid 6m+3$ and $3 \nmid 2^{6m+3} - 1$, which is obvious. We use the same procedure for $n \equiv 4, 5 \pmod 6$. □

THEOREM 6. *Let $n \geqslant 2$ be any positive integer. Then*

$$\begin{aligned} 2 \mid H_n(2) &\Longleftrightarrow n \equiv 0, 1, 2 \pmod 4, \\ 3 \mid H_n(2) &\Longleftrightarrow n \equiv 1, 2 \pmod 3, \\ 5 \mid H_n(2) &\Longleftrightarrow n \equiv 0, 1, 2, 9, 18 \pmod{20}, \\ 7 \mid H_n(2) &\Longleftrightarrow n \equiv 0, 1, 2, 11, 13, 17, 26 \pmod{42}, \\ 11 \mid H_n(2) &\Longleftrightarrow n \equiv 0, 1, 2, 21, 42 \pmod{55}. \end{aligned}$$

PROOF. The proof is similar to the proof of Theorem 5. □

THEOREM 7. *Let $n \geqslant 2$ be any positive integer. Then*

$$2 \mid H_n(3) \iff n \equiv 1, 2 \pmod 4,$$
$$3 \mid H_n(3) \iff n \equiv 0, 1, 2 \pmod 9,$$
$$5 \mid H_n(3) \iff n \equiv 0, 1, 2 \pmod{10},$$
$$7 \mid H_n(3) \iff n \equiv 0, 1, 2, 4, 12, 17 \pmod{21},$$
$$11 \mid H_n(3) \iff n \equiv 0, 1, 2, 15, 36 \pmod{55}.$$

PROOF. Again the proof is similar to the proof of Theorem 5. □

We used Theorem 5, Theorem 6 and Theorem 7 for the computer testing of primality of the numbers $H_n(1)$, $H_n(2)$ and $H_n(3)$ in the following way.

The conditions of divisibility by the numbers 2, 3 and 5 lead to the fact that every prime $H_n(1)$ must be in the form

$$H_{60k+4}(1),\ H_{60k-20}(1),\ H_{60k-8}(1),\ H_{60k-9}(1),\ H_{60k+3}(1),\ H_{60k+11}(1)$$

or $H_{60k+23}(1)$, every prime $H_n(2)$ in the form

$$H_{60k+3}(2),\quad H_{60k+15}(2),\quad H_{60k+27}(2),\quad H_{60k+39}(2),\quad H_{60k+51}(2)$$

and every prime $H_n(3)$ must be in the form

$H_{210k+3}(3)$, $H_{210k+4}(3)$, $H_{210k+7}(3)$, $H_{210k+8}(3)$, $H_{210k+15}(3)$, $H_{210k+16}(3)$, $H_{210k+23}(3)$, $H_{210k+24}(3)$, $H_{210k\pm35}(3)$, $H_{210k+39}(3)$, $H_{210k\pm43}(3)$, $H_{210k+44}(3)$, $H_{210k+48}(3)$, $H_{210k+59}(3)$, $H_{210k\pm67}(3)$, $H_{210k+68}(3)$, $H_{210k+75}(3)$, $H_{210k+76}(3)$, $H_{210k+79}(3)$, $H_{210k+84}(3)$, $H_{210k\pm87}(3)$, $H_{210k+88}(3)$, $H_{210k\pm95}(3)$, $H_{210k+96}(3)$, $H_{210k\pm103}(3)$, $H_{210k+104}(3)$, $H_{210k-94}(3)$, $H_{210k-86}(3)$, $H_{210k-71}(3)$, $H_{210k-63}(3)$, $H_{210k-62}(3)$, $H_{210k-54}(3)$, $H_{210k-51}(3)$, $H_{210k-42}(3)$, $H_{210k-34}(3)$, $H_{210k-31}(3)$.

We have found by computer that $H_4(1)$, $H_{15}(1)$, $H_{143}(1)$, $H_{855}(1)$, $H_{8788}(1)$, $H_{243}(2)$, $H_4(3)$, $H_7(3)$ and $H_8(3)$ are the only primes with the index lesser than 10000.

Acknowledgements. This paper was supported by Czech Science Foundation Grant GACR 201/01/0471.

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