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Prace Naukowe Uniwersytetu Śląskiego nr 2142, Katowice

MATRIX TRANSFORMATIONS IN THE SEQUENCE SPACES $L_\infty^V(P,S)$ AND $C_0^V(P,S)$

Tunay Bilgin

Abstract. The object of this paper is to obtain necessary and sufficient conditions to characterize the matrices in classes $(l_\infty^v(p,s), l_\infty(q))$, $(c_0^v(p,s), l_\infty(q))$, $(l_\infty^v(p,s), c_0(q))$, and $(c_0^v(p,s), c_0(q))$ which will fill up a gap in the existing literature.

1. Introduction

Let $p=(p_n)$ be a bounded sequence of strictly positive real numbers and $v=(v_n)$ any fixed sequence of non–zero complex numbers such that

$$\liminf_{n\to\infty} |v_n|^{1/n} = r, \qquad (0<r<\infty).$$

We define (Bilgin [2]) the sequence spaces $c_0^v(p,s)$ and $t_\infty^v(p,s)$ as follows;

$$c_0^v(p,s) = \{x=(x_n):\ n^{-s}|x_nv_n|^{p_n} \to \infty,\ s\geqslant 0\}$$

and

$$l_\infty^v(p,s) = \{x=(x_n):\ \sup_n n^{-s}|x_nv_n|^{p_n} < \infty,\ s\geqslant 0\}.$$

When $s=0, v_n=1$ and $p_n=1$ for every n, the spaces $c_0^v(p,s)$ and $l_\infty^v(p,s)$ turn out to be, respectively, the scalar sequence spaces c_0 and l_∞.

When $s=0, v_n=1$ for every n, these spaces are, respectively, the well known spaces $c_0(p)$ and $l_\infty(p)$ defined by Maddox [8] and Simons [11].

When $v_n=1$ for every n, these spaces are, respectively, the spaces $c_0(p,s)$ and $l_\infty(p,s)$ defined by Başarir [1].

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When $s = 0$, these spaces are, respectively, $(c_0(p))$ and $(l_\infty(p))$ defined by Colak and al. [5].

$c_0^v(p,s)$ is paranormed space by $g(x) = \sup_k (k^{-s}|x_k v_k|^{p_k})^{1/M}$, where $M = \max(I, H)$ and $H = \sup_k p_k$. Also $l_\infty^v(p,s)$ is paranormed by $g(x)$ if and only if $\inf p_k > 0$.

If (X, g) is a paranormed space with paranorm g, then we denote by X^* the continuous dual of X, i.e. the set of all continuous linear functionals on X. If E is any set of complex sequences $x = (x_n)$ then E^α will denote the α–dual of E:

$$E^\alpha = \left\{ a : \sum_k |a_k x_k| < \infty \qquad \text{for all} \quad x \in E \right\}.$$

In the following lemmas we have the α – and continuous duals of $c_0^v(p,s)$ and α–dual of $l_\infty^v(p,s)$ (see Bilgin [2]).

LEMMA 1. *Let* $0 < p_k \leqslant \sup_k p_k < \infty$. *Then*
(i) $(c_0^v(p,s))^\alpha = M_0^v(p,s)$, *where*

$$M_0^v(p,s) = \bigcup_{N>1} \left\{ a = (a_k) : \sum_k \left| \frac{a_k}{v_k} \right| k^{s/p_k} N^{-1/p_k} < \infty, s \geqslant 0 \right\};$$

(ii) $(c_0^v(p,s))^*$ *is isomorphic to* $M_0^v(p,s)$.

LEMMA 2. $(l_\infty^v(p,s)) = M_\infty^v(p,s)$, *where*

$$M_\infty^v(p,s) = \bigcap_{N>1} \left\{ a = (a_k) : \sum_k \left| \frac{a_k}{v_k} \right| k^{s/p_k} N^{1/p_k} < \infty, s \geqslant 0 \right\}.$$

2. Matrix transformations

Let X and Y be any two nonempty subsets of s, the set of all sequences of real or complex numbers, and let $A = (a_{nk})$ be the infinite matrix of complex numbers a_{nk} $(n, k = 1, 2, \ldots)$. For every $x = (x_k) \in X$ and every integer n, we write

$$(1) \qquad A_n(x) = \sum_k a_{nk} x_k.$$

The sum without limits in (1) is always taken from $k=1$ to $k=\infty$. The sequence $Ax=(A_n(x))$, if it exists, is called the transformation of $x=(x_k)$ by the matrix A. We write $A\in(X,Y)$ if and only $Ax\in Y$ whenever $x\in X$. Necessary and sufficient conditions for a matrix $A=(a_{nk})$ to be in the class (X,Y) for different sequence spaces X and Y are given by several authors.

Our results in this note characterize some of the classes like $(l_\infty^v(p,s), l_\infty(q))$, $(c_0^v(p,s), l_\infty(q))$, $(l_\infty^v(p,s), c_0(q))$, and $(c_0^v(p,s), c_0(q))$.

The following two theorems give the characterizations of the matrix in the classes $(l_\infty^v(p,s), l_\infty(q))$ and $(l_\infty^v(p,s), c_0(q))$.

THEOREM 3. $A\in(l_\infty^v(p,s), l_\infty(q))$ *if and only if*

(2) $$\sup_n\left(\sum|a_{nk}/v_k|\,k^{s/p_k}N^{1/p_k}\right)^{q_n}<\infty \qquad \textit{for every integer}\quad N>1.$$

PROOF. Sufficiency: Let $x=(x_k)\in l_\infty^v(p,s)$. Choose an integer N such that $N>\max\,(1,\sup_k k^{-s}|v_kx_k|^{p_k})$. Then

$$\begin{aligned}\sup_n|A_n(x)|^{q_n} &\leqslant \sup_n\left(\sum_k|a_{nk}x_k|\right)^{q_n}\\ &\leqslant \sup_n\left(\sum_k|a_{nk}/v_k|k^{s/p_k}\left(k^{-s}|x_kv_k|^{p_k}\right)^{1/p_k}\right)^{q_n}\\ &\leqslant \sup_n\left(\sum_k|a_{nk}/v_k|k^{s/p_k}N^{1/p_k}\right)^{q_n}<\infty\end{aligned}$$

Hence $A(x)\in l_\infty(q)$ and $A\in(l_\infty^v(p,s), l_\infty(q))$.

Necessity. Let $A\in(l_\infty^v(p,s), l_\infty(q))$. If condition (2) is not satisfied, then there exists $N>1$ such that

$$\sup_n\left(\sum_k|a_{nk}/v_k|k^{s/p_k}N^{1/p_k}\right)^{q_n}=\infty.$$

So the matrix $B=\left(|a_{nk}/v_k|k^{s/p_k}N^{1/p_k}\right)\notin(l_\infty, l_\infty(q))$. Hence there exists an $x=(x_k)$ with $\sup_k|x_k|=1$ such that $B(x)\notin l_\infty(q)$.

Now choose a sequence $y=(y_k)$, where $y_k=(x_k/v_k)k^{s/p_k}N^{1/p_k}$. Then $\sup_k k^{-s}|v_ky_k|^{p_k}=\sup_k|x_k|^{p_k}N<\infty$. That is, $y\in l_\infty^v(p,s)$. But

$$A_n(y)=\sum_k a_{nk}y_k=\sum_k a_{nk}(x_k/v_k)k^{s/p_k}N^{1/p_k},$$

so that

$$\sup_n |A_n(y)|^{q_n} = \sup_n \left(\sum_k a_{nk}(x_k/v_k)k^{s/p_k}N^{1/p_k}\right)^{q_n} = \infty$$

That is, $A(y) \notin l_\infty(q)$, contradicting $A \in (l_\infty^v(p,s), l_\infty(q))$.

COROLLARY 4 (Bilgin [1998]). *$A \in (l_\infty(p,s), l_\infty(q))$ if and only if*

$$\sup_n \left(\sum_k |a_{nk}|k^{s/p_k}N^{1/p_k}\right)^{q_n} < \infty \qquad \textit{for every integer} \quad N > 1.$$

PROOF. Follows from Theorem 3, taking $v_k = 1$ for each k.

COROLLARY 5 (Sirajudeen [1981]). *$A \in (l_\infty(p), l_\infty(q))$ if and only if*

$$\sup_n \left(\sum_k |a_{nk}|N^{1/p_k}\right)^{q_n} < \infty \qquad \textit{for every integer} \quad N > 1.$$

PROOF. Follows from Theorem 3, taking $s = 0$ and $v_k = 1$ for each k.

COROLLARY 6 (Basarir [1995]). *Let p be bounded. Then $A \in (l_\infty(p,s), l_\infty)$ if and only if*

$$\sup_n \left(\sum_n |a_{nk}|k^{s/p_k}N^{1/p_k}\right) < \infty \qquad \textit{for every integer} \quad N > 1.$$

PROOF. Follows from Theorem 3, taking $v_k = 1$ and $q_k = 1$ for each k.

COROLLARY 7 (Lascarides and Maddox [1970]). *Let p be bounded. Then $A \in (l_\infty(p), l_\infty)$ if and only if*

$$\sup_n \left(\sum_k |a_{nk}|N^{1/p_k}\right) < \infty \qquad \textit{for every integer} \quad N > 1.$$

PROOF. Follows from Theorem 3, taking $s = 0$ and $v_k = q_k = 1$ for each k.

THEOREM 8. $A \in (l_\infty^v(p,s), c_0(q))$ *if and only if*

$$\left(\sum_k |a_{nk}/v_k| k^{s/p_k} N^{1/p_k}\right)^{q_n} \to 0 \quad as \quad n \to \infty \qquad \textit{for every integer} \quad N > 1.$$

PROOF. Sufficiency. Let $x \in l_\infty^v(p,s)$. So that $\sup_k k^{-s}|v_k x_k|^{p_k} < \infty$. Choose $N > \max(1, \sup_k k^{-s}|v_k x_k|^{p_k})$. Then

$$\begin{aligned}|A_n(x)|^q n &\leqslant \left(\sum_k |a_{nk}/v_k|\ |v_k x_k|\right)^{q_n} \\ &\leqslant \left(\sum_k |a_{nk}/v_k| k^{s/p_k} N^{1/p_k}\right)^{q_n} \to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

Hence $A_n(x) \in c_0(q)$ and $A \in (l_\infty^v(p,s), c_0(q))$.

Necessity. The necessity of the condition is obtained in a similar manner as done in Theorem 9(ii) ([4]), by choosing a sequence $x = (x_k) \in l_\infty^v(p,s)$ as:

$$\begin{aligned} x_k &= (N+1)^{-1/p_k} v_k^{-1} k^{s/p_k} \mathrm{Sgn}(a_{nk}/v_k) && \text{for all } n \text{ and for } 1 \leqslant k \leqslant k_j \\ &= (N+j)^{-1/p_k} v_k^{-1} k^{s/p_k} \mathrm{Sgn}(a_{nk}/v_k) && \text{for all } n \text{ and } k_{j-1} \leqslant k \leqslant k_j; \\ & && j = 2, 3, \ldots \end{aligned}$$

COROLLARY 9 (Bilgin [1998]). $A \in (l_\infty(p,s), c_0(q))$ *if and only if*

$$\left(\sum_k |a_{nk}| k^{s/p_k} N^{1/p_k}\right)^{q_n} \to 0 \ as\ n \to \infty \qquad \textit{for every integer} \quad N > 1.$$

PROOF. Follows from Theorem 8, taking $v_k = 1$ for each k.

COROLLARY 10 (Willey [1973]). $A \in (l_\infty, c_0(q))$ *if and only if*

$$\left(\sum_k |a_{nk}|\right)^{q_n} = o(1).$$

PROOF. Follows from theorem 8, taking $s = 0$ and $v_k = p_k = 1$, $k = 1, 2, \ldots$.

We now characterize the matrix transformation in $c_0^v(p,s)$.

THEOREM 11. $A \in (c_0^v(p,s), l_\infty(q))$ *if and only if*

$$T = \sup_n \left(\sum_k |a_{nk}/v_k| k^{s/p_k} N^{-1/p_k} \right)^{q_n} < \infty \qquad \textit{for some} \quad N > 1.$$

PROOF. Sufficiency. Let $x = (x_k) \in c_0^v(p,s)$. Then there exists k_0 such that $|v_k x_k| < k^{s/p_k} N^{-1/p_k}$ for some $N > 1$ and $k > k_0$. Hence for every n we have

$$|A_n(x)|^{q_n} \leqslant L \left| \sum_{k=0}^{k_0} a_{nk} x_k \right|^{q_n} + L \left| \sum_{k>k_0} a_{nk} x_k \right|^{q_n} = L(S_1 + S_2),$$

where $L = \max(1, 2^{H-1})$, $H = \sup_n q_n$.

$$\begin{aligned} S_1 &= \left(\left| \sum_{k=0}^{k_0} a_{nk} x_k \right| \right)^{q_n} = \left(\left| \sum_{k=0}^{k_0} (a_{nk}/v_k) v_k x_k \right| \right)^{q_n} \\ &\leqslant \left(\sum_{k \leqslant k_0} |a_{nk}/v_k| k^{s/p_k} N^{-1/p_k} \max_{k \leqslant k_0} |v_k x_k| N^{1/p_k} k^{-s/p_k} \right)^{q_n} < \infty. \end{aligned}$$

For the sum S_2, we have,

$$S_2^{1/q_n} = \left| \sum_{k>k_0} a_{nk} x_k \right| = \left| \sum_{k>k_0} (a_{nk}/v_k) v_k x_k \right| \leqslant \sum_{k>k_0} |a_{nk}/v_k| k^{s/p_k} N^{-1/p_k}.$$

Hence $S_2 \leqslant T$. Thus $A_n(x) \in l_\infty(q)$ and hence $A \in (c_0^v(p,s), l_\infty(q))$.

Necessity. Using the same kind of argument to that in [4], the necessity of the condition is obtained in a similar manner as done in Theorem 1, by choosing a sequence $x \in c_0^v(p,s)$:

$$x_k^m = \delta^{M/p_k} / v_k k^{s/p_k} \,(\mathrm{sgn} a_{nk}/v_k) \qquad \text{if} \quad 1 \leqslant k \leqslant m$$

and

$$x_k^m = 0 \qquad \text{if} \quad k > m, \qquad \text{where} \quad \delta < 1.$$

COROLLARY 12 (Bilgin [2002]). *$A \in (c_0^v(p,s), l_\infty)$ if and only if*

$$\sup_n \sum_k |a_{nk}/v_k| k^{s/p_k} N^{-1/p_k} < \infty \qquad \text{for some} \quad N > 1.$$

PROOF. Follows from Theorem 11, taking $q_k = 1$ for each k.

COROLLARY 13 (Bilgin [1998]). *$A \in (c_0(p,s), l_\infty(q))$ if and only if*

$$\sup_n \left(\sum_k \left(|a_{nk}| k^{s/p_k} N^{-1/p_k} \right)^{q_n} \right) < \infty \qquad \text{for every integer} \quad N > 1.$$

PROOF. Follows from Theorem 11, taking $v_k = 1$ for each k.

COROLLARY 14 (Başarir [1995]). *$A \in (c_0(p,s), l_\infty)$ if and only if there exists $B > 1$ such that*

$$\sup_n \sum_k |a_{nk}| k^{s/p_k} B^{-1/p_k} < \infty.$$

PROOF. Follows from Theorem 11 taking $v_k = q_k = 1$ for each k.

COROLLARY 15 (Lascarides [1971]). *$A \in (c_0(p), l_\infty(q))$ if and only if there exists $B > 1$ such that*

$$\sup_n \left(\sum_k |a_{nk}| B^{-1/p_k} \right)^{q_n} < \infty.$$

PROOF. Follows from Theorem 11, taking $s = 0$ and $v_k = 1$ for each k.

COROLLARY 16 (Roles [1970]). *$A \in (c_0(p), l_\infty)$ if and only if there exists $M > 1$ such that*

$$\sup_n \sum_k |a_{nk}| M^{-1/p_k} < \infty.$$

PROOF. Follows from Theorem 11, taking $s = 0$ and $v_k = q_k = 1$ for each k.

THEOREM 17. $A \in (c_0^v(p,s), c_0(q))$, *if and only if*

(i) $$|a_{nk}/v_k|^{q_n} \to 0 \text{ as } n \to \infty \quad \text{for each } k,$$

and

(ii) $$\lim_N \limsup_n \left(\sum_k |a_{nk}/v_k| k^{s/p_k} N^{-1/p_k}\right)^{q_n} = 0.$$

PROOF. Sufficiency. Let $\varepsilon > 0$ and $x = (x_k) \in c_0^v(p,s)$. Now by (ii) there exists integer $N > 1$ such that

(3) $$\limsup_n \left(\sum_k |a_{nk}/v_k| k^{s/p_k} N^{-1/p_k}\right)^{q_n} < \varepsilon$$

Since $x = (x_k) \in c_0^v(p,s)$, so there exists an integer k_0 such that

$$|v_k x_k| < k^{s/p_k} N^{-1/p_k} \qquad \text{for} \quad k > k_0$$

$$\begin{aligned} |A_n(x)|^{q_n} &= \left|\sum_{k=1}^{\infty} a_{nk} x_k\right|^{q_n} \\ &\leqslant L\left[\left(\sum_{k=1}^{k_0} |a_{nk} x_k|\right)^{q_n} + \left(\sum_{k>k_0} |a_{nk} x_k|\right)^{q_n}\right] \\ &\leqslant L \max_{k \leqslant k_0} |v_k x_k|^{q_n} \left(\sum_{k=1}^{k_0} |a_{nk}/v_k|^{q_n/H}\right)^H \\ &\quad + L\left(\sum_{k>k_0} a_{nk}/v_k| k^{s/p_k} N^{-1/p_k}\right)^{q_n} \end{aligned}$$

where $L = \max(1, 2^{H-1})$, $H = \sup_n q_n$. By taking limsup as $n \to \infty$, by (i) and (3) we see that $A(x) \in c_0(q)$. Hence $A \in (c_0^v(p,s), c_0(q))$.

For the necessity of (i), taking $x = (0, 0, \ldots, 0, 1/|v_k|, 0, \ldots)$ with $1/|v_k|$ at the k–th place and 0 elsewhere. We get $|a_{nk}/v_k|^{q_n} \to 0$ as $n \to \infty$. The necessity of (ii) is obtained in a similar manner as done in Theorem 8.

COROLLARY 18 (Bilgin [1997]). $A \in (c_0(p,s), c_0(q))$ *if and only if*

(i) $|a_{nk}|^{q_n} \to 0$ *as* $n \to \infty$ *for each* k, *and*

(ii) $\lim_N \limsup_n \left(\sum_k |a_{nk}| k^{s/p_k} N^{-1/p_k}\right)^{q_n} = 0.$

COROLLARY 19 (Maddox) [1972]). $A \in (c_0(p), c_0(q))$ *if and only if*

(i) $|a_{nk}|^{q_n} \to 0$ *as* $n \to \infty$ *for each* k, *and*

(ii) $\lim_N \limsup_n \left(\sum_k |a_{nk}| N^{-1/p_k}\right)^{q_n} = 0.$

PROOF. Follows from Theorem 17, taking $s = 0$ and $v_k = 1$ for each k.

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Yüzüncü Yıl University
Faculty of Education
Department of Mathematics
Van–Turkey

e-mail: tbilgin@yyu.edu.tr