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ON A PROBLEM OF WU WEI CHAO

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Abstract. Answering a question posed by Wu Wei Chao [2], we determine all solutions $f : \mathbb{R} \to \mathbb{R}$ of the equation

$$f(x^2 + y + f(y)) = f(x)^2 + 2y, \qquad x, y \in \mathbb{R}.$$

In volume 108 (No 10, December 2001) of the American Mathematical Monthly, Wu Wei Chao [2] posed the following problem: find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the functional equation

$$f(x^2 + y + f(y)) = f(x)^2 + 2y, \qquad x, y \in \mathbb{R}. \tag{1}$$

We will show that the only solution of (1) is an identity function i.e. $f(x) = x$, $x \in \mathbb{R}$. Our proof is based upon two lemmas.

LEMMA 1. *If $f : \mathbb{R} \to \mathbb{R}$ is a solution of the functional equation* (1), *then it is odd.*

PROOF. Firstly we shall show that

$$c := f(0) = 0. \tag{2}$$

By (1) we have

$$f(c) = c^2; \tag{3}$$

$$f(x^2 + c) = f(x)^2, \qquad x \in \mathbb{R}; \tag{4}$$

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and

$$f(c+c^2)=c^2+2c. \tag{5}$$

By virtue of (3) and (4), $f(c+c^2)=c^4$, which together with (5) implies

$$c(c^3-c-2)=0. \tag{6}$$

Putting $x=0$ in (1), we get

$$f(y+f(y))=c^2+2y, \qquad y\in\mathbb{R}, \tag{7}$$

and whence $f(\mathbb{R})=\mathbb{R}$.

Let y_0 be chosen such that $f(y_0)=0$. It follows from (7) that $0=c^2+2y_0$ or, equivalently,

$$y_0=-\frac{c^2}{2}. \tag{8}$$

Putting y_0 instead of x in (1), we obtain

$$f(\frac{c^4}{4}+y+f(y))=2y, \qquad y\in\mathbb{R}.$$

Therefore $f(\frac{c^4}{4}+c)=0$, and using (8) we get $c(c^3+2c+4)=0$. According to (6), we infer that $c=0$, which finishes the proof of condition (2).

Putting $y=0$ in (1) and on account of (2), we get

$$f(x^2)=f(x)^2, \qquad x\in\mathbb{R}. \tag{9}$$

This implies that

$$f(u)\geqslant 0, \qquad u\in[0,\infty). \tag{10}$$

According to (9), we have the following alternative

$$f(x)=f(-x) \qquad \text{or} \qquad f(-x)=-f(x), \qquad x\in\mathbb{R}.$$

Assume that for some $z>0$ we have $f(z)=f(-z)$. It follows from (1) and (9) that

$$f(x^2-z+f(z))=f(x^2)-2z, \qquad x\in\mathbb{R}. \tag{11}$$

Choose x such that $x^2 = z$. By virtue of (10)

$$0 \leqslant f(f(z)) = f(z) - 2z.$$

Taking into account (11) with $x = 0$ and using (2), we obtain

$$f(-z + f(z)) = -2z,$$

and, therefore (cf. (10)) $f(z) \leqslant z$. Thus

$$0 \leqslant f(z) - 2z \leqslant z - 2z = -z,$$

which means that $z \leqslant 0$. This contradicts our assumption that $z > 0$ and proves that $f(-x) = -f(x)$ for each $x \in \mathbb{R}$. This ends the proof of Lemma 1.

LEMMA 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function fulfilling the equation* (1). *Then function $F : \mathbb{R} \to \mathbb{R}$ given by the formula $F(x) := f(x) + x$, $x \in \mathbb{R}$, satisfies the following conditions:*

$$\text{(12)} \qquad F \text{ is odd};$$

$$\text{(13)} \qquad F(x^2 + F(y)) = F(x^2) + F(y) + 2y, \qquad x, y \in \mathbb{R};$$

$$\text{(14)} \qquad F((0,\infty)) = (0,\infty) \qquad \text{and} \qquad F((-\infty,0)) = (-\infty,0).$$

PROOF. Condition (12) is a consequence of the definition of F and Lemma 1. Rewrite (1) in the form

$$f(x^2 + y + f(y)) + x^2 + y + f(y) = f(x)^2 + 2y + x^2 + y + f(y), \quad x, y \in \mathbb{R}.$$

By definition of F and on account of (9), we obtain (13). Now, let us put $y = -x^2$ in (13). Then, by (12) we get

$$\text{(15)} \qquad F(u - F(u)) = -2u, \qquad u \in \mathbb{R},$$

and whence $F(\mathbb{R}) = \mathbb{R}$. It follows from (10) and the definition of F that $\operatorname{sgn}(u)(u - F(u)) \leqslant 0$ for each $u \neq 0$. Thus, (14) follows directly from (15). The proof of Lemma 2 is completed.

THEOREM. *If $f : \mathbb{R} \to \mathbb{R}$ satisfies the functional equation* (1) *then $f(x) = x$ for every $x \in \mathbb{R}$.*

Proof. Let F be function defined in Lemma 2. It follows from (13) and (12) that

$$F(x^2 - F(y)) = F(x^2) - F(y) - 2y, \qquad x, y \in \mathbb{R}.$$

Summing up this equality and (13), we get

$$F(x^2 + F(y)) + F(x^2 - F(y)) = 2F(x^2), \qquad x, y \in \mathbb{R}.$$

Since $F((0,\infty)) = (0,\infty)$ (comp. (14), the above equality means that F satisfies Jensen's functional equation

$$F\left(\frac{u+v}{2}\right) = \frac{F(u) + F(v)}{2}, \qquad u, v \in (0,\infty),$$

and therefore F, being bounded below on $(0,\infty)$, has to be of the following form

$$F(u) = ku + b, \qquad u \in (0,\infty),$$

where $k, b \geqslant 0$ are constants (cf. [1], pages 315, 316, for example). Since F is odd and $F(0) = 0$, b has to be equal to zero and $F(u) = ku$ for every $u \in \mathbb{R}$. Moreover, k has to be equal to 2, because F is a solution of (13). Now, our assertion follows immediately from the definition of F.

References

[1] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities, Cauchy's Equation and Jensen's Inequality*, Prace Naukowe Uniwersytetu Śląskiego w Katowicach nr 489, Polish Scientific Publishers, Warszawa–Kraków–Katowice, 1985.

[2] Wu Wei Chao, *Problem 10908*, Amer. Math. Monthly, 108, 10 December (2001).

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