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ON SOME CONDITIONAL FUNCTIONAL EQUATIONS

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Abstract. Let X, Y be real linear spaces. We are looking for a function $G : X^2 \to \mathbb{R}$ such that the equation

$$f(x+y) = G(x,y)[f(x)+f(y)]$$

is equivalent to the orthogonal Cauchy equation

$$x \perp y \Rightarrow f(x+y) = f(x) + f(y).$$

Several kinds of orthogonalities are considered. The quotient $\frac{\|x-y\|}{\|x+y\|}$ closely connected with the James orthogonality plays here a distinguished role. Similar problems are considered for the Ptolemaic equation

$$x \perp y \Rightarrow f(x+y)f(x-y) = f(x)^2 + f(y)^2.$$

As a result a characterization of inner product spaces is obtained.

1. Introduction

In the present paper we intend to examine the properties of some conditional equations. Namely we are going to deal with so called orthogonal equations. That means equations which are assumed to be satisfied only for orthogonal pairs of vectors. We shall consider functions defined on normed spaces and in the case of inner product spaces we shall deal with the usual orthogonality defined by an inner product. However in the case of a normed space in which the norm does not come from an inner product we shall have to define another orthogonality.

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Since it is impossible to define a nontrivial orthogonality in a linear space with dimension equal to 1, we assume in the whole paper that the domain of considered functions is at least 2-dimensional.

We are going to consider the well known orthogonal Cauchy equation

$$x \perp y \Rightarrow f(x+y) = f(x) + f(y) \tag{1}$$

and the Ptolemaic equation

$$x \perp y \Rightarrow f(x)^2 + f(y)^2 = f(x+y)f(x-y). \tag{2}$$

Now we would like to find unconditional equations which preserve the properties (solutions) of the above equations. The simplest way to do it is to remove the conditions and to leave the equations unchanged. However it does not seem to be a good idea. Have a look at the equation (1). It is known that in an inner product space the square of the norm is a solution of this equation. However the square of the norm is clearly nonadditive. That means that after removing the condition we loose some important solutions of this equation. Further let us consider the modified version of (2)

$$\|x-y\| = \|x+y\| \Rightarrow f(x)^2 + f(y)^2 = f(x+y)f(x-y). \tag{3}$$

We are able to consider this version of the Ptolemaic equation in any normed space. We shall prove later in the paper that the function $f(x) = \|x\|$ satisfies the equation (3) if and only if X is an inner product space. Once the condition is removed we obtain immediately that this equation has no nonzero solutions. We not only loose the solutions, we loose also the above mentioned characterization of inner product spaces. It is now clear that if we want to find unconditional equations which preserve the properties of conditional equations we have to modify them in a way.

In [7] the following version of Jensen equation was considered

$$f\left(\frac{x+y}{2}\right) = \gamma\left(\frac{\|x-y\|}{\|x+y\|}\right)[f(x)+f(y)]. \tag{4}$$

We shall try to modify the equations of Cauchy and Ptolemeus in a similar way. However dealing with (4) we use the quotient $\frac{\|x-y\|}{\|x+y\|}$ which is equal to 1 if and only if x and y are James orthogonal (we exclude here the case of $x = y = 0$). We would like to generalize these considerations to cover for instance some other conditional equations involving different types of orthogonality.

2. The Cauchy Equation

Before starting with further considerations it is convenient to recall a definition of some kind of orthogonalities. Jürg Rätz (see [5]) introduced the following definition. Let us consider a relation "$\perp$" defined on pairs of elements of an at least 2-dimensional real linear space. We shall say that this relation is an orthogonality in a sense of Rätz if and only if it satisfies the following conditions:

$$
(\mathrm{R})\begin{cases}
1. & x\perp 0, 0\perp x; x\in X\\
2. & [x\neq 0\neq y, x\perp y]\Rightarrow x,y \text{ are linearly independent}\\
3. & [x,y\in X, x\perp y]\Rightarrow \alpha x\perp\beta y \text{ for all } \alpha,\beta\in\mathbb{R}\\
4. & \text{Let } P \text{ be a linear subspace of } X \text{ such that } \dim P=2.\\
 & \text{If } x\in P, \lambda\in\mathbb{R}^+, \text{ then there exists } y\in P \text{ such that } x\perp y\\
 & \text{and } x+y\perp\lambda x-y.
\end{cases}
$$

LEMMA 1. *Let X be a real linear space in which we have an orthogonality satisfying conditions* (R) *and let Y be a real linear space. Let further $D\subset X\times X$ be such that*

$$x\perp y, (x,y)\neq(0,0)\Rightarrow(x,y)\in D.$$

If $f:X\to Y$ is a solution of the following modified version of the Cauchy equation

$$f(x+y)=G(x,y)[f(x)+f(y)],\qquad (x,y)\in D, \tag{5}$$

where $G: X\times X\supset D\to\mathbb{R}$ is some function satisfying the condition

$$x\perp y\Rightarrow G(x,y)=c \qquad (*)$$

for all $(x,y)\in D$ and some $c\in\mathbb{R}\setminus\{0\}$,then f is constant and $c=\frac{1}{2}$
or
f is given by the following formula

$$f(x)=a(x)+h(x)$$

where a is an additive function, h is a quadratic mapping, and $c=1$. If we additionally assume that X is an inner product space, then in this case we have $f(x)=a(x)+b(\|x\|^2)$ with some additive functions $a:X\to\mathbb{R}$ and $b:[0,\infty)\to\mathbb{R}$.

PROOF. Fix a vector $x \in X, x \neq 0$. Then $x \perp 0$ implies that $(x, 0) \in D$ and

$$f(x) = c[f(x) + f(0)];$$

thus $(1-c)f(x) = cf(0)$. If now $c = 1$, then we have $f(0) = 0$ and consequently

$$x \perp y \Rightarrow f(x+y) = f(x) + f(y)\,, \quad x, y \in X,$$

which means that function f is orthogonally additive and we obtain the desired form of this function using suitable results contained in [5].

Now, assuming that $c \neq 1$, we infer that

$$f(x) = \frac{c}{1-c} f(0), \qquad \text{for} \quad x \in X \setminus \{0\}. \tag{6}$$

Without loss of generality we may assume that $f(0) \neq 0$ (otherwise $f = 0$). We are going to show that in this case $c = \frac{1}{2}$ which, in view of (6), means that f is constant. Indeed, fixing $x \in X$ and using axiom 4 from conditions (R) one can find a $y \in X \setminus \{0\}$ that is orthogonal to x. Consequently we get

$$f(x+y) = c[f(x) + f(y)].$$

Now applying (6) to $f(x+y)$, $f(x)$ and $f(y)$ and substituting it to the above equation we have

$$\frac{c}{1-c} f(0) = 2c \frac{c}{1-c} f(0)$$

and, consequently, $c = \frac{1}{2}$. Summarizing, we have shown that in this case function f is constant and $G(x, y) = \frac{1}{2}$ (in the case where $f \neq 0$). □

DEFINITION 1. Let $(X, \|\cdot\|)$ be a normed space. We define the function $s : X \times X \to \mathbb{R}$ by the following formula

$$s(x, y) = \begin{cases} \inf_{\lambda \in \mathbb{R}} \frac{\|x + \lambda y\|}{\|x\|} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

The following remark gives a few simple properties of the function s.

REMARK 1. Let $(X, \|\cdot\|)$ be a real normed space. Function s takes its values in the interval $[0, 1]$. For all real numbers $\alpha, \beta \neq 0$ and all $x, y \in X$ we have $s(\alpha x, \beta y) = s(x, y)$. Further $s(x, y) = 0$ if and only if x, y are linearly dependent.

If additionally X is an inner product space then

$$x \perp y \Leftrightarrow s(x, y) = 1 \quad \text{and} \quad s(x, y) = s(y, x).$$

Moreover, in this case

$$s(x,y)=\sqrt{1-\frac{(x|y)^2}{\|x\|^2\|y\|^2}}\qquad (x,y)\neq(0,0).$$

PROOF. The assertions concerning the case of any normed space are obvious. Assume that X is an inner product space and take two orthogonal vectors $x,y\in X$. Then for every $\lambda\neq 0$ we have $\|x+\lambda y\|>\|x\|$, which means that the minimal value is achieved for $\lambda=0$. On the other hand if $s(x,y)=1$ then for every $\lambda\in\mathbb{R}$ one has $\|x+\lambda y\|\geqslant\|x\|$. That means that

$$(x|x)+2\lambda(x|y)+\lambda^2(y|y)\geqslant(x|x),\qquad \lambda\in\mathbb{R},$$

and

$$\lambda(2(x|y)+\lambda(y|y))\geqslant 0,\qquad \lambda\in\mathbb{R},$$

which is possible in the case when $(x|y)=0$ exclusively.

To prove the last assertion we fix two vectors $x,y\in X\setminus\{0\}$. Put $a:=\|y\|^2, b:=(x|y)$, and $c:=\|x\|^2$. We shall determine a $\lambda_0\in\mathbb{R}$ such that the expression $\|x+\lambda y\|$ is minimal at $\lambda=\lambda_0$. Since the function $\lambda\to\|x+\lambda y\|$ is nonnegative, it takes its minimum at the same point as its square. Since

$$\|x+\lambda y\|^2=c+2\lambda b+\lambda^2 a,$$

it is obvious that $\lambda_0=-\frac{b}{a}$. Thus

$$\begin{aligned}s(x,y)&=\sqrt{\frac{c+2\lambda_0 b+\lambda_0^2 a}{c}}=\sqrt{\frac{c-2\frac{b}{a}b+\frac{b^2}{a^2}a}{c}}\\&=\sqrt{1-\frac{\frac{b^2}{a}}{c}}=\sqrt{1-\frac{b^2}{ac}}=\sqrt{1-\frac{(x|y)^2}{\|x\|^2\|y\|^2}}.\end{aligned}$$

□

Function s is strictly connected with the Birkhoff—James orthogonality defined by the formula

$$x\perp_{BJ}y\iff\bigwedge_{\beta\in\mathbb{R}}\|x+\beta y\|\geqslant\|x\|.$$

Recall that the Birkhoff—James orthogonality satisfies the conditions (R) (see [5]).

REMARK 2. Let $(X, \|\cdot\|)$ be a real normed space. Then

$$x \perp_{BJ} y \iff s(x,y) = 1.$$

Now, using function s, we can try to get an unconditional equation which is connected with Birkhoff—James orthogonal additivity.

REMARK 3. Let X be a real normed (inner product) space. Then function $f : X \to \mathbb{R}$ satisfying the equation

$$f(x+y) = g(s(x,y))[f(x) + f(y)] \tag{7}$$

with some function g, is either constant or is of the form $f(x) = a(x) + h(x)$, $x \in X$, for some additive mapping a and quadratic h. (Respectively, $f(x) = a(x) + b(\|x\|^2)$ for some additive $a : X \to \mathbb{R}$ and $b : \mathbb{R} \to \mathbb{R}$.)

For the proof it suffices to note that all the assumptions of the Lemma 1 are satisfied.

However this equations fails to be equivalent to the equation of orthogonal additivity. More precisely, in an inner product space all continuous solutions of this equation are constant or additive; no quadratic terms occur.

THEOREM 1. *Let $(X, (\cdot|\cdot))$ be a real inner product space. Then a continuous function $f : X \to \mathbb{R}$ satisfies equation* (7) *with some function $g : [0,1] \to \mathbb{R}$ if and only if f is constant or additive.*

PROOF. An additive function clearly satisfies the equation considered with function $g := 1$. Constant function satisfies our equation with $g = \frac{1}{2}$.

Let us assume that f is a continuous solution of (7). From Remark 3 we infer that

$$f(x) = a(x) + b(\|x\|^2), \quad x \in X,$$

for some additive functions a and b. The continuity of f allows us to write the equality

$$f(x) = a(x) + k\|x\|^2, \quad x \in X,$$

where k is some real number. We are going to show that the following alternative is true

$$f(x) = a(x), \ x \in X, \quad \text{or} \quad f(x) = k\|x\|^2, \ x \in X.$$

For indirect proof let us assume that $a \neq 0, k \neq 0$. Substituting this form of f to the equation we obtain

$$a(x+y)+k\|x+y\|^2 = g(s(x,y))[a(x)+k\|x\|^2 + a(y)+k\|y\|^2], \; x,y \in X$$

Let us consider two cases:

1^0 one can find a pair of nonorthogonal vectors x, y such that $g(s(x,y)) = 1$. Then we get the equality

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2,$$

which is obviously false.

2^0 For all nonorthogonal x, y we have $g(s(x,y)) \neq 1$. Then we obtain

$$(1-g(s(x,y))a(x+y) = k[g(s(x,y))(\|x\|^2+\|y\|^2) - \|x+y\|^2], \; x,y \in X,$$

and substituting here $\frac{x}{2}$ in place of x and y we get

$$(1-g(0))a(x) = k\left(\frac{g(0)}{2}-1\right)\|x\|^2,$$

for all $x \neq 0$ which means that $a = 0$, a contradiction.

We have shown that $f(x) = a(x)$, $x \in X$, or $f(x) = k\|x\|^2$, $x \in X$. The remaining part of the proof is to show that the second of these functions is not a solution of our equation. To this end suppose that

$$k\|x+y\|^2 = g(s(x,y))k(\|x\|^2+\|y\|^2), \qquad x \in X,$$

for some function $g : [0,1] \to \mathbb{R}$. Taking here $x = y$ we may write

$$\|2x\|^2 = g(0)2\|x\|^2, \qquad x \in X,$$

i.e. $g(0) = 2$. Writing now $2x$ instead of x and x in the place of y we achieve

$$\|2x+x\|^2 = g(0)(\|2x\|^2+\|x\|^2), \qquad x \in X,$$

which gives us $g(0) = \frac{9}{5}$, a contradiction. □

Now we are going to introduce the quotient connected with James orthogonality. Elements x, y of a normed space are called James orthogonal if and only if $\|x-y\| = \|x+y\|$. Now, we are going to use the quotient $\frac{\|x-y\|}{\|x+y\|}$ in similar way as we previously used the function s. Namely, let us consider the following equation:

$$f(x+y) = g\left(\frac{\|x-y\|}{\|x+y\|}\right)[f(x)+f(y)]. \tag{8}$$

It is known that in normed spaces which are not inner product spaces the James orthogonality does not satisfy conditions (R).

That means that in the general case we cannot use Lemma 1 to prove a result similar to that of Remark 3. But we can formulate the following result.

THEOREM 2. *Let* $(X,(\cdot|\cdot))$ *be a real inner product space and let* $f: X \to \mathbb{R}$ *be a continuous function. Then* f *is a solution of* (8) *with some function* $g:[0,\infty) \to \mathbb{R}$ *such that* $g(1) \neq 0$, *if and only if* f *is constant, additive or* $f(x) = k\|x\|^2$.

Further, the equation (5), *where* G *is some function satisfying the condition* $(*)$ *has a nonadditive, not constant and continuous solutions* $f: X \to \mathbb{R}$ *if and only if*

$$G(x,y) = \begin{cases} g\left(\frac{\|x-y\|}{\|x+y\|}\right) & x \neq -y \\ 0 & x = -y \neq 0 \\ \alpha & x = y = 0 \end{cases}$$

with some function $g:[0,\infty) \to \mathbb{R}$ *and* $\alpha \in \mathbb{R}$. *Moreover, if in this case* f *is a continuous nonadditive and not constant solution of* (5), *then the corresponding function* g *is given by the formula* $g(a) = \frac{2}{1+a^2}$, $\alpha \in [0,\infty)$.

PROOF. Let us start the proof with the latter of the above two statements. If f is not constant then we obtain (similarly as in the above theorem) the following form of this function $f(x) = a(x) + k\|x\|^2$. Also in the same manner as before we can prove that

$$f(x) = a(x),\ x \in X, \quad \text{or} \quad f(x) = k\|x\|^2,\ x \in X.$$

That means that every continuous and nonadditive solution of this equation must be of the form $k\|x\|^2$. Let us substitute this function to the equation (5)

$$k\|x+y\|^2 = G(x,y)(k\|x\|^2 + k\|y\|^2).$$

If now $x = -y \neq 0$ and $k \neq 0$, then $G(x,y)$ must be equal to zero, $G(0,0)$ may take any value. Consequently there is no loss of generality in assumption that $\|x+y\| \neq 0$. It allows us to write

$$G(x,y) = \frac{\|x+y\|^2}{\|x\|^2 + \|y\|^2} = \frac{1}{\frac{\|x+y\|^2 + \|x-y\|^2}{2\|x+y\|^2}} = \frac{2}{1 + \left(\frac{\|x-y\|}{\|x+y\|}\right)^2}$$

for all $x, y \in X, x \neq -y$.

Proceeding to the second part of our theorem we note that in a real inner product space the James orthogonality coincides with the usual one.

Since the orthogonality defined by an inner product satisfies conditions (R), we are able to use Lemma 1. If now f is not constant then, similarly as before, we obtain

$$f(x) = a(x),\ x \in X, \quad \text{or} \quad f(x) = k\|x\|^2,\ \ x \in X.$$

On the other hand in the first part of the proof it has been shown that both of these functions satisfy equation (8). □

Studying the functions s and the James quotient $\frac{\|x-y\|}{\|x+y\|}$ it is worthwhile to note that the function s is nonzero only in the case of spaces of dimension at least 2 and function $\frac{\|x-y\|}{\|x+y\|}$ may be considered also in one-dimensional case. Considering equations of this type on a real line we obtained (under some assumptions) a characterization of multiplicative functions (see [6] and [7]).

Furthermore although we were considering mainly continuous functions, this assumptions is not essential. Namely, using a result contained in [7], it can be proved that every solution of (8) which is not additive must be continuous.

3. The Ptolemaic Equation

The Theorem of Ptolemeus states that in a quadrilateral inscribed in a circle the product of the diagonals is equal to the sum of products of the opposite sides. If we consider the parallelograms then the only parallelogram which can be inscribed in a circle is a rectangle That means that we obtain the following property

$$x \perp y \Rightarrow \|x+y\|\|x-y\| = \|x\|^2 + \|y\|^2.$$

In a natural way this leads to a conditional functional equation which is satisfied by the norm coming from an inner product. In the sequel the corresponding equation

$$x \perp y \Rightarrow f(x+y)f(x-y) = f(x)^2 + f(y)^2$$

will be called the Ptolemaic equation. Trying to generalize this equation to the case of normed spaces we shall once more use the James orthogonality

$$\|x+y\| = \|x-y\| \Rightarrow f(x)^2 + f(y)^2 = f(x+y)f(x-y). \tag{9}$$

Using this equation we shall present a characterization of inner product spaces. A proof of the next theorem can be found in Dan Amir monograph [1] but since that theorem can be proved very shortly, for the sake of completeness we shall present our approach.

THEOREM 3. *Let* $(X, \|\cdot\|)$ *be a normed space. Then* X *is an inner product space if and only if the function* $f(x) = \|x\|$ *satisfies equation* (9).

PROOF. Let $(X, (\cdot|\cdot))$ be an inner product space. Then

$$\|x-y\| = \|x+y\| \Rightarrow (x|y) = 0 \Rightarrow \|x\|^2 + \|y\|^2 = \|x+y\|^2,$$

which means that our equation is satisfied.

On the other hand, assume that the function $f(x) = \|x\|$, $x \in X$, satisfies (9). Fix a pair $(x, y) \in X \times X$ satisfying the equalities

$$\|x\| = \|y\| = 1.$$

and put

$$u := x + y, \quad v := x - y.$$

Then $\|u+v\| = \|u-v\|$, which means that u, v are orthogonal in the sense of James, whence using the equation (9), we obtain

$$\|u+v\| \, \|u-v\| = \|u\|^2 + \|v\|^2.$$

Thus we have shown that

$$\|x\| = \|y\| = 1 \Rightarrow \|2x\| \, \|2y\| = \|x+y\|^2 + \|x-y\|^2$$

i.e.

$$\|x\| = \|y\| = 1 \Rightarrow 4 = \|x+y\|^2 + \|x-y\|^2$$

and further

$$\|x\| = \|y\| = 1 \Rightarrow 2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2. \tag{10}$$

Thus we have the parallelogram identity on the unit sphere. It is well known (see [2] or [1] p.47) that this condition forces a normed space to be an inner product space. □

As we have seen, equation (9) can be used to characterize inner product spaces. However, if we assume this equation to hold for all pairs of x and y we infer that this equation has no nonzero solutions. Therefore we shall modify it in a similar way as we have done it in the case of the Cauchy equation. Namely, we shall multiply the right-hand side of this equation by a function which depends on the James quotient, getting

$$f(x+y)f(x-y) = g\left(\frac{\|x-y\|}{\|x+y\|}\right)[f(x)^2 + f(y)^2], \qquad x \neq -y. \tag{11}$$

REMARK 4. Let $(X, \|\cdot\|)$ be a normed space. Then the function $f(x) = \|x\|$ satisfies equation (11) if and only if X is an inner product space.

PROOF. Assume that the function $f(x) = \|x\|$ satisfies equation (11). Take two James orthogonal vectors x, y. Then

$$\|x-y\|\,\|x+y\| = g(1)(\|x\|^2 + \|y\|^2),$$

which is obviously true also for $x = y = 0$. Taking here $y = 0$, we obtain $g(1) = 1$ which means that

$$\|x+y\| = \|x-y\| \Rightarrow \|x+y\|\,\|x-y\| = \|x\|^2 + \|y\|^2, \quad x, y \in X,$$

i.e. the norm satisfies equation (9). In view of Theorem 3, this means that X is an inner product space.

Let now X be an inner product space. We are going to show that the function $f(x) = \|x\|$ satisfies the equation (11) with function

$$g(a) := \frac{2a}{a^2+1}, \quad a \in [0, \infty). \tag{12}$$

The desired equality

$$\|x+y\|\,\|x-y\| = g\left(\frac{\|x-y\|}{\|x+y\|}\right)(\|x\|^2 + \|y\|^2)$$

is equivalent to the following one

$$g\left(\frac{\|x-y\|}{\|x+y\|}\right) = \frac{\|x+y\|\,\|x-y\|}{\|x\|^2 + \|y\|^2}.$$

Using here the form of g from (12) we obtain the following equation

$$\frac{2\frac{\|x-y\|}{\|x+y\|}}{\left(\frac{\|x-y\|}{\|x+y\|}\right)^2 + 1} = \frac{\|x+y\|\,\|x-y\|}{\|x\|^2 + \|y\|^2}$$

which has to be proved. Multiplying the denominator and the nominator of the left-hand side of this equality by $\|x+y\|^2$ we get

$$\frac{2\|x-y\|\,\|x+y\|}{\|x-y\|^2+\|x+y\|^2}=\frac{\|x+y\|\,\|x-y\|}{\|x\|^2+\|y\|^2},$$

which is true since X is an inner product space. □

Under some assumptions concerning the functions considered, the Ptolemaic equation has been solved in a paper of M. Fochi [4]. This equation was considered in connection with the orthogonal d'Alembert's equation (see [3]). Using this result we shall solve our unconditional equation (under the same assumptions). It will be seen that also in this case we do not loose the solutions of orthogonal equation.

THEOREM 4. *Let $(X,(\cdot|\cdot))$ be a real inner product space of dimension at least* 3. *Then a nonnegative (nonpositive) function $f : X \to \mathbb{R}$ satisfies equation* (11) *and the condition*

$$f(2x)=2f(x) \tag{12}$$

if and only if

$$f(x)=c^2\|x\|$$

(respectively $f(x)=-c^2\|x\|$) for all $x\in X$ and some $c\in\mathbb{R}$.

PROOF. As we have already checked, functions of the form described above yield a solution to equation (11). It is clear that they also satisfy the additional assumptions. Consequently it suffices to show that every solution of equation (11) satisfying the above conditions is of the form $f(x)=k\|x\|, x\in X$.

Let us assume that a function f satisfies equation (11) with some function g. We have

$$((x,y)\neq(0,0), x\perp y)\Rightarrow f(x+y)f(x-y)=g(1)[f(x)^2+f(y)^2].$$

If now $g(1)=1$ then f satisfies equation (9). Margherita Fochi [4] has shown that a constant sign solution of equation (9) satisfying (13) is of the desired form.

Consider the case of $g(1)\neq 1$. Then, taking $y=0$, we get

$$f(x)^2=g(1)[f(x)^2+f(0)^2].$$

Now, from the condition (13), we have $f(0) = 0$ which together with the last equation gives us $f = 0$. Thus the theorem has been proved. □

References

[1] D. Amir, *Characterizations of Inner Product Spaces*, Birkhäuser Verlag, Basel–Boston–Stuttgart 1986.

[2] M. M. Day, *Some characterizations of inner product spaces*, Trans. Amer. Math. Soc. **62** (1947), 320–337.

[3] M. Fochi, *D'Alembert's functional equation on restricted domains*, Aequationes Math. **52** (1996), 246–253.

[4] M. Fochi, *Characterization of special classes of solutions for some functional equations on orthogonal vectors*, Aequationes Math. **59** (2000), 150–159.

[5] J. Rätz, *On orthogonally additive mappings*, Aequationes Math. **28** (1985), 35–49.

[6] T. Szostok, *On a modified version of Jensen Inequality*, J. of Inequal. Appl. (1999), **3**, 331–347.

[7] T. Szostok, *Modified version of Jensen equation and orthogonal additivity*, Publ. Math. Debrecen **58/3** (2001), 491–504.

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