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DIAGONALIZING THE TRACE FORM IN SOME NUMBER FIELDS

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Abstract. In the present paper the trace form on the ring of integers of a number field is considered. All quadratic fields are determinated for which the form can be diagonalized, i.e. the quadratic fields with an integral basis orthogonal with respect to the trace. There are also given examples of fields of higher degree with the same property.

0. Introduction

In [B] Eva Bayer-Fluckiger investigated lattices with an integral bilinear symmetric form

$$b:\ I\times I \longrightarrow \mathbb{Z}$$

$$b(x,y)=\mathrm{Tr}(ax\overline{y}),$$

where I is a fractional ideal of a number field F, $a\in F$ is an appropriate scaling factor, $\mathrm{Tr}=\mathrm{Tr}_{F/\mathbb{Q}}$ is the absolute trace, and $y\mapsto\overline{y}$ is an involution of F.

In particular, she asked which lattices can occur in this way, and she got a partial answer in the case when the involution is nontrivial.

In the present paper we consider the case when the involution is trivial, $a=1$, and $I=\mathcal{O}_F$ is the ring of integers of a number field F. We ask when the trace form can be diagonalized, i.e. when in $\mathcal{O}_F$ there is an integral basis orthogonal with respect to the trace. We determine all quadratic number fields with this property, next we extend our results to some composita of quadratic fields.

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At the end of the paper we give a list of orthogonal integral bases in quadratic imaginary fields with discriminants $-d \equiv 1 \pmod 4$, $3 \leq d < 500$, provided such a basis exists. These examples have been computed using the package GP/PARI.

1. Notation

Let $F = \mathbb{Q}(\sqrt{d})$, where d is a squarefree integer, be a quadratic number field. Let $\circ : F \times F \longrightarrow \mathbb{Q}$ be a pairing defined by

$$\alpha \circ \beta = \mathrm{Tr}(\alpha\beta), \quad \text{for} \quad \alpha, \beta \in F,$$

where $\mathrm{Tr} = \mathrm{Tr}_{F/\mathbb{Q}}$ is the trace.

Denote

$$\omega = \begin{cases} \sqrt{d}, & \text{if } d \equiv 2,3 \pmod 4, \\ \frac{1+\sqrt{d}}{2}, & \text{if } d \equiv 1 \pmod 4. \end{cases}$$

It is known that $1, \omega$ is an integral basis in F. Obviously, $1 \circ \omega = 0$, if $d \equiv 2, 3 \pmod 4$, i.e. in this case the integral basis $1, \omega$ is orthogonal.

If $d \equiv 1 \pmod 4$ then $1 \circ \omega = \mathrm{Tr}(\omega) = 1 \neq 0$, thus this integral basis is not orthogonal.

In the present note we shall characterize all $d \equiv 1 \pmod 4$ such that in $F = \mathbb{Q}(\sqrt{d})$ there is an orthogonal integral basis.

2. Main Results

In theorems below we give some conditions equivalent to the existence of an orthogonal integral basis in the quadratic number field with the discriminant $d \equiv 1 \pmod 4$

THEOREM 1. *Let $F = \mathbb{Q}(\sqrt{d})$, where $d \equiv 1 \pmod 4$ is squarefree. Then the following conditions are equivalent:*

(i) There is an orthogonal integral basis in F,

(ii) There are $p, q, r, s \in \mathbb{Z}$ satisfying

$$ps - qr = 1 \tag{1}$$

and

$$(2p+q)(2r+s) + qsd = 0, \tag{2}$$

(iii) There are $p, q \in \mathbb{Z}$ such that for $t = 2p + q$ and $\Delta = t^2 + dq^2$ the numbers

$$r = -\frac{t + dq}{\Delta}, \quad s = \frac{2t}{\Delta}, \tag{3}$$

are integers.

PROOF. Let $p, q, r, s \in \mathbb{Z}$. Then

$$\beta_1 = p + q\omega,$$
$$\beta_2 = r + s\omega,$$

is an integral basis iff the matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ is invertible in $\mathbb{Z}$. We may even assume that (1) holds changing the sign of β_2 if necessary.

Computing the trace we get

$$\beta_1 \circ \beta_2 = \mathrm{Tr}(pr + (ps + qr)\omega + qs\omega^2) = \frac{1}{2}((2p + q)(2r + s) + qsd).$$

Thus β_1, β_2 is an orthogonal integral basis if and only if there exist $p, q, r, s \in \mathbb{Z}$ satisfying (1) and (2).

Using Cramer's rule we determine r and s from (1) and (2) and we get (3).

THEOREM 2. *Under assumptions of Theorem* 1 *the following conditions are equivalent:*

(i) There is an orthogonal integral basis in F,

(iv) There is $\Delta' \mid d$ such that 2 *is represented over $\mathbb{Z}$ by the form $\Delta' X^2 + \Delta'' Y^2$, where $\Delta'' = d/\Delta'$.*

PROOF. We use the above notation, and we shall prove that the equivalent conditions of Theorem 1 imply (iv).

Taking the equality (2) modulo 4, it follows easily from (1) and (2) that qs is odd. Hence $t = 2p + q$ is odd and $\Delta = t^2 + dq^2 \equiv 2 (\mathrm{mod}\ 4)$. Put $\Delta' = \Delta/2$, then Δ' is odd and from (3) we get $\Delta' \mid t + dq$, $\Delta' \mid t$. Since $gcd(p, q) = 1$ by (1), then $gcd(t, q) = 1$ and from the above divisibilities it follows that $\Delta' \mid dq$ and $\Delta' \mid t$, hence $\Delta' \mid d$. Denote $\Delta'' = d/\Delta'$.

Now, $2\Delta' = \Delta = t^2 + dq^2 = (s\Delta')^2 + \Delta'\Delta'' q^2 = \Delta'(\Delta' s^2 + \Delta'' q^2)$, i.e. $2 = \Delta' s^2 + \Delta'' q^2$ and (iv) holds.

Conversely, if $2 = \Delta' x^2 + \Delta'' y^2$, where $\Delta'\Delta'' = d$ and $x, y \in \mathbb{Z}$, then evidently x, y are odd (since d is odd). Then $p = \dfrac{\Delta' x - y}{2}$ and $q = y$ are integers and we shall prove that r, s defined by (3) are integers. In fact, $t = 2p + q = \Delta' x$ and hence

$$\Delta = t^2 + dq^2 = (\Delta' x)^2 + \Delta'\Delta'' y^2 = \Delta'(\Delta' x^2 + \Delta'' y^2) = 2\Delta',$$

and

$$t+dq=\Delta' x+dy=\Delta'(x+\Delta'' y).$$

Then $t+dq$ is divisible by $2\Delta'=\Delta$, i.e. $r=-\frac{t+dq}{\Delta}$ is an integer. Similarly $s=\frac{2t}{\Delta}=\frac{t}{\Delta'}=x$ is an integer.

Corollary 1. *If $F=\mathbb{Q}(\sqrt{d})$ where $d\equiv 1 (\mathrm{mod}\ 4)$ is squarefree and $d>1$ then in F there is no orthogonal integral basis.*

Proof. From the assumption it follows that $d\geq 5$. Then 2 cannot be represented by the form $\Delta' X^2+\Delta'' Y^2$, where $\Delta'\Delta''=d$, since Δ',Δ'' have the same sign and $\Delta'\Delta''\geq 5$.

Let $\varepsilon=u+v\sqrt{a}$ be the fundamental unit of the field $\mathbb{Q}(\sqrt{a})$ where $a\equiv 3(\mathrm{mod}\ 4)$, $a>0$ is squarefree. Then $N\varepsilon=u^2-av^2=1$, since $a\equiv 3(\mathrm{mod}\ 4)$. Denote $\varepsilon^n=u_n+v_n\sqrt{a}$, for $n\in\mathbb{Z}$. It is easy to observe that u is odd iff v is even iff all u_n are odd. Thus if u_n is even for some n then u is even.

Theorem 3. *Let $F=\mathbb{Q}(\sqrt{d})$ where $d\equiv 1(\mathrm{mod}\ 4)$ is squarefree and $d<0$. Let $\varepsilon=u+v\sqrt{a}$ be the fundamental unit of the field $\mathbb{Q}(\sqrt{a})$, where $a=-d>0$. Then the following conditions are equivalent:*
(i) There is an orthogonal integral basis in F,
(v) u is even.

Proof. $(v)\Rightarrow(i)$. Assume that u is even. Then from $1=N\varepsilon=u^2-av^2$ it follows that $(u+1)(u-1)=av^2$ and $gcd(u+1,u-1)=1$. Consequently

$$\begin{aligned}u+1&=a_1x^2,\\ u-1&=a_2y^2\end{aligned}$$

where $a_1a_2=a$ and $xy=v$. Subtracting we obtain $2=a_1x^2-a_2y^2$ and taking $\Delta'=a_1$, $\Delta''=-a_2$ we get (iv) since $\Delta'\Delta''=-a_1a_2=-a=d$. The claim follows from Theorem 2.

$(i)\Rightarrow(v)$. In view of Theorem 2 there are $\Delta'\mid d$ and $x,y\in\mathbb{Z}$ satisfying $\Delta' x^2+\Delta'' y^2=2$, where $\Delta''=d/\Delta'$. Then $\Delta'\Delta''=-a$. Let us observe that $U=\Delta' x^2-1=1-\Delta'' y^2$ and $V=xy$ satisfy

$$U^2-aV^2=(\Delta' x^2-1)(1-\Delta'' y^2)+\Delta'\Delta'' x^2y^2=\Delta' x^2+\Delta'' y^2-1=1.$$

Moreover U is even. Therefore from the observation before Theorem 3 it follows that (v) holds.

3. Examples

First we consider the case where $-d$ is a prime number.

THEOREM 4. *Let $p \equiv 3 (\mathrm{mod}\ 4)$ be a prime number, and let $\varepsilon = u + v\sqrt{p}$ be the fundamental unit of the field $\mathbb{Q}(\sqrt{p})$. Then u is even.*

PROOF. As we have observed above, $N\varepsilon = 1$, hence $(u+1)(u-1) = pv^2$. If u is odd, then v is even, thus $u = 2u_1 + 1$, $v = 2v_1$. Then $(u_1+1)u_1 = pv_1^2$. Hence

$$\begin{aligned} u_1 + 1 &= py^2 \\ u_1 &= x^2 \end{aligned} \qquad \text{or} \qquad \begin{aligned} u_1 + 1 &= x^2 \\ u_1 &= py^2 \end{aligned}$$

for some positive x, y satisfying $xy = v_1$.

The first case is impossible modulo 4. In the second case subtracting we get

$$x^2 - py^2 = 1$$

where $x \le u_1 + 1 < u$. This contradicts the minimality of u.

SECOND PROOF It is known that the class number of the field $\mathbb{Q}(\sqrt{p})$ is odd, and 2 ramifies in $\mathbb{Q}(\sqrt{p})$, i.e. $(2) = \mathfrak{p}^2$, $N\mathfrak{p} = 2$. Then the ideal class containing $\mathfrak{p}$ has order ≤ 2, thus $\mathfrak{p}$ is principal, $\mathfrak{p} = (x + y\sqrt{p})$. Hence taking norms we get $2 = N\mathfrak{p} = |x^2 - py^2|$ and the condition (iv) of Theorem 2 is satisfied.

COROLLARY 2. *If $p \equiv 3 (\mathrm{mod}\ 4)$ is a prime number, then in $\mathbb{Q}(\sqrt{-p})$ there is an orthogonal integral basis.*

THEOREM 5. *Let p, q be prime numbers, $pq \equiv 3 (\mathrm{mod}\ 4)$, let $\varepsilon = u + v\sqrt{pq}$ be the fundamental unit of the field $\mathbb{Q}(\sqrt{pq})$. If $\left(\frac{p}{q}\right) = -1$ then u is even.*

PROOF. We have $N\varepsilon = u^2 - pqv^2 = 1$, where $u, v > 0$. We may assume that $p \equiv 1 (\mathrm{mod}\ 4)$, $q \equiv 3 (\mathrm{mod}\ 4)$.

Suppose that u is odd. Then v is even, $u = 2u_1 + 1$, $v = 2v_1$. Consequently $u_1(u_1 + 1) = pqv_1^2$.

There are four possibilities:

$$u_1 + 1 = pqx^2, x^2, px^2, qx^2 \text{ and respectively, } u_1 = y^2, pqy^2, qy^2, py^2,$$

where x, y are positive integers of different parity and $xy = v_1$. Subtracting we get respectively

$$1 = pqx^2 - y^2, \quad 1 = x^2 - pqy^2, \quad 1 = px^2 - qy^2, \quad 1 = qx^2 - py^2.$$

The first and the last equalities are impossible modulo 4. From the third one it follows that p is a quadratic residue modulo q, contrary to the assumption. From the second equality we get $N(x+y\sqrt{pq})=1$, then by the minimality of u we have $u \le x \le v_1 \le u_1 < u$, contradiction.

COROLLARY 3. *If p, q are prime numbers satisfying $pq \equiv 3 (\mathrm{mod}\ 4)$, $\left(\frac{p}{q}\right) = -1$ then in $\mathbb{Q}(\sqrt{-pq})$ there is an orthogonal integral basis.*

4. Quartic fields

Basing on the above results it is easy to give examples of quartic bicyclic fields with orthogonal integral bases.

We fix the following notation. For $j=1,2$, let $K_j=\mathbb{Q}(\sqrt{d_j})$ be the quadratic number field of discriminant d_j, where $d_1 \ne d_2$. Then $K=K_1K_2$ is a quartic bicyclic field. Denote $\mathrm{Tr}_j = \mathrm{Tr}_{K_j/\mathbb{Q}}$, $\mathrm{Tr}_j{}' = \mathrm{Tr}_{K/K_j}$ and $\mathrm{Tr} = \mathrm{Tr}_{K/\mathbb{Q}}$.

Suppose that $\beta_1^{(j)}, \beta_2^{(j)}$ is an integral basis of K_j, and consider the set

$$B=\{\beta_i^{(1)}\beta_k^{(2)} \ : \ 1 \le i,k \le 2\}.$$

We shall use the following Theorem 88 of Hilbert:

THEOREM 6. *Under the above notation if $gcd(d_1,d_2)=1$ then the discriminant of K equals $(d_1d_2)^2$, and B is an integral basis of K.*

PROOF. See [H], Theorem 88.

THEOREM 7. *Under the above notation if $\beta_1^{(j)}, \beta_2^{(j)}$ is an orthogonal integral basis of K_j, for $j=1,2$, then B is orthogonal with respect to* Tr.

If moreover $gcd(d_1,d_2)=1$ then B is an orthogonal integral basis of K.

PROOF. Let $i,k,l,m \in \{1,2\}$. Since $\mathrm{Tr} = \mathrm{Tr}_1 \circ \mathrm{Tr}_1{}'$, $\mathrm{Tr}_1{}' \,|\, K_2 = \mathrm{Tr}_2$, and $\mathrm{Tr}_1{}'$ is K_1-linear, then

$$\mathrm{Tr}\left(\beta_i^{(1)}\beta_k^{(2)}\cdot\beta_l^{(1)}\beta_m^{(2)}\right) = \mathrm{Tr}_1\left(\mathrm{Tr}_1{}'\left(\beta_i^{(1)}\beta_l^{(1)}\cdot\beta_k^{(2)}\beta_m^{(2)}\right)\right)$$

$$= \mathrm{Tr}_1\left(\beta_i^{(1)}\beta_l^{(1)}\cdot\mathrm{Tr}_1{}'\left(\beta_k^{(2)}\beta_m^{(2)}\right)\right) = \mathrm{Tr}_1\left(\beta_i^{(1)}\beta_l^{(1)}\cdot\mathrm{Tr}_2\left(\beta_k^{(2)}\beta_m^{(2)}\right)\right)$$

$$= \mathrm{Tr}_1\left(\beta_i^{(1)}\beta_l^{(1)}\right)\cdot\mathrm{Tr}_2\left(\beta_k^{(2)}\beta_m^{(2)}\right) = 0$$

unless $i = l$ and $k = m$. Thus B is orthogonal with respect to Tr.

Now, the second part of the theorem follows from the theorem of Hilbert.

COROLLARY 4. *If d, d' are relatively prime squarefree integers $\neq 1$ and $d \equiv 1 (\mathrm{mod}\ 4)$, and in the fields $\mathbb{Q}(\sqrt{d}), \mathbb{Q}(\sqrt{d'})$ there are orthogonal integral bases, then in the field $\mathbb{Q}(\sqrt{d}, \sqrt{d'})$ there is an orthogonal integral basis.*

PROOF. Let $d_1 = d$ and $d_2 = d'$, resp. $4d'$ if $d' \equiv 1 (\mathrm{mod}\ 4)$ resp. $d' \equiv 2, 3 (\mathrm{mod}\ 4)$. Then d_1 and d_2 satisfy the assumptions of the second part of Theorem 7.

5. Remarks

1) There are fields $\mathbb{Q}(\sqrt{-pq})$ not satisfying the assumptions of Corollary 3 with an orthogonal integral basisĖ.g. for $p = 17$, $q = 19$ we have the fundamental unit $\varepsilon = 18 + \omega$, in $\mathbb{Q}(\sqrt{pq})$. Then by Theorem 3 in the field $\mathbb{Q}(\sqrt{-pq})$ there is an orthogonal integral basis. We have also $\left(\frac{17}{19}\right) = 1$.

2) It is easy to see that in $\mathbb{Q}(\sqrt{d})$, for $d \equiv 2, 3 (\mathrm{mod}\ 4)$, the orthogonal integral basis is unique up to a permutation and sign changes. On the other hand due to the infinity of solutions of the Pell equation, in the case $d \equiv 1 (\mathrm{mod}\ 4)$ if there exists an orthogonal integral basis, then the number of such bases is infinite.

3) We do not know any cubic field with an orthogonal integral basis. We do not know if there is an orthogonal integral basis in a quartic field not satisfying the assumptions of Theorem 7.

4) One can generalize the case of quartic fields as follows. Let $d_1, \ldots, d_r$ be pairwise relatively prime squarefree integers $\neq 1$ satisfying $d_j \equiv 1 (\mathrm{mod}\ 4)$ for $j = 1, 2, \ldots, r - 1$. If in the field $K_j = \mathbb{Q}(\sqrt{d_j})$, for $j = 1, 2, \ldots, r$ there is an orthogonal integral basis, then in the field $K = K_1 K_2 \cdots K_r$ there is an orthogonal integral basis.

In view of Corollary 2 this gives examples of fields with an orthogonal integral basis of arbitrary large degrees.

5) On the other hand, if we consider the hermitian pairing (see e.g. [B]): $\alpha \circ \beta = \mathrm{Tr}(\alpha\overline{\beta})$, where $\overline{\beta}$ is the complex conjugate of β, then it is easy to see that in $\mathbb{Q}(\sqrt{d})$ there is an orthogonal integral basis if and only if $d \equiv 2, 3 (\mathrm{mod}\ 4)$. Then $1, \omega$ is an orthogonal integral basis. Namely, for $d \equiv 1 (\mathrm{mod}\ 4)$ the equality (1) and the orthogonality condition analogous to (2): $(2p + q)(2r + s) - qsd = 0$ give a contradiction modulo 4.

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informing me on his results. In particular he has proved independently Theorem 3 above.

6. The table

In the table below we give an orthogonal integral basis in $\mathbb{Q}(\sqrt{d})$, where $-500 < d < 0$, $d \equiv 1 (\mathrm{mod}\ 4)$ is squarefree, provided such a basis exists. The table has been computed using the package GP/PARI, version 1.39 (see [BBCO]).

For every d in question there are given (in the above notation) $-d$, $[\Delta', \Delta'']$, $[x, y]$ satisfying $x^2\Delta' + y^2\Delta'' = 2$ and an orthogonal integral basis $[\beta_1, \beta_2] = [p + q\omega, r + s\omega]$. We can always assume that $\Delta'' < 0$. If for some d such a line is empty then in $\mathbb{Q}(\sqrt{d})$ there is no orthogonal integral basis. E.g. it is the case for $d = -39, -55, -95$, etc.

Table

$-d$	$[\Delta', \Delta'']$	$[x, y]$	$[\beta_1, \beta_2]$
3	$[3, -1]$	$[1, 1]$	$[\omega + 1, \omega]$
7	$[1, -7]$	$[3, 1]$	$[\omega + 1, 3\omega + 2]$
11	$[11, -1]$	$[1, 3]$	$[3\omega + 4, \omega + 1]$
15	$[5, -3]$	$[1, 1]$	$[\omega + 2, \omega + 1]$
19	$[19, -1]$	$[3, 13]$	$[13\omega + 22, 3\omega + 5]$
23	$[1, -23]$	$[5, 1]$	$[\omega + 2, 5\omega + 9]$
31	$[1, -31]$	$[39, 7]$	$[7\omega + 16, 39\omega + 89]$
35	$[7, -5]$	$[1, 1]$	$[\omega + 3, \omega + 2]$
39			
43	$[43, -1]$	$[9, 59]$	$[59\omega + 164, 9\omega + 25]$
47	$[1, -47]$	$[7, 1]$	$[\omega + 3, 7\omega + 20]$
51	$[51, -1]$	$[1, 7]$	$[7\omega + 22, \omega + 3]$
55			
59	$[59, -1]$	$[3, 23]$	$[23\omega + 77, 3\omega + 10]$
67	$[67, -1]$	$[27, 221]$	$[221\omega + 794, 27\omega + 97]$
71	$[1, -71]$	$[59, 7]$	$[7\omega + 26, 59\omega + 219]$
79	$[1, -79]$	$[9, 1]$	$[\omega + 4, 9\omega + 35]$
83	$[83, -1]$	$[1, 9]$	$[9\omega + 37, \omega + 4]$
87	$[29, -3]$	$[1, 3]$	$[3\omega + 13, \omega + 4]$
91	$[7, -13]$	$[15, 11]$	$[11\omega + 47, 15\omega + 64]$
95			
103	$[1, -103]$	$[477, 47]$	$[47\omega + 215, 477\omega + 2182]$
107	$[107, -1]$	$[3, 31]$	$[31\omega + 145, 3\omega + 14]$
111			
115	$[23, -5]$	$[7, 15]$	$[15\omega + 73, 7\omega + 34]$
119	$[1, -119]$	$[11, 1]$	$[\omega + 5, 11\omega + 54]$
123	$[123, -1]$	$[1, 11]$	$[11\omega + 56, \omega + 5]$
127	$[1, -127]$	$[2175, 193]$	$[193\omega + 991, 2175\omega + 11168]$
131	$[131, -1]$	$[9, 103]$	$[103\omega + 538, 9\omega + 47]$
139	$[139, -1]$	$[747, 8807]$	$[8807\omega + 47513, 747\omega + 4030]$
143	$[13, -11]$	$[1, 1]$	$[\omega + 6, \omega + 5]$
151	$[1, -151]$	$[41571, 3383]$	$[3383\omega + 19094, 41571\omega + 234631]$

$-d$	$[\Delta', \Delta'']$	$[x, y]$	$[\beta_1, \beta_2]$
155			
159	$[53, -3]$	$[5, 21]$	$[21\,\omega + 122, 5\,\omega + 29]$
163	$[163, -1]$	$[627, 8005]$	$[8005\,\omega + 47098, 627\,\omega + 3689]$
167	$[1, -167]$	$[13, 1]$	$[\omega + 6, 13\,\omega + 77]$
179	$[179, -1]$	$[153, 2047]$	$[2047\,\omega + 12670, 153\,\omega + 947]$
183			
187	$[187, -1]$	$[3, 41]$	$[41\,\omega + 260, 3\,\omega + 19]$
191	$[1, -191]$	$[2999, 217]$	$[217\,\omega + 1391, 2999\,\omega + 19224]$
195	$[15, -13]$	$[1, 1]$	$[\omega + 7, \omega + 6]$
199	$[1, -199]$	$[127539, 9041]$	$[9041\,\omega + 59249, 127539\,\omega + 835810]$
203			
211	$[211, -1]$	$[36321, 527593]$	$[527593\,\omega + 3568069, 36321\,\omega + 245636]$
215	$[5, -43]$	$[3, 1]$	$[\omega + 7, 3\,\omega + 20]$
219	$[3, -73]$	$[5, 1]$	$[\omega + 7, 5\,\omega + 34]$
223	$[1, -223]$	$[15, 1]$	$[\omega + 7, 15\,\omega + 104]$
227	$[227, -1]$	$[1, 15]$	$[15\,\omega + 106, \omega + 7]$
231	$[77, -3]$	$[1, 5]$	$[5\,\omega + 36, \omega + 7]$
235	$[47, -5]$	$[1, 3]$	$[3\,\omega + 22, \omega + 7]$
239	$[1, -239]$	$[2489, 161]$	$[161\,\omega + 1164, 2489\,\omega + 17995]$
247	$[13, -19]$	$[81, 67]$	$[67\,\omega + 493, 81\,\omega + 596]$
251	$[251, -1]$	$[121, 1917]$	$[1917\,\omega + 14227, 121\,\omega + 898]$
255	$[17, -15]$	$[1, 1]$	$[\omega + 8, \omega + 7]$
259			
263	$[1, -263]$	$[373, 23]$	$[23\,\omega + 175, 373\,\omega + 2838]$
267	$[267, -1]$	$[3, 49]$	$[49\,\omega + 376, 3\,\omega + 23]$
271	$[1, -271]$	$[340551, 20687]$	$[20687\,\omega + 159932, 340551\,\omega + 2632813]$
283	$[283, -1]$	$[699, 11759]$	$[11759\,\omega + 93029, 699\,\omega + 5530]$
287	$[1, -287]$	$[17, 1]$	$[\omega + 8, 17\,\omega + 135]$
291	$[291, -1]$	$[1, 17]$	$[17\,\omega + 137, \omega + 8]$
295			
299			
303	$[101, -3]$	$[5, 29]$	$[29\,\omega + 238, 5\,\omega + 41]$
307	$[307, -1]$	$[537, 9409]$	$[9409\,\omega + 77725, 537\,\omega + 4436]$
311	$[1, -311]$	$[4109, 233]$	$[233\,\omega + 1938, 4109\,\omega + 34177]$
319	$[29, -11]$	$[667, 1083]$	$[1083\,\omega + 9130, 667\,\omega + 5623]$
323	$[19, -17]$	$[1, 1]$	$[\omega + 9, \omega + 8]$
327			
331	$[331, -1]$	$[2900979, 52778687]$	$[52778687\,\omega + 453722681, 2900979\,\omega + 24938854]$
335	$[5, -67]$	$[11, 3]$	$[3\,\omega + 26, 11\,\omega + 95]$
339	$[339, -1]$	$[17, 313]$	$[313\,\omega + 2725, 17\,\omega + 148]$
347	$[347, -1]$	$[43, 801]$	$[801\,\omega + 7060, 43\,\omega + 379]$
355			
359	$[1, -359]$	$[19, 1]$	$[\omega + 9, 19\,\omega + 170]$
367	$[1, -367]$	$[137913, 7199]$	$[7199\,\omega + 65357, 137913\,\omega + 1252060]$
371			
379	$[379, -1]$	$[5843427, 113759383]$	$[113759383\,\omega + 1050449725, 5843427\,\omega + 53957978]$
383	$[1, -383]$	$[137, 7]$	$[7\,\omega + 65, 137\,\omega + 1272]$
391	$[1, -391]$	$[2709, 137]$	$[137\,\omega + 1286, 2709\,\omega + 25429]$
395			
399	$[21, -19]$	$[1, 1]$	$[\omega + 10, \omega + 9]$
403	$[31, -13]$	$[147, 227]$	$[227\,\omega + 2165, 147\,\omega + 1402]$

$-d$	$[\Delta', \Delta'']$	$[x, y]$	$[\beta_1, \beta_2]$
407			
411	$[411, -1]$	$[11, 223]$	$[223\,\omega + 2149, 11\,\omega + 106]$
415	$[5, -83]$	$[1919, 471]$	$[471\,\omega + 4562, 1919\,\omega + 18587]$
419	$[419, -1]$	$[803, 16437]$	$[16437\,\omega + 160010, 803\,\omega + 7817]$
427	$[7, -61]$	$[3, 1]$	$[\omega + 10, 3\,\omega + 29]$
431	$[1, -431]$	$[12311, 593]$	$[593\,\omega + 5859, 12311\,\omega + 121636]$
435	$[3, -145]$	$[7, 1]$	$[\omega + 10, 7\,\omega + 69]$
439	$[1, -439]$	$[21, 1]$	$[\omega + 10, 21\,\omega + 209]$
443	$[443, -1]$	$[1, 21]$	$[21\,\omega + 211, \omega + 10]$
447	$[149, -3]$	$[1, 7]$	$[7\,\omega + 71, \omega + 10]$
451	$[451, -1]$	$[321, 6817]$	$[6817\,\omega + 68977, 321\,\omega + 3248]$
455	$[65, -7]$	$[1, 3]$	$[3\,\omega + 31, \omega + 10]$
463	$[1, -463]$	$[15732537, 731153]$	$[731153\,\omega + 7500692, 15732537\,\omega + 161395651]$
467	$[467, -1]$	$[59, 1275]$	$[1275\,\omega + 13139, 59\,\omega + 608]$
471			
479	$[1, -479]$	$[1729, 79]$	$[79\,\omega + 825, 1729\,\omega + 18056]$
483	$[23, -21]$	$[1, 1]$	$[\omega + 11, \omega + 10]$
487	$[1, -487]$	$[7204587, 326471]$	$[326471\,\omega + 3439058, 7204587\,\omega + 75893395]$
491	$[491, -1]$	$[13809, 305987]$	$[305987\,\omega + 3237116, 13809\,\omega + 146089]$
499	$[499, -1]$	$[3, 67]$	$[67\,\omega + 715, 3\,\omega + 32]$

References

[BBCO] D. Batut, C. Bernardi, H. Cohen, M. Olivier, *GP-PARI, a computer package, version* 1.39.

[B] E. Bayer–Fluckiger, *Lattices and number fields*, Contemp. Math. **241**, 1999, 69–84.

[H] D. Hilbert, *Die Theorie der algebraischen Zahlkörper*, Jahresber. Deutsch. Math. Verein. **4**, 1897, 175–546 (= *Ges. Abhandlungen* vol. I, p. 63–362, Chelsea, New York, 1963).

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