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ON A FUNCTIONAL EQUATION CONNECTED WITH PTOLEMAIC INEQUALITY

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In the first part of the present paper we consider Ptolemaic inequality and give some necessary conditions for its solutions. The other part is devoted to solution of some functional equation which, simultaneously, satisfies the Ptolemaic inequality.

1. Let $(X,+)$ be an Abelian group. In what follows we assume that a real function $f : X \to \mathbb{R}$ satisfies Ptolemaic inequality:

$$f(x-y)f(z) \leq f(y-z)f(x) + f(x-z)f(y) \tag{I}$$

for all $x, y, z \in X$.

REMARK 1. *The function f satisfies the following conditions:*

(1) *if* $f(0) > 0$, *then* $f(x) \geq 0$ *for every* $x \in X$;

(2) *if* $f(0) < 0$, *then* $f(x) \leq 0$ *for every* $x \in X$;

(3) *if* $f(0) = 0$, *then* $f(x) = f(-x)$ *for every* $x \in X$.

PROOF. If $y = z = 0$, then from (I) it follows that

$$f(x)f(0) \leq f(0)f(x) + f(x)f(0)$$

for every $x \in X$. Hence $f(x)f(0) \geq 0, x \in X$ and, consequently, conditions (1) and (2) hold.

However, if $f(0) = 0, x = z$ and $y = 0$, then (I) implies the inequality:

$$f^2(x) \leq f(-x)f(x), \quad x \in X,$$

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and, consequently,

$$f^2(-x) \leq f(x)f(-x), \quad x \in X.$$

Hence if follows that

$$0 \geq f(x)^2 + f(-x)^2 - 2(f(x)f(-x)) = (f(x) - f(-x))^2, \; x \in X,$$

and, consequently, $f(x) = f(-x)$ for every $x \in X$.

THEOREM 1. *If $f(0) = 0$ and $f(a) = 0$ for some $a \in X$, then $f(x-a) = f(x) = f(x+a)$ for every $x \in X$. Moreover, f is either non–negative or f is a non–positive function.*

PROOF. From condition (3) it follows that f is even. Moreover, (I) implies that

$$f(x-a)f(z) \leq f(a-z)f(x), \quad x, z \in X.$$

This jointly with the evenness of f implies that

$$f(x-a)f(z) \leq f(z-a)f(x) \leq f(x-a)f(z), \quad x, z \in X.$$

Thus

$$f(x-a)f(z) = f(z-a)f(x) \quad \text{for all} \quad x, z \in X.$$

We can assume that f does not vanish identically. Let $b \in X$ be such that $f(b) \neq 0$. Since

$$f(x-a)f(b) = f(b-a)f(x), \quad x \in X,$$

we have

$$f(x-a) = \frac{f(b-a)}{f(b)} f(x), \quad x \in X.$$

Put

$$c := \frac{f(b-a)}{f(b)}.$$

If $c = 0$, then $f(x-a) = 0$ for every $x \in X$, and, consequently, $f(b) = f((b+a)-a) = 0$; a contradiction. Therefore $c \neq 0$ and

$$f(x-a) = cf(x), \quad x \in X.$$

Moreover

$$f(x+a) = f(-x-a) = cf(-x) = cf(x), \quad x \in X,$$

whence

$$f(x-a) = cf(x) = f(x+a), \quad x \in X. \tag{4}$$

Moreover, we observe that

$$f(x) = f((x+a)-a) = cf(x+a) = c^2 f(x), \quad x \in X;$$

in particular,

$$f(b) = c^2 f(b),$$

and, consequently, $c^2 = 1$. Now, (I) and the evenness of f imply that

$$\begin{aligned} f(x-y)f(z) &\le f(y-z)f(x) + f(x-z)f(y) \\ &\le f(z-x)f(y) + f(y-x)f(z) + f(x-z)f(y) \\ &= 2f(x-z)f(y) + f(x-y)f(z) \end{aligned}$$

for all $x, y, z \in X$ and, therefore,

$$f(x-z)f(y) \ge 0, \quad x, y, z \in X. \tag{5}$$

Now, from conditions (4) and (5) it follows that

$$0 \le f(b-a)f(b) = cf(b)^2.$$

From here we infer that $c \ge 0$, and, finally, $c = 1$. Finally, (5) implies that

$$f(x)f(b) \ge 0, \quad x \in X,$$

whence,

$$\operatorname{sgn} f(x) = \operatorname{sgn} f(b) \quad \text{or} \quad f(x) = 0$$

for every $x \in X$. This completes the proof.

Remark 1 and Theorem 1 imply Corollary 1.

COROLLARY 1. *If $f : X \to \mathbb{R}$ satisfies (I), then either f is non–negative or f is a non–positive function.*

THEOREM 2. *Let $G := \{x \in X : f(x) = 0\}$. If $G \neq \emptyset$, then $(G, +)$ yields a subgroup of the group $(X, +)$.*

PROOF. Suppose that $G \neq \emptyset$ and fix arbitrarily an $x \in G$. Then

$$0 \le f(0)^2 = f(x-x)f(0) \le f(x-0)f(x) + f(x-0)f(x) = 2f(x)^2 = 0.$$

Hence $f(0) = 0$ and $0 \in G$. Therefore, by virtue of Remark 1 f is even, and, consequently, $-x \in G$. Moreover, if $x, y \in G$, then, by Theorem 1, we have

$$f(x-y)f(z) \leq f(y-z)f(x) + f(x-z)f(y) = 0$$

for every $z \in X$. Putting $z = x - y$ we infer that $f(x-y) = 0$ and $x - y \in G$. Since $-y \in G$, we have $x + y = x - (-y) \in G$ which finishes the proof.

REMARK 2. *If a function $f : X \to \mathbb{R}$ satisfies inequality (I) and $c \in \mathbb{R}$, then functions cf and $|f|$ satisfy (I) as well.*

THEOREM 3. *Let $f : X \to \mathbb{R}$ be an even and bounded function satisfying (I). If f is non-negative, then f is subadditive function. If f is non-positive, then f is superadditive function.*

PROOF. Suppose that f is a non-negative and non-zero function. Let $M := \sup f(X), 0 < M < +\infty$. For every $n \in \mathbb{N}$ there exists a $z_n \in X$ such that

$$M - \frac{1}{n} < f(z_n) \leq M.$$

In view of (I) we obtain

$$\begin{aligned} f(x-y)f(z_n) &\leq f(y-z_n)f(x) + f(x-z_n)f(y) \\ &\leq Mf(x) + Mf(y) \end{aligned}$$

for all $x, y \in X$, $n \in N$. Hence

$$Mf(x-y) = \lim_{n\to\infty} f(x-y)f(z_n) \leq Mf(x) + Mf(y)$$

for all $x, y \in X$ and, consequently,

$$f(x-y) \leq f(x) + f(y), \ x, y \in X.$$

From here and from the evenness of f it follows that

$$f(x+y) = f(x-(-y)) \leq f(x) + f(-y) = f(x) + f(y)$$

for all $x, y \in X$, i.e. f is subadditive on X.

Obviously, the zero–function is subadditive, too.

If f is a non–positive and non–zero function, the $-f$ is non–negative and non–zero. Hence f is superadditive because $-f$ subadditive.

In view of the well known inequality:

$$(x+y)^p \leq x^p + y^p, \ x, y \geq 0, \ 0 < p \leq 1, \tag{6}$$

and by Corollary 1 and Remark 2 we obtain

REMARK 3. *If* $p \in (0,1]$ *and a function* $f : X \to \mathbb{R}$ *satisfies (I), then so does* $|f|^p$.

Now, we give some examples of solutions of the Ptolemaic inequality.

EXAMPLE 1. It is well known (see [1], [2], [3]) that if $(X, \|\cdot\|)$ is an inner product space, then $\|\cdot\|$ satisfies (I) and, consequently, so does $\|\cdot\|^p$, where $0 < p \leq 1$ is arbitrarily fixed.

EXAMPLE 2. If $a : X \to \mathbb{R}$ is an additive function and $0 < p \leq 1$, then $|a|^p$ satisfies (I). Moreover, if $(X, +)$ is a uniquely 2–divisible Abelian group, then the following functions:

$$\begin{aligned} f(x) &= |\sin(a(x))|^p, \ x \in X, \\ g(x) &= |\sin h(a(x))|^p, \ x \in X, \\ h(x) &= |\cos h(a(x))|^p, \ x \in X, \end{aligned}$$

yield solutions of (I).

EXAMPLE 3. If $(X, +)$ is a uniquely 2–divisible Abelian group and f is a real or complex solution of Wilson's sine functional equation

$$\text{(II)} \qquad f(x)^2 - f(y)^2 = f(x+y)f(x-y), \ x, y \in X,$$

then $|f|$ satisfies (I) on X.

EXAMPLE 4. Let $(G, +)$ be a subgroup of $(X, +)$ and let $f : X \to \mathbb{R}$ be a function defined by the formula:

$$f(x) = \begin{cases} 0 & \text{for} \quad x \in G \\ c & \text{for} \quad x \in X \backslash G, \end{cases}$$

where $c \in \mathbb{R}$ is arbitrarily fixed. Then f satisfies (I).

Now, we present some necessary conditions for a given function to be a solution of the Ptolemaic inequality.

THEOREM 4. *Let* $f : X \to \mathbb{R}$ *be an even solution of (I). Then the following conditions are satisfied;*

$$\text{(7)} \qquad \left| \frac{f(x+y) - f(x-y)}{f(y)} \right| \leq \frac{f(2x)}{f(x)} \leq \frac{f(x+y) + f(x-y)}{f(y)},$$

provided that $f(x) \neq 0, f(y) \neq 0,\ x, y \in X$;

(8) $\quad | f(x)^2 - f(y)^2 | \leq f(x+y)f(x-y) \leq f(x)^2 + f(y)^2,\ x, y \in X.$

Moreover, if $(X, +)$ *is a uniquely 2–divisible Abelian group, then condition (8) is equivalent to the following one:*

(9)
$$\left| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 \right| \leq f(x)f(y) \leq f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x-y}{2}\right)^2,\ x, y \in X.$$

PROOF. By Corollary 1, we may suppose that f is non–negative. The assumptions imply that

$$f(x-y)f(x) = f(x-y)f(-x) \leq f(y+x)f(x) + f(2x)f(y),$$

for all $x, y \in x$. Hence

$$(f(x-y) - f(x+y))f(x) \leq f(2x)f(y),\ x, y \in X.$$

Now, replacing here y by $-y$, we obtain

$$(f(x+y) - f(x-y))f(x) \leq f(2x)f(y),\ x, y \in X.$$

Thus

(10) $$| f(x+y) - f(x-y) |\ f(x) \leq f(2x)f(y)\ x, y \in X.$$

Moreover

$$\begin{aligned} f(2x)f(y) = f(x-(-x))f(y) &\leq f(-x-y)f(x) + f(x-y)f(-x) \\ &= f(x+y)f(x) + f(x-y)f(x) \end{aligned}$$

for all $x, y \in X$, whence

(11) $$f(2x)f(y) \leq (f(x+y) + f(x-y))f(x),\ x, y \in X.$$

From (10) and (11) we have (7), whenever $f(x) \neq 0, f(y) \neq 0, x, y \in X$.

If f is non–positive, then we can replace f by $-f$ getting (7) in that case.

In order to prove (8) we put $z = x + y$ in (I) to get

(12) $f(x-y)f(x+y) \leq f(-x)f(x)+f(-y)f(y) = f(x)^2+f(y)^2, x, y \in X.$

However, if $x = y + z$, then, by (I), it follows that

(13) $$f(z)^2 \leq f(y-z)f(y+z) + f(y)^2, \ y, z \in X,$$

and, consequently,

(14) $$f(y)^2 \leq f(z-y)f(z+y) + f(z)^2, \ y, z \in X.$$

Conditions (13) and (14) jointly with the evenness of f imply that

(15) $$| f(y)^2 - f(z)^2 | \leq f(y+z)f(y-z), \ y, z \in X.$$

Now, (12) and (15) imply (8).

Obviously, conditions (8) and (9) are equivalent, provided that $(X, +)$ is a uniquely 2–divisible Abelian group.

2. In this section we assume that $(X, +)$ is a uniquely 2–divisible Abelian group and a function $f : X \to \mathbb{R}$ satisfies the following functional equation:

(III) $$| f(x)^2 - f(y)^2 | = f(x+y)f(x-y), \ x, y \in X,$$

or its equivalent form

(IV) $$\left| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 \right| = f(x)f(y), \ x, y \in X.$$

THEOREM 5. *If a function $f : X \to \mathbb{R}$ satisfies functional equation (III), then f satisfies Ptolemaic inequality (I).*

PROOF. From (IV) it follows that

$$\begin{aligned} &f(y-z)f(x) + f(x-z)f(y) \\ &\quad = \left| f\left(\tfrac{x+y-z}{2}\right)^2 - f\left(\tfrac{x-y+z}{2}\right)^2 \right| + \left| f\left(\tfrac{x+y-z}{2}\right)^2 - f\left(\tfrac{x-y-z}{2}\right)^2 \right| \\ &\quad \geq \left| f\left(\tfrac{x-y+z}{2}\right)^2 - f\left(\tfrac{x-y-z}{2}\right)^2 \right| = f(x-y)f(z) \end{aligned}$$

for all $x, y, z \in X$, which finishes the proof.

THEOREM 6. *If $f : X \to \mathbb{R}$ satisfies (III), then $f(0) = 0$, f is even and f is either non–negative on X or f is non–positive on X.*

PROOF. If $x = y = 0$, then, by (III), it follows that $f(0)^2 = 0$, and, consequently, $f(0) = 0$. Now, Theorem 5, Remark 1 and Corollary 1 imply the assertion.

In the sequel we shall restrict ourselves to real solutions of equation (III) on X, because any complex solution of (III) proves to be real. Indeed, if f is a complex solution of (III) on X, then $f(0) = 0$. Moreover, putting $y = 0$, we infer that
$| f(x)^2 | = f(x)^2 \in \mathbb{R}$ for every $x \in X$. Let $x \in X$ be fixed, $w = f(x) = u + iv \in \mathbb{C}$, where $u = \mathrm{Re}w$, $v = \mathrm{Im}w$ ($\mathbb{C}$ – denotes here the field of all complex numbers). Then $w^2 = |w^2|$ and, therefore

$$u^2 - v^2 + 2iuv = u^2 + v^2 \in \mathbb{R}.$$

Thus $uv = 0$ and $u^2 - v^2 = u^2 + v^2$, whence $v = 0$ and finally $w = f(x) \in \mathbb{R}$.

REMARK 4. *If $f : X \to \mathbb{R}$ is a solution of (III) and $c \in \mathbb{R}$, then so is cf.*

REMARK 5. *If $F : X \to \mathbb{R}$ is a solution of Wilson's sine functional equation (II), then $|F|$ satisfies (III).*

In what follows we shall show that for every non–negative solution f of (III) on X there exists a solution F of (II) on X such that $f = |F|$.

At first we shall prove some lemmas, which will be needed to find all real solutions of equation (III) on X.

LEMMA 1. *If $f : X \to \mathbb{R}$ satisfies (III), $x \in X$, and $f(2x) \neq 0$, then $f(x) \neq 0$ and*

$$\frac{f(4x)}{f(2x)} = \left|2 - \frac{f(2x)^2}{f(x)^2}\right|. \tag{16}$$

PROOF. Suppose that $f(2x) \neq 0$. By Theorem 5 and Theorem 1 it follows that $f(x) \neq 0$.

On account of (III) we have

$$|f(x)^2 - f(2x)^2| = f(3x)f(x), \quad f(3x) = \frac{|f(x)^2 - f(2x)^2|}{f(x)}$$

and

$$|f(x)^2 - f(3x)^2| = f(4x)f(2x).$$

Hence

$$f(4x)f(2x) = \left|f(x)^2 - \frac{(f(x)^2 - f(2x)^2)^2}{f(x)^2}\right| = \left|\frac{2f(x)^2f(2x)^2 - f(2x)^4}{f(x)^2}\right|$$
$$= f(2x)^2\left|\frac{2f(x)^2 - f(2x)^2}{f(x)^2}\right| = f(2x)^2\left|2 - \frac{f(2x)^2}{f(x)^2}\right|.$$

From here we obtain that

$$\frac{f(4x)}{f(2x)} = \left|2 - \frac{f(2x)^2}{f(x)^2}\right|.$$

LEMMA 2. *If $f \geq 0$ satisfies (III), $x \in X$ and $f(x) \neq 0$, then $f\left(\frac{x}{2^n}\right) \neq 0$ for every $n \in \mathbb{N}$ and the following conditions are satisfied:*

$$\text{(17)} \qquad \text{if } \frac{f(2x)}{f(x)} = 2, \quad \text{then } \frac{f(2^n x)}{f(2^{n-1}x)} = 2, \ \frac{f(\frac{x}{2^{n-1}})}{f(\frac{x}{2^n})} = 2$$

for $n \in \mathbb{N}$, $\lim_{n\to\infty} f(2^n x) = +\infty$, $\lim_{n\to\infty} f(\frac{x}{2^n}) = 0$;

$$\text{(18)} \qquad \text{if } \frac{f(2x)}{f(x)} > 2, \quad \text{then } \frac{f(2^n x)}{f(2^{n-1}x)} > 2, \quad \frac{f(\frac{x}{2^{n-1}})}{f(\frac{x}{2^n})} > 2$$

for $n \in \mathbb{N}$, $\lim_{n\to\infty} f(2^n x) = +\infty$, $\lim_{n\to\infty} f(\frac{x}{2^n}) = 0$;

$$\text{(19)} \qquad \text{if } \frac{f(2x)}{f(x)} < 2, \quad \text{then } \frac{f(\frac{x}{2^{n-1}})}{f(\frac{x}{2^n})} < 2, \quad \text{for } n \in \mathbb{N}$$

and $\frac{f(2^n x)}{f(2^{n-1}x)} < 2$ for $n \in \mathbb{N}$, provided $f(2^n x) \neq 0$ for $n \in \mathbb{N}$.

PROOF. The assertions result from Lemma 1 by induction.

THEOREM 7. *If $f \geq 0$ satisfies (III), then f satisfies exactly one of the following two conditions:*

$$\text{(20)} \qquad \bigwedge_{x\in X}\left(f(x) \neq 0 \Longrightarrow \frac{f(2x)}{f(x)} > 2\right);$$

$$\text{(21)} \qquad \bigwedge_{x\in X}\left(f(x) \neq 0 \Longrightarrow \frac{f(2x)}{f(x)} \leq 2\right).$$

PROOF. We first prove that f satisfies the following condition:

$$\left(\bigvee_{\substack{y\in X\\ f(y)\neq 0}} \frac{f(2y)}{f(y)} > 2\right) \Longrightarrow \left(\bigwedge_{\substack{x\in X\\ f(x)\neq 0}} \frac{f(2x)}{f(x)} > 2\right). \tag{22}$$

In fact, let $y \in X$ be an element such that $f(y) \neq 0$ i $\frac{f(2y)}{f(y)} > 2$. Assume, that there exists an $x \in X$ such that $f(x) \neq 0$ and $\frac{f(2x)}{f(x)} \leq 2$. Condition (18) of Lemma 2 implies that $\lim\limits_{n\to\infty} f(\frac{y}{2^n}) = 0$. Thus there exists a $k \in \mathbb{N}$ such that $f(\frac{y}{2^k}) < f(x)$. Put $z := \frac{y}{2^k}$. Then $0 < f(z) < f(x)$. Moreover, by (18), $\frac{f(2z)}{f(z)} > 2$. On account of inequalities (7) from Theorem 4 we have

$$\begin{aligned}
4 &\geq \tfrac{f(2x)^2}{f(x)^2} \geq \left(\tfrac{f(x+z)-f(x-z)}{f(z)}\right)^2\\
&= \tfrac{(f(x+z)+f(x-z))^2}{f(x)^2}\cdot\tfrac{f(x)^2}{f(z)^2} - \tfrac{4f(x+z)f(x-z)}{f(z)^2}\\
&\geq \tfrac{f(2z)^2}{f(z)^2}\cdot\tfrac{f(x)^2}{f(z)^2} - \tfrac{4|f(x)^2-f(z)^2|}{f(z)^2} > 4\cdot\tfrac{f(x)^2}{f(z)^2} - 4\cdot\tfrac{f(x)^2-f(z)^2}{f(z)^2} = 4,
\end{aligned}$$

a contradiction. Hence condition (22) holds true. Observe that (22) is equivalent to the following condition

$$\left(\bigvee_{\substack{x\in X\\ f(x)\neq 0}} \frac{f(2x)}{f(x)} \leq 2\right) \Longrightarrow \left(\bigwedge_{\substack{y\in X\\ f(y)\neq 0}} \frac{f(2y)}{f(y)} \leq 2\right). \tag{23}$$

Now, (22) and (23) imply (20) and (21).

LEMMA 3. *Suppose that $f : X \to \mathbb{R}$ is a non-zero and non-negative function satisfying (III), $y \in X$, i $f(y) \neq 0$. Let $F : X \to \mathbb{R}$ be a function defined by the formula:*

(24) $F(x) := \frac{f(\frac{x+y}{2})^2 - f(\frac{x-y}{2})^2}{f(y)}$, $x \in X$. *The F satisfies the following conditions:*

(25) $F(y) = f(y)$;

(26) $F(0) = 0$;

(27) $|F(x)| = f(x), x \in X$;

(28) $F(-x) = -F(x), x \in X$;

(29) $F(x)F(y) = F\left(\frac{x+y}{2}\right)^2 - F\left(\frac{x-y}{2}\right)^2, x \in X$;

(30) $F(x)^2F(z)^2 = \left(F\left(\frac{x+z}{2}\right)^2 - F\left(\frac{x-z}{2}\right)^2\right)^2$, $x, z \in X$;

(31) *if $z \in X$ is an element such that $F(z) = 0$, then $F(2z) = 0$ and $F(x+z)^2 = F(x)^2 = F(x-z)^2$ for every $x \in X$.*

PROOF. By Theorem 6 we have $f(0) = 0$ and therefore $F(y) = f(y)$. Formula (24) jointly with then evenness of f implies (26), whereas (IV) implies (27). Condition (28) holds, because f is even. Moreover, by (24), (25) and (27) we have (29). Condition (27) and equation (IV) imply (30). Finally, if $z \in X$ and $F(z) = 0$, then (27), Theorem 5 and Theorem 1 imply (31).

LEMMA 4. *If $f \geq 0$ satisfies (III) and $y \in X$ is an element such that $f(y) \neq 0$ and $f(2y) = 0$, then the function F defined by (24) with the aid of this y satisfies Wilson's sine functional equation (II) on X.*

PROOF. By (27) we have $F(2y) = 0$. Consequently (30) implies that

$$0 = F(2x)^2 F(2y)^2 = \left(F(x+y)^2 - F(x-y)^2\right)^2, \ x \in X,$$

whence

$$F(x+y)^2 - F(x-y)^2 = 0, \ x \in X. \tag{32}$$

On account of (28), (29), (30) and (32) we get

$$\begin{aligned}
0 &= \left(F(2x+y)^2 - F(2x-y)^2\right)\left(F(2z+y)^2 - F(2z-y)^2\right) \\
&= \left(F(x+z+y)^2 - F(x-z)^2\right)^2 - \left(F(x+z)^2 - F(x-z+y)^2\right)^2 \\
&- \left(F(x+z)^2 - F(x-z-y)^2\right)^2 + \left(F(x+z-y)^2 - F(x-z)^2\right)^2 \\
&= \left(\left(F(x+z+y)^2 - F(x-z)^2\right) - \left(F(x+z-y)^2 - F(x-z)^2\right)\right)^2 \\
&+2\left(F(x+z+y)^2 - F(x-z)^2\right)\left(F(x+z-y)^2 - F(x-z)^2\right) \\
&- \left(\left(F(x+z)^2 - F(x-z+y)^2\right) - \left(F(x+z)^2 - F(x-z-y)^2\right)\right)^2 \\
&-2\left(F(x+z)^2 - F(x-z+y)^2\right)\left(F(x+z)^2 - F(x-z-y)^2\right) \\
&= 2\big(F(x+z+y)^2F(x+z-y)^2 - F(x+z+y)^2F(x-z)^2 \\
&-F(x-z)^2F(x+z-y)^2 + F(x-z)^4 - F(x+z)^4 \\
&+F(x+z)^2F(x-z-y)^2 + F(x-z+y)^2F(x+z)^2 - \\
&F(x-z+y)^2F(x-z-y)^2\big) \\
&= 2\Bigg(\left(F(x+z)^2 - F(y)^2\right)^2 - \left(F\left(\tfrac{2x+y}{2}\right)^2 - F\left(\tfrac{2z+y}{2}\right)^2\right)^2 \\
&- \left(F\left(\tfrac{2x-y}{2}\right)^2 - F\left(\tfrac{2z-y}{2}\right)^2\right)^2 + F(x-z)^4 \\
&-F(x+z)^4 + \left(F\left(\tfrac{2x-y}{2}\right)^2 - F\left(\tfrac{2z+y}{2}\right)^2\right)^2 \\
&+ \left(F\left(\tfrac{2x+y}{2}\right)^2 - F\left(\tfrac{2z-y}{2}\right)^2\right)^2 - \left(F(x-z)^2 - F(y)^2\right)^2\Bigg) \\
&= 2\left(-2F(y)^2\left(F(x+z)^2 - F(x-z)^2\right)\right.
\end{aligned}$$

$$+2\left(F\left(\tfrac{2x+y}{2}\right)^2-F\left(\tfrac{2x-y}{2}\right)^2\right)\left(F\left(\tfrac{2z+y}{2}\right)^2-F\left(\tfrac{2z-y}{2}\right)^2\right)\Big)$$
$$=4F(y)^2(F(2x)F(2z)-(F(x+z)^2-F(x-z)^2))$$

for all $x,z\in X$. Hence

$$F(2x)F(2z)=F(x+z)^2-F(x-z)^2,\ x,z\in X \tag{33}$$

Putting $x:=\frac{s+t}{2}, z:=\frac{s-t}{2}$ in (33), we obtain

$$F(s+t)F(s-t)=F(s)^2-F(t)^2,\ s,t\in X.$$

This completes the proof.

Lemma 5. *If $f:X\to\mathbb{R}$ is a non-zero and non-negative function satisfying (III) and $y\in X$ is an element such that $f(y)\neq 0$ and*

$$4\neq\frac{f(y)^2}{f(\frac{y}{2})^2}\cdot\frac{f(2x)}{f(x)}\cdot\frac{f(2z)}{f(z)} \tag{34}$$

for all $x,z\in X$, provided that $x\neq z, f(x)\neq 0, f(z)\neq 0$, then the function $F:X\to\mathbb{R}$ defined by (24) with the aid of that y satisfies Wilson's equation (II) on X.

Proof. On account of (28), (29), (30) we have

$$F(y)^2F(2x)F(2z)$$
$$=\left(F\left(\tfrac{2x+y}{2}\right)^2-F\left(\tfrac{2x-y}{2}\right)^2\right)\left(F\left(\tfrac{2z+y}{2}\right)^2-F\left(\tfrac{2z-y}{2}\right)^2\right)$$
$$=\left(F\left(\tfrac{x+z+y}{2}\right)^2-F\left(\tfrac{x-z}{2}\right)^2\right)^2-\left(F\left(\tfrac{x+z}{2}\right)^2-F\left(\tfrac{x-z+y}{2}\right)^2\right)^2$$
$$-\left(F\left(\tfrac{x+z}{2}\right)^2-F\left(\tfrac{x-z-y}{2}\right)^2\right)^2+\left(F\left(\tfrac{x+z-y}{2}\right)^2-F\left(\tfrac{x-z}{2}\right)^2\right)^2$$
$$=\left(F\left(\tfrac{x+z+y}{2}\right)^2-F\left(\tfrac{x+z-y}{2}\right)^2\right)^2-\left(F\left(\tfrac{x-z+y}{2}\right)^2-F\left(\tfrac{x-z-y}{2}\right)^2\right)^2$$
$$+2\left(F\left(\tfrac{x+z+y}{2}\right)^2-F\left(\tfrac{x-z}{2}\right)^2\right)\left(F\left(\tfrac{x+z-y}{2}\right)^2-F\left(\tfrac{x-z}{2}\right)^2\right)$$
$$-2\left(F\left(\tfrac{x+z}{2}\right)^2-F\left(\tfrac{x-z-y}{2}\right)^2\right)\left(F\left(\tfrac{x+z}{2}\right)^2-F\left(\tfrac{x-z-y}{2}\right)^2\right)$$
$$=F(y)^2F(x+z)^2-F(y)^2F(x-z)^2+2\Bigg(\left(F\left(\tfrac{x+z}{2}\right)^2-F\left(\tfrac{y}{2}\right)^2\right)^2$$
$$-\left(F\left(\tfrac{2x+y}{4}\right)^2-F\left(\tfrac{2z+y}{4}\right)^2\right)^2-\left(F\left(\tfrac{2x-y}{4}\right)^2-F\left(\tfrac{2z-y}{4}\right)^2\right)^2$$
$$+F\left(\tfrac{x-z}{2}\right)^4-F\left(\tfrac{x+z}{4}\right)^4+\left(F\left(\tfrac{2x+y}{4}\right)^2-F\left(\tfrac{2z-y}{4}\right)^2\right)^2$$
$$+\left(F\left(\tfrac{2x-y}{4}\right)^2-F\left(\tfrac{2z+y}{4}\right)^2\right)^2-\left(F\left(\tfrac{x-z}{2}\right)^2-F\left(\tfrac{y}{2}\right)^2\right)^2\Bigg)$$

$$= F(y)^2(F(x+z)^2 - F(x-z)^2)$$
$$+4\left(\left(F\left(\tfrac{2x+y}{4}\right)^2 - F\left(\tfrac{2x-y}{4}\right)^2\right)\left(F\left(\tfrac{2z+y}{4}\right)^2 - F\left(\tfrac{2z-y}{4}\right)^2\right)\right.$$
$$\left.-F\left(\tfrac{y}{2}\right)^2\left(F\left(\tfrac{x+z}{2}\right)^2 - F\left(\tfrac{x-z}{2}\right)^2\right)\right)$$

for all $x, z \in X$. Hence

$$\begin{aligned}(35)\quad & F(y)^2(F(2x)F(2z) - (F(x+z)^2 - F(x-z)^2)) \\ & = 4\left(\left(F\left(\tfrac{2x+y}{4}\right)^2 - F\left(\tfrac{2x-y}{4}\right)^2\right)\left(F\left(\tfrac{2z+y}{4}\right)^2 - F\left(\tfrac{2z-y}{4}\right)^2\right)\right. \\ & \left.-F\left(\tfrac{y}{2}\right)^2\left(F\left(\tfrac{x+z}{2}\right)^2 - F\left(\tfrac{x-z}{2}\right)^2\right)\right)\end{aligned}$$

for all $x, z \in X$.

Now, observe that condition (30) implies the following two equalities:

$$(36)\qquad |F(2x)F(2z)| = |F(x+z)^2 - F(x-z)^2| \quad , x, z \in X,$$

and

$$\begin{aligned}(37)\quad & \left|\left(F\left(\tfrac{2x+y}{4}\right)^2 - F\left(\tfrac{2x-y}{4}\right)\right)^2\left(F\left(\tfrac{2z+y}{4}\right)^2 - F\left(\tfrac{2z-y}{4}\right)^2\right)\right| \\ & = F\left(\tfrac{y}{2}\right)^2\left|F(x)F(z)\right| = F\left(\tfrac{y}{2}\right)^2\left|F\left(\tfrac{x+z}{2}\right)^2 - F\left(\tfrac{x-z}{2}\right)^2\right|\end{aligned}$$

for all $x, z \in X$.

Suppose that there exist $a, b \in X$ such that

$$(38)\qquad F(2a)F(2b) \neq F(a+b)^2 - F(a-b)^2.$$

Then, by (31), it follows that $F(a) \neq 0, F(b) \neq 0$ and, consequently, $f(a) \neq 0$, $f(b) \neq 0$, because (27) is satisfied. Obviously, $a \neq b$, by (26).

Now, in view of (35) and (38) we infer that

$$\begin{aligned}(39)\quad & \left(F\left(\tfrac{2a+y}{4}\right)^2 - F\left(\tfrac{2a-y}{4}\right)^2\right)\left(F\left(\tfrac{2b+y}{4}\right)^2 - F\left(\tfrac{2b-y}{4}\right)^2\right) \\ & \neq F\left(\tfrac{y}{2}\right)^2\left(F\left(\tfrac{a+b}{2}\right)^2 - F\left(\tfrac{a-b}{2}\right)^2\right).\end{aligned}$$

Hence, by (36), (37), (38) and (39) it follows that

$$-(F(a+b)^2 - F(a-b)^2) = F(2a)F(2b)$$

and

$$-F\left(\tfrac{y}{2}\right)^2\left(F\left(\tfrac{a+b}{2}\right)^2-F\left(\tfrac{a-b}{2}\right)^2\right)$$
$$=\left(F\left(\tfrac{2a+y}{4}\right)^2-F\left(\tfrac{2a-y}{4}\right)^2\right)\left(F\left(\tfrac{2b+y}{4}\right)^2-F\left(\tfrac{2b-y}{4}\right)^2\right).$$

This jointly with (35) implies that

$$2F(y)^2F(2a)F(2b)$$
$$=8\left(F\left(\tfrac{2a+y}{4}\right)^2-F\left(\tfrac{2a-y}{4}\right)^2\right)\left(F\left(\tfrac{2b+y}{4}\right)^2-F\left(\tfrac{2b-y}{4}\right)^2\right).$$

From here and from (27) and (30) we have

$$f(y)^2f(2a)f(2b)=|F(y)^2F(2a)F(2b)|$$
$$=4\left|F\left(\tfrac{2a+y}{4}\right)^2-F\left(\tfrac{2a-y}{4}\right)^2\right|\left|F\left(\tfrac{2b+y}{4}\right)^2-F\left(\tfrac{2b-y}{4}\right)^2\right|$$
$$=4f\left(\tfrac{y}{2}\right)^2f(a)f(b).$$

Hence we obtain

$$4=\frac{f(y)^2}{f(\frac{y}{2})^2}\cdot\frac{f(2a)}{f(a)}\cdot\frac{f(2b)}{f(b)},$$

which contradicts (34). Finally, F satisfies (33) and, consequently, F yields a solution to Wilson's equation (II) on X.

THEOREM 8. *Suppose that a non-zero and non-negative function $f : X \to \mathbb{R}$ satisfies (III). If $f(2x) \le 2f(x)$ for every $x \in X$, then for an arbitrary $y \in X$ such that $f(y) \ne 0$ the function $F : X \to \mathbb{R}$ defined by formula (24) with the aid of that y satisfies Wilson's equation (II) on X.*

PROOF. For arbitrarily fixed element $y \in X$ such that $f(y) \ne 0$ we define the function $F_y : X \to \mathbb{R}$ by formula (24), i.e.

$$F_y(x) := \frac{f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2}{f(y)}, \quad x \in X.$$

Then, every function F_y satisfies conditions (25)–(31).

We assume first that for every $y \in X$ such that $f(y) \ne 0$ the function F_y does not satisfy (II) on X.

On account of this assumption and Lemma 4 we infer that $f(2y) \ne 0$ and $f(4y) \ne 0$ provided that $f(y) \ne 0, y \in X$. However, our assumption jointly with Lemma 5 implies that

$$\bigwedge_{\substack{y\in X\\ f(y)\neq 0}} \bigvee_{\substack{x,z\in X\\ f(x)\neq 0\ f(z)\neq 0}} 4=\frac{f(y)^2f(2x)f(2z)}{f\left(\frac{y}{2}\right)^2f(x)f(z)}. \tag{40}$$

Since $f(2t) \leq 2f(t)$ for every $t \in X$, condition (40) implies that

$$4 \leq \frac{f(y)^2}{f\left(\frac{y}{2}\right)^2} \cdot 4$$

provided that $f(y) \neq 0$, $y \in X$. This implies that

$$1 \leq \frac{f(y)}{f\left(\frac{y}{2}\right)} \tag{41}$$

for every $y \in X$ such that $f(y) \neq 0$.

By (41) and (16) of Lemma 1 we have

$$1 \leq \frac{f(y)}{f\left(\frac{y}{2}\right)} = \left| 2 - \frac{f\left(\frac{y}{2}\right)^2}{f\left(\frac{y}{4}\right)^2} \right|$$

provided that $f(y) \neq 0, y \in X$. Hence, if $y \in X$ and $f(y) \neq 0$, then

$$\sqrt{3} \leq \frac{f\left(\frac{y}{2}\right)}{f\left(\frac{y}{4}\right)} \quad \text{or} \quad 1 \geq \frac{f\left(\frac{y}{2}\right)}{f\left(\frac{y}{4}\right)}$$

Putting here $a := \frac{y}{4}$, we infer that

$$\sqrt{3} \leq \frac{f(2a)}{f(a)} \quad \text{or} \quad 1 \geq \frac{f(2a)}{f(a)}$$

for every $a \in X$ such that $f(4a) \neq 0$. This jointly with (41) gives

$$\sqrt{3} \leq \frac{f(2a)}{f(a)} \quad \text{or} \quad 1 = \frac{f(2a)}{f(a)} \tag{42}$$

provided that $f(4a) \neq 0, a \in X$.

Now we shall show that the inequality $f(4a) \neq 0$ forces $\frac{f(2a)}{f(a)}$ to be different from 1.

Assume the contrary: there exists a $b \in X$ such that $f(4b) \neq 0$ and $\frac{f(2b)}{f(b)} = 1$. Obviously, $f(b) \neq 0$ and $f(2b) \neq 0$, by Lemma 2. Hence, by (40), there exist $x, z \in X$ such that $f(x) \neq 0, f(z) \neq 0$ and

$$4 = \frac{f(2b)}{f(b)} \cdot \frac{f(2x)}{f(x)} \cdot \frac{f(2z)}{f(z)} = \frac{f(2x)f(2z)}{f(x)f(z)}.$$

On account of the assertion of this theorem we have $\frac{f(2x)}{f(x)} \leq 2$, $\frac{f(2z)}{f(z)} \leq 2$ and infer that $\frac{f(2x)}{f(x)} = 2 = \frac{f(2z)}{f(z)}$. Now, by Lemma 2 it follows that $f(\frac{x}{2^n}) \neq 0$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} f(\frac{x}{2^n}) = 0$. Hence there exists a $k \in \mathbb{N}$ such that $f(\frac{x}{2^k}) < f(b)$. Put $c := \frac{x}{2^k}$. Then $0 < f(c) < f(b)$. By Lemma 2 we have $\frac{f(2c)}{f(c)} = 2$. Now, Theorem 5, Theorem 6 and condition (7) of Theorem 4 imply that

$$\begin{aligned}
4 = \frac{f(2c)^2}{f(c)^2} &\leq \left(\frac{f(c+b)+f(c-b)}{f(b)}\right)^2 \\
&= \frac{(f(c+b)-f(c-b))^2}{f(c)^2}\cdot\frac{f(c)^2}{f(b)^2} + \frac{4f(c+b)f(c-b)}{f(b)^2} \\
&\leq \frac{f(2b)^2}{f(b)^2}\cdot\frac{f(c)^2}{f(b)^2} + \frac{4|f(b)^2-f(c)^2|}{f^2(b)} \\
&= \frac{f(c)^2}{f(b)^2} + \frac{4(f(b)^2-f(c)^2)}{f(b)^2} = 4 - 3\frac{f(c)^2}{f(b)^2}.
\end{aligned}$$

Consequently, we get $\frac{f(c)^2}{f(b)^2} \leq 0$, a contradiction because $f(c) > 0$.

Finally, $\frac{f(2b)}{f(b)} \neq 1$, provided that $f(4b) \neq 0, b \in X$. This jointly with (42) implies that

$$\frac{f(2a)}{f(a)} \geq \sqrt{3} \tag{43}$$

for every $a \in X$ such that $f(4a) \neq 0$.

Since $f(a) \neq 0$ implies $f(4a) \neq 0$, we have (43) for every $a \in X$ such that $f(a) \neq 0$.

Therefore, for all $x, y, z \in X$ such that $f(x) \neq 0, f(y) \neq 0, f(z) \neq 0$ we obtain

$$\frac{f(y)^2 f(2x) f(2z)}{f(\frac{y}{2})^2 f(x) f(z)} \geq 9,$$

which contradicts (40).

Finally, we infer that there exists $y \in X$ such that $f(y) \neq 0$ and F_y satisfies (II) on X.

We shall show that for every $z \in X$ such that $f(z) \neq 0$ the function F_z satisfies (II) on X.

Let $z \in X$ such that $f(z) \neq 0$ be fixed. By the definition of F_z and F_y and in view of the equality (27) in Lemma 3 we have

$$F_z(x) = \frac{f\left(\frac{x+z}{2}\right)^2 - f\left(\frac{x-z}{2}\right)^2}{f(z)} = \frac{F_y\left(\frac{x+z}{2}\right)^2 - F_y\left(\frac{x-z}{2}\right)^2}{|F_y(z)|} = \frac{F_y(x)F_y(z)}{|F_y(z)|}$$

for every $x \in X$. Put $\lambda := \frac{F_y(z)}{|F_y(z)|}$; obviously $\lambda \in \{-1, 1\}$. We obtain

$$F_z(x) = \lambda F_y(x) \;\; x \in X,$$

whence F_z satisfies (II) on X as well. This completes the proof.

THEOREM 9. *Suppose that a non-zero and non-negative function $f : X \to \mathbb{R}$ satisfies (III). If $f(2x) > 2f(x)$ for every $x \in X$ such that $f(x) \neq 0$, then for arbitrary $y \in X$ such that $f(y) \neq 0$ the function $F : X \to \mathbb{R}$ defined by formula (24) with the aid of that y satisfies Wilson's equation (II) on X.*

PROOF. Let $y \in X, f(y) \neq 0$ and let $F : X \to \mathbb{R}$ be defined by (24). Since

$$\frac{f(y)^2 f(2x) f(2z)}{f\left(\frac{y}{2}\right)^2 f(x) f(z)} > 16$$

for all $x, z \in X$ such that $f(x) \neq 0, f(z) \neq 0$, Lemma 5 implies that F satisfies (II) on X.

Theorems 7, 8, 9 lead now to the following

COROLLARY 2. *Let $f : X \to \mathbb{R}$ be a non-zero and non-negative function satisfying equation (III) on X. Then there exist exactly two different functions*
$H_1, H_2 : X \to \mathbb{R}$ satisfying Wilson's sine functional equation (II) on X and the condition $|H_1(x)| = |H_2(x)| = f(x)$ for every $x \in X$. Moreover $H_1(x) = -H_2(x)$,
$x \in X$.

PROOF. Let $y \in X$ such that $f(y) \neq 0$ be fixed. Let $F : X \to \mathbb{R}$ be defined by (24) and let $H : X \to \mathbb{R}$ be a function satisfying (II) on X and the condition: $|H(x)| = f(x), x \in X$. Then $H(y) \neq 0$ and

$$\begin{aligned} H(x) &= \frac{H(x)H(y)}{H(y)} = \frac{H\left(\frac{x+y}{2}\right)^2 - H\left(\frac{x-y}{2}\right)^2}{H(y)} \\ &= \frac{f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2}{f(y)} \cdot \frac{f(y)}{H(y)} = \frac{f(y)}{H(y)} \cdot F(x), \end{aligned}$$

for every $x \in X$. Hence $H = F$ or $H = -F$, which was to be proved.

References

[1] C. Alsina, J. L. Garcia–Roig, *On a functional equation related to the Ptolemaic inequality*, Aequationes Math. **34** (1987), 298–303.

[2] C. Alsina, J. L. Garcia–Roig, R. Ger, *How to keep the quadrilateral inscribed*, Abh. Math. Sem. Univ. Hamburg **65** (1995), 123–133.

[3] D. Amir, *Characterizations of inner product spaces*, Birkhäuser Verlag, Basel–Boston–Stuttgart 1986.