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ON A TWO POINT BOUNDARY VALUE PROBLEM FOR LINEAR DIFFERENTIAL EQUATIONS OF THE FOURTH ORDER IN THE COLOMBEAU ALGEBRA.

JAN LIGĘZA

Dedicated to Professor Tadeusz Dłotko on the occasion on his seventieth birthday

Abstract. The existence and uniqueness of solutions of the two point boundary value problem for ordinary linear differential equations of fourth order in the Colombeau algebra are considered.

1. Introduction

We examine the following problem

$$(1.0)\quad L(x) \equiv x''''(t)+p_1(t)x'''(t)+p_2(t)x''(t)+p_3(t)x'(t)+p_4(t)x(t) = p_5(t),$$

$$(1.1)\quad \begin{aligned} &L_1(x) \equiv x(0) = d_1, \quad L_2(x) \equiv x(T) = d_2, \quad L_3(x) \equiv x'(0) = d_3, \\ &L_4(x) \equiv x'(T) = d_4, \quad d_i \in \overline{R}, \quad 0 < T < \infty; \quad i = 1,2,3,4. \end{aligned}$$

We assume that p_j ($j = 1,2,3,4,5$) are elements of the Colombeau algebra $\mathcal{G}(\mathbb{R})$, d_i ($i = 1,\dots,4$) are elements of the Colombeau algebra $\overline{\mathbb{R}}$ of generalized real numbers; $x(0), x'(0), x(T), x'(T)$ are understood as the value of the generalized functions x and x' at the points 0 and T respectively (see [1]). The elements p_j ($j = 1,2,\dots,5$) are given. The multiplication, the

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derivative, the sum and the equality are meant in the Colombeau algebra sense. We prove theorems on the existence and uniquence of solutions of problem (1.0)–(1.1). Proved theorems generalize in some cases results given in [5].

2. Notation

Let $\mathcal{D}(\mathbb{R})$ be the set of all C^∞ functions $\mathbb{R} \to \mathbb{R}$ with compact support. For $q = 1, 2, \dots$ we denote by $\mathcal{A}_q$ the set of all functions $\varphi \in \mathcal{D}(\mathbb{R})$ such that the relations

$$\int_{-\infty}^{\infty} \varphi(t)dt = 1, \qquad \int_{-\infty}^{\infty} t^k \varphi(t)dt = 0, \quad 1 \le k \le q \tag{2.1}$$

hold.

Next, $\mathcal{E}[\mathbb{R}]$ is the set of all functions $R : \mathcal{A}_1 \times \mathbb{R} \to \mathbb{R}$ such that $R(\varphi, t) \in C^\infty$ for every fixed $\varphi \in \mathcal{A}_1$.

If $R \in \mathcal{E}[\mathbb{R}]$, then $D_k R(\varphi, t)$ for any fixed φ denotes a differential operator in t (i.e. $D_k R(\varphi, t) = \frac{d^k}{dt^k}(R(\varphi, t))$ for $k \ge 1$ and $D_0 R(\varphi, t) = R(\varphi, t)$).

For given $\varphi \in \mathcal{D}(\mathbb{R})$ and $\varepsilon > 0$ we define φ_ε by

$$\varphi_\varepsilon(t) = \frac{1}{\varepsilon}\varphi\left(\frac{t}{\varepsilon}\right). \tag{2.2}$$

An element R of $\mathcal{E}[\mathbb{R}]$ is moderate if: for every compact interval K of $\mathbb{R}$ and every differential operator D_k there is $N \in \mathbb{N}$ such that the following conditions holds: for every $\varphi \in \mathcal{A}_N$ there are $c > 0$, $\varepsilon_0 > 0$ such that

$$\sup_{t \in K} |D_k R(\varphi_\varepsilon, t)| \le c\varepsilon^{-N} \quad \text{if} \quad 0 < \varepsilon < \varepsilon_0. \tag{2.3}$$

We denote by $\mathcal{E}[\mathbb{R}]$ the set of all moderate elements of $\mathcal{E}[\mathbb{R}]$.

By Γ we denote the set of all the increasing functions α from $\mathbb{N}$ into $\mathbb{R}^+$ such that $\alpha(q) \to \infty$ if $q \to \infty$.

We define an ideal $\mathcal{N}[\mathbb{R}]$ in $\mathcal{E}_M[\mathbb{R}]$ as follows: $R \in \mathcal{N}[\mathbb{R}]$ if for every compact interval K of $\mathbb{R}$ and every differential operation D_k there are $N \in \mathbb{N}$ and $\alpha \in \Gamma$ such that the following condition holds: for every $q \ge N$ and $\varphi \in \mathcal{A}_q$ there are $c > 0$ and $\varepsilon_0 > 0$ such that

$$\sup_{t \in K} |D_k R(\varphi_\varepsilon, t)| \le c\varepsilon^{\alpha(q) - N} \quad \text{if} \quad 0 < \varepsilon < \varepsilon_0. \tag{2.4}$$

The algebra $\mathcal{G}(\mathbb{R})$ (the Colombeau algebra of generalized functions) is defined as quotient algebra of $\mathcal{E}_M[\mathbb{R}]$ with respect to $\mathcal{N}[\mathbb{R}]$ (see [1]).

For $R \in \mathcal{E}_M[\mathbb{R}]$ the coresponding class $G = R + \mathcal{N}[\mathbb{R}] \in \mathcal{G}(\mathbb{R})$ is denoted by $[R]$, i.e. $G = [R]$. Vice versa, if $G \in \mathcal{G}(\mathbb{R})$, then its representative in $\mathcal{E}_M[\mathbb{R}]$ is usually denoted by R_G. If $G_i = [R_{G_i}] \in \mathcal{G}(\mathbb{R}), i = 1, 2$, then we define $G_1G_2 := [R_{G_1}R_{G_2}]$. (This definition does not depend on the choice of R_{G_1} and R_{G_2}.)

We denote by $\mathcal{E}_0$ the set of all functions from $\mathcal{A}_1$ into $\mathbb{R}$. Next, we denote by $\mathcal{E}_M$ the set of all the so–called moderate elements of $\mathcal{E}_0$ defined by

(2.5) $\mathcal{E}_M = \{R \in \mathcal{E}_0$: there is $N \in \mathrm{N}$ such that for every $\varphi \in \mathcal{A}_N$ there are $\varepsilon > 0,\ \eta_0 > 0$ such that $|R(\varphi_\varepsilon)| \leq c\varepsilon^{-N}$ for $0 < \varepsilon < \eta_0\}$.

The ideal $\mathcal{N}$ of $\mathcal{E}_M$ is defined by

(2.6) $\mathcal{N} = \{R \in \mathcal{E}_0$: there are $N \in \mathrm{N}$ and $\alpha \in \Gamma$ such that for every $q \geq N$ and $\varphi \in \mathcal{A}_q$ there are $c > 0,\ \eta_0 > 0$ such that $|R(\varphi_\varepsilon) \leq \varepsilon^{\alpha(q)-N}$ for $0 < \varepsilon < \eta_0\}$

and

$$\overline{\mathbb{R}} = \frac{\mathcal{E}_M}{\mathcal{N}} \quad \text{(see [1])}.$$

It is known that $\overline{\mathbb{R}}$ is an algebra while it is not a field. Its elements are called generalized real numbers.

If $R \in \mathcal{E}_M[\mathbb{R}]$ is a representative of $G \in \mathcal{G}(\mathbb{R})$, then for a fixed t the map $Y : \varphi \to R(\varphi, t) \in \mathbb{R}$ is defined on $\mathcal{A}_1$ and $Y \in \mathcal{E}_M$. This class is denoted by $G(t)$ and is called the value of the generalized function G at the point t (see [1]).

We say that $G \in \mathcal{G}(\mathbb{R})$ is a constant generalized function on $\mathbb{R}$ if it admits a representative $R(\varphi, t)$ which is independent on t. With any $Z \in \overline{\mathbb{R}}$ we associate a constant generalized function which admits $R(\varphi, t) = Z(\varphi)$ as its representative, provided we denote by Z a representative of Z (see [1]).

We denote by $R_p(\varphi, t),\ R_x(\varphi, t), R_{x^{(i)}}(\varphi, t), R_{x(t_0)}(\varphi)$ and $R_{x^{(i)}(t_0)}(\varphi)$ the representatives of elements $p_j, x, x^{(i)}, x(t_0)$ and $x^{(i)}(t_0)$, respectively.

Throughont the paper K denotes a compact interval in $\mathbb{R}$ containing zero and $[0, T]$ is the compact interval (i.e. $-\infty < 0 \leq t \leq T < \infty$).

For $x \in C^\infty$ we put

$$||D_n(x)||_K^0 = \max_{t \in K} |D_n(x)(t)|, \quad ||x||_K^n = \sum_{i=0}^n ||D_i(x)||_K^0.$$

We say that $x \in \mathcal{G}(\mathbb{R})$ is a solution of the equation (1.0) if there is

$\eta \in \mathcal{N}[\mathbb{R}]$ such that for any representative R_x of x, the relations

$$\begin{aligned} L_\varphi(R_x(\varphi,t) \equiv D_4R_x(\varphi,t) + R_{p_1}(\varphi,t)D_3R_x(\varphi,t) \\ + R_{p_2}(\varphi,t)D_2R_x(\varphi,t) + R_{p_3}(\varphi,t)D_1R_x(\varphi,t)+ \\ + R_{p_4}(\varphi,t)R_x(\varphi,t) = R_{p_5}(\varphi,t) + \eta(\varphi,t) \end{aligned}$$

are satisfied for all $\varphi \in \mathcal{A}_1$ and $t \in \mathbb{R}$.

3. The main results

First we shall give two hypotheses.

Hypothesis H_1

(3.0) $$p_i \in \mathcal{G}(\mathbb{R}) \quad \text{for} \quad i = 1, 2, \ldots, 5;$$

the elements $p_v \in \mathcal{G}(\mathbb{R})$ (for $v = 1,2,3,4$) admit representatives $R_{p_v}(\varphi,t)$ with the following properties: for every K there is $N \in \mathbb{N}$ such that for every $\varphi \in \mathcal{A}_N$ there are constants $c > 0$ and ε_0 such that

(3.1) $$\sup_{t\in K}\Big|\int_0^t |R_{p_v}(\varphi_\varepsilon,s)|ds\Big| \le c \qquad \text{if} \quad 0 < \varepsilon < \varepsilon_0 \quad \text{and} \quad v = 1,2,3,4.$$

REMARK 3.0. *Let δ denote the generalized function which admits as a representative the function $R_\delta(\varphi,t) = \varphi(t)$, where $\varphi \in \mathcal{A}_1$. Then δ has property (3.1).*

Hypothesis H_2

The elements $p_v \in \mathcal{G}(\mathbb{R})$ $(v = 1,2,3,4)$ admit representatives $R_{p_v}(\varphi,t)$ with the following property: there is $N \in \mathbb{N}$ such that for every $\varphi \in \mathcal{A}_N$ there are constants $\varepsilon_0 > 0$ and $\gamma > 0$ satisfying at least one of the following six conditions:

(3.2) $$\begin{aligned} I_0(p_1,p_2,p_3,p_4)_\varepsilon = b\Big(\int_0^T |R_{p_1}(\varphi_\varepsilon,t)|dt + \int_0^T |R_{p_2}(\varphi_\varepsilon,t)|dt \\ + \int_0^T |R_{p_3}(\varphi_\varepsilon,t)|dt + \int_0^T |R_{p_4}(\varphi_\varepsilon,t)|dt\Big) \le 1 - \gamma, \end{aligned}$$

where

$$b = \frac{T^3}{192} + \frac{39\sqrt{13} - 138)}{162}T^2 + \frac{1}{8}T + 1 \qquad \text{and} \quad 0 < \varepsilon < \varepsilon_0;$$

(3.3)
$$I_4(p_4)_\varepsilon = a_4 \int_0^T |R_{p_4}(\varphi_\varepsilon, t)|dt \leq 1 - \gamma,$$

where

$$a_4 = \frac{T^3}{192} \qquad \text{and} \quad 0 < \varepsilon < \varepsilon_0,$$

(3.4)
$$I_3(p_3)_\varepsilon = a_3 \int_0^T |R_{p_3}(\varphi_\varepsilon, t)|dt \leq 1 - \gamma,$$

where

$$a_3 = \left(\frac{39\sqrt{13} - 138}{162}\right)T^2 \qquad \text{and} \quad 0 < \varepsilon < \varepsilon_0,$$

(3.5)
$$I_2(p_2)_\varepsilon = a_2 \int_0^T |R_{p_2}(\varphi_\varepsilon, t)|dt \leq 1 - \gamma,$$

where

$$a_2 = \frac{4}{27}T \qquad \text{and} \quad 0 < \varepsilon < \varepsilon_0,$$

(3.6)
$$I_1(p_1)_\varepsilon = \int_0^T |R_{p_1}(\varphi_\varepsilon, t)|dt \leq 1 - \gamma \qquad \text{for} \quad \varepsilon \in (0, \varepsilon_0),$$

(3.7)
$$R_{p_4}(\varphi_\varepsilon, t) \geq 0 \qquad \text{for} \;\; t \in [0, T] \;\; \text{and} \;\; \varepsilon \in (0, \varepsilon_0).$$

Now we will give theorems on the existence and uniqueness of the solution of the problem (1.0)–(1.1). Apart from the problem (1.0)-(1.1) we will consider the homogeneous problem of the form

(3.8)
$$L(x) = 0, \qquad L_i(x) = 0, \qquad i = 1, 2, 3, 4.$$

THEOREM 3.1. *We assume that conditions (3.0)–(3.1) are satisfied and* $x = 0$ *is the unique solution of the problem (3.8) in* $\mathcal{G}(\mathbb{R})$. *Then the problem (1.0)–(1.1) has unique solution in* $\mathcal{G}(\mathbb{R})$.

REMARK 3.1. *If* $p_i \in \mathcal{G}(\mathbb{R})$ $(i = 1, 2, 3, 4)$ *have property (3.1), then the problem*

$$L(x) = p_5(t), \tag{3.9}$$

(3.10) $x^{(j)}(t_0) = r_j,\ t_0 \in \mathbb{R};\ r_j \in \overline{\mathbb{R}},\ j = 0, 1, 2, 3,$
has a unique solution $x \in \mathcal{G}(\mathbb{R})$ *(see [9]). Moreover every solution* x *of the equation (3.9) has a representation*

$$x = c_0\psi_0 + c_1\psi_1 + c_2\psi_2 + c_3\psi_3 + Q, \tag{3.11}$$

where ψ_j $(j = 0, 1, 2, 3)$ *are solutions of the problems*
(3.12)
$$L(\psi_j) = 0, \quad \psi_j^{(j)}(t_0) = 1, \quad \psi_j^{(r)}(t_0) = 0 \quad \text{for} \quad j \neq r; \quad j, r = 0, 1, 2, 3,$$

Q *is a particular solution of the equation (3.9) and* c_0, c_1, c_2, c_3 *are generalized constant functions on* $\mathbb{R}$. *The solution* x *is the class of solutions of the problems*

$$L_\varphi(x) = R_{p_5}(\varphi, t)\,. \tag{3.13}$$

$$x^{(j)}(t_0) = R_{r_j}(t_0), \quad \varphi \in \mathcal{A}_1, \quad j = 0, 1, 2, 3; \tag{3.14}$$

where

$$\begin{aligned} L_\varphi(t) = x''''(t) + R_{p_1}(\varphi, t)x'''(t) + R_{p_2}(\varphi, t)x''(t) \\ + R_{p_3}(\varphi, t)x'(t) + R_{p_4}(\varphi, t)x(t) \qquad \text{(see [9]).} \end{aligned} \tag{3.15}$$

THEOREM 3.2. *We assume that all assumptions of Theorem 3.1 are satisfied. Then the problem*

$$L_{\varphi_\varepsilon}(x) = R_{p_5}(\varphi_\varepsilon, t), \tag{3.16}$$

$$L_i(x) = R_{d_i}(\varphi_\varepsilon), \quad i = 1, 2, 3, 4, \tag{3.17}$$

has exactly one solution $x(\varphi_\varepsilon, t)$ *(for sufficiently large* N *and for small* $\varepsilon > 0$*),* $x(\varphi, t) \in \mathcal{E}_M[\mathbb{R}]$ *and* $x = [R_x(\varphi, t)]$ *is a solution of the problem*

(1.0)–(1.1). *(We put $x(\varphi_\varepsilon, t) = 0$ if the problem* (3.16)–(3.17) *has no solution).*

THEOREM 3.3. *We assume conditions (3.1)–(3.2). Then the problem* (3.8) *has only the trivial solution x in $\mathcal{G}(\mathbb{R})$.*

THEOREM 3.4. *We assume conditions (3.1) and (3.3). Then the problem*

$$x''''(t) = -p_4(t)x(t), \quad L_i(x) = 0, \quad i = 1,2,3,4 \tag{3.18}$$

has only the trivial solution x in $\mathcal{G}(\mathbb{R})$.

REMARK 3.2. *Let p_4 denote the generalized function defined by*

$$R_{p_4}(\varphi, t) = \frac{a_4\varphi(t)}{\int\limits_{-\infty}^{+\infty} |\varphi(t)|dt},$$

where $\varphi \in \mathcal{A}_1$ and $a_4 < 1$. Then p_4 has properties (3.1) and (3.3).

THEOREM 3.5. *We assume conditions (3.1) and (3.4). Then the problem*

$$x''''(t) = -p_3(t)x'(t), \quad L_i(x) = 0, \quad i = 1,2,3,4 \tag{3.19}$$

has only the trivial solution x in $\mathcal{G}(\mathbb{R})$.

THEOREM 3.6. *We assume conditions (3.1) and (3.5). Then the problem*

$$x''''(t) = -p_2(t)x''(t), \quad L_i(x) = 0, \quad i = 1,2,3,4 \tag{3.20}$$

has only the trivial solution x in $\mathcal{G}(\mathbb{R})$.

THEOREM 3.7. *We assume conditions (3.1) and (3.6). Then the problem*

$$x''''(t) = -p_3(t)x'''(t), \quad L_i(x) = 0, \quad i = 1,2,3,4 \tag{3.21}$$

has only the trivial solution x in $\mathcal{G}(\mathbb{R})$.

THEOREM 3.8. *We assume conditions (3.1) and (3.7). Then the problem (3.18) has only the trivial solution x in $\mathcal{G}(\mathbb{R})$.*

4. Proofs.

PROOF OF THEOREM 3.1. We examine the following systems of equations

$$Hc = b, \tag{4.1}$$

where

$$H = (\mathrm{H}_{ir}), \quad \mathrm{H}_{ir} = L_i\left(\psi_{r-1}^{(i-1)}\right), \quad b = (b_1, \ldots, b_4)^T, \tag{4.2}$$

$$b_i = d_i - L_i(Q); \quad i, r = 1, 2, 3, 4;$$

$\psi_0, \psi_1, \psi_2, \psi_3$ are solutions of the problem (3.12) and T denotes the transpose.

Assumptions of Theorem 3.1 and Theorem from [10] imply that det H is an invertible element of $\overline{\mathbb{R}}$ which completes the proof.

PROOF OF THEOREM 3.2. Let $R_{\psi_j}(\varphi_\varepsilon, t)$ $(j = 0, 1, 2, 3)$ be solutions of the problems

$$L_{\varphi_\varepsilon}(\psi_i) = 0, \quad R_{\psi_{i-1}}^{(i-1)}(\varphi_\varepsilon, 0) = 1, \quad R_{\psi_{r-1}}^{(i-1)}(\varphi_\varepsilon, 0) = 0 \quad \text{for } i \neq r, \quad i, r = 1, 2, 3, 4. \tag{3.12'}$$

Then every solution $x(\varphi_\varepsilon, t)$ of the equation (3.16) has the representation

$$x(\varphi_\varepsilon, t) = \sum_{j=0}^{3} c_j(\varphi_\varepsilon) R_{\psi_j}(\varphi_\varepsilon, t) + Q(\varphi_\varepsilon, t), \tag{4.3}$$

where

$$Q(\varphi_\varepsilon, t) = \int_0^t W^{-1}(\varphi_\varepsilon, s)\Big(\sum_{j=0}^{3}(R_{\psi_j}(\varphi_\varepsilon, t) \cdot D_{4j+1}(s)_\varepsilon)\Big) R_{p_5}(\varphi_\varepsilon, s)ds, \tag{4.4}$$

$$W(\varphi_\varepsilon, t) = \exp\Big(-\int_0^t R_{p_1}(\varphi_\varepsilon, s)ds\Big), \tag{4.5}$$

$D_{4j+1}(s)_\varepsilon$ denote the cofactor of $a_{4j+1}(s)_\varepsilon$ of the matrix $U_\varepsilon = (a_{ir}(s))_\varepsilon$ provided

$$a_{ir}(s) = R_{\psi_{r-1}}^{(i-1)}(\varphi_\varepsilon, s). \tag{4.6}$$

We consider the equation (3.16) with the following conditions

$$L_i(x(\varphi_\varepsilon,t)) = R_{d_i}(\varphi_\varepsilon), \quad i=1,2,3,4. \tag{4.7}$$

By (3.17), (4.3) and (4.7) we obtain the systems of equations

$$H(\varphi_\varepsilon)c(\varphi_\varepsilon) = b(\varphi_\varepsilon), \tag{4.8}$$

where

$$H(\varphi_\varepsilon) = (H_{ir}(\varphi_\varepsilon)), \quad H_{ir}(\varphi_\varepsilon) = L_i(\psi_{r-1}^{(i-1)}(\varphi_\varepsilon,s)), \tag{4.9}$$

$b_i(\varphi_\varepsilon) = R_{d_i}(\varphi_\varepsilon) - L_i(Q(\varphi_\varepsilon))$, $c(\varphi_\varepsilon) = (c_1(\varphi_\varepsilon),\ldots,c_4(\varphi_\varepsilon))^\tau$ $(i,r=1,2,3,4)$.

Applying assumptions of Theorem 3.2 and relations (4.2)–(4.9) we conclude that there is $N \in \mathbb{N}$ such that: for every $\varphi \in \mathcal{A}_N$ there are $c>0$ and $\varepsilon_0 > 0$ such that

$$|\det H(\varphi_\varepsilon)| \geq c\varepsilon^N \quad \text{for} \quad 0<\varepsilon<\varepsilon_0 \tag{4.10}$$

(because det H is an invertible element of $\overline{\mathbb{R}}$).

Using (4.3)–(4.10) we deduce that problem (3.16)–(3.17) has exactly one solution $x(\varphi_\varepsilon,t)$ (for $\varphi \in \mathcal{A}_q$, $q \geq N$ and $0<\varepsilon<\varepsilon_0$). By (4.8)–(4.10) we get

$$c(\varphi_\varepsilon) = H^{-1}(\varphi_\varepsilon)b(\varphi_\varepsilon) \tag{4.11}$$

(for $\varphi \in \mathcal{A}_N$ and $0<\varepsilon<\varepsilon_0$).

Relation (4.11) and Remark 3.1 yield (we put $c_i(\varphi_\varepsilon)=0$, $x(\varphi_\varepsilon,t)=0$ if det $H(\varphi_\varepsilon)=0$)

$$c_j(\varphi) \in \mathcal{E}_M \quad (j=0,1,2,3). \tag{4.12}$$

Since

$$R_{\psi_j}(\varphi,t) \in \mathcal{E}_M[\mathbb{R}] \quad (\text{for } j=0,1,2,3) \tag{4.13}$$

therefore $x(\varphi,t) \in \mathcal{E}_M[\mathbb{R}]$, which completes the proof of Theorem 3.2.

Before giving the proof of Theorem 3.3 we will formulate a lemma.

LEMMA 4.1. *Let $G(t,s)$ be a function defined by*

$$G(t,s) = \begin{cases} G_1(t,s), & 0 \leq t \leq s \leq T, \\ G_2(t,s), & 0 \leq s \leq t \leq T, \end{cases} \tag{4.14}$$

where

$$(4.15)\qquad G_1(t,s) = \left(-\frac{1}{3}\frac{s^3}{T^3} + \frac{1}{2T^2}s^2 - \frac{1}{6}\right)t^3 + \left(\frac{1}{2T^2}s^3 - \frac{s^2}{T} + \frac{1}{2}s\right)t^2$$

and

$$(4.16)\qquad G_2(t,s) = \left(-\frac{1}{3}\frac{s^3}{T^3} + \frac{1}{2}\frac{s^2}{T^2}\right)t^3 + \left(\frac{1}{2}\frac{s^3}{T^2} - \frac{s^2}{T}\right)t^2 + \frac{1}{2}s^2t - \frac{1}{6}s^3.$$

Then

$$(4.17)\qquad \sup_{t,s\in(0,T)} |G(t,s)| = \left|G\left(\frac{T}{2},\frac{T}{2}\right)\right| = \frac{T^3}{192} \equiv a_4,$$

$$(4.18)\qquad \sup_{t,s\in(0,T)} \left|\frac{\partial G}{\partial t}(t,s)\right| = \left|\frac{\partial G}{\partial t}(t_2,s_2)\right| = \frac{(39\sqrt{13}-138)T^2}{162} \equiv a_3,$$

where

$$(4.19)\qquad t_2 = T\left(\frac{5-\sqrt{13}}{6}\right),\quad s_2 = T\left(\frac{\sqrt{13}-1}{6}\right),$$

$$(4.20)\qquad \sup_{t,s\in(0,T)} \left|\frac{\partial^2 G}{\partial t^2}(t,s)\right| = \left|\frac{\partial^2 G}{\partial t^2}\left(0,\frac{T}{3}\right)\right| = \frac{4}{27}T \equiv a_2,$$

$$(4.21)\qquad \sup_{t,s\in(0,T)} \left|\frac{\partial^3 G}{\partial t^3}(t,s)\right| = 1$$

and the derivatives are understood in the classical sense.

PROOF OF THEOREM 3.3. If $x = [R_x(\varphi,t)]$ is a solution of the problem (3.8), then

$$(4.22)\qquad L_\varphi(R_x(\varphi,t)) = \eta(\varphi,t)$$

and

$$(4.23)\qquad L_i(R_x(\varphi,t)) = \eta_i(\varphi),$$

where

$$\eta(\varphi,t)\in\mathcal{N}[\mathbb{R}],\quad \eta_i(\varphi)\in\mathcal{N},\quad \varphi\in\mathcal{A}_1,\quad i=1,2,3,4. \tag{4.24}$$

Hence

(4.25)

$$R_x(\varphi,t)=-\int_0^T G(t,s)M_x(\varphi,s)ds+A_3(\varphi)t^3+A_2(\varphi)t^2+A_1(\varphi)t+A_0(\varphi),$$

where

$$\begin{aligned} M_x(\varphi,s)&=R_{p_1}(\varphi,s)R_{x'''}(\varphi,s)+R_{p_2}(\varphi,s)R_{x''}(\varphi,s)\\ &\quad+R_{p_3}(\varphi,s)R_{x'}(\varphi,s)+R_{p_4}(\varphi,s)R_x(\varphi,s)-\eta(\varphi,s)\end{aligned} \tag{4.26}$$

and

$$A_j(\varphi)\in\mathcal{N}\qquad \text{for}\quad j=0,1,2,3. \tag{4.27}$$

By virtue of relations (4.17)–(4.27) and (3.2) we have

$$\begin{aligned} \|R_x(\varphi_\varepsilon,t)\|^0_{[0,T]}&\le a_4\tilde{I}(p_1,p_2,p_3,p_4)_\varepsilon\cdot\|R_x(\varphi_\varepsilon,t)\|^3_{[0,T]}\\ &\quad+a_4\int_0^T|\eta(\varphi_\varepsilon,s)|ds+\sum_{j=0}^{3}A_j(\varphi_\varepsilon)T^j,\end{aligned} \tag{4.28}$$

$$\begin{aligned} \|R_{x'}(\varphi_\varepsilon,t)\|^0_{[0,T]}&\le a_3\tilde{I}(p_1,p_2,p_3,p_4)_\varepsilon\cdot\|R_x(\varphi_\varepsilon,t)\|^3_{[0,T]}\\ &\quad+a_3\int_0^T|\eta(\varphi_\varepsilon,s)|ds++3|A_3(\varphi_\varepsilon)|T^2+2|A_2(\varphi_\varepsilon)|T+|A_1(\varphi_\varepsilon)|,\end{aligned} \tag{4.29}$$

$$\begin{aligned} \|R_{x''}(\varphi_\varepsilon,t)\|^0_{[0,T]}&\le a_2\tilde{I}(p_1,p_2,p_3,p_4)_\varepsilon\cdot\|R_x(\varphi_\varepsilon,t)\|^3_{[0,T]}\\ &\quad+a_2\int_0^T|\eta(\varphi_\varepsilon,s)|ds+6|A_3(\varphi_\varepsilon)|^T+2|A_2(\varphi_\varepsilon)|\end{aligned} \tag{4.30}$$

and

$$\begin{aligned} \|R_{x'''}(\varphi_\varepsilon,t)\|^0_{[0,T]}&\le \tilde{I}(p_1,p_2,p_3,p_4)_\varepsilon\cdot\|R_x(\varphi_\varepsilon,t)\|^3_{[0,T]}\\ &\quad+\int_0^T|\eta(\varphi_\varepsilon,s)|ds+6|A_3(\varphi_\varepsilon)|,\end{aligned} \tag{4.31}$$

where

$$\tilde{I}(p_1,p_2,p_3,p_4)_\varepsilon = \frac{I_0(p_1,p_2,p_3,p_4)_\varepsilon}{b}, \tag{4.32}$$

ε is sufficiently small and $\varphi \in \mathcal{A}_N$ (N is sufficiently large).
Taking into account (4.28)–(4.32) we get

$$\|R_x(\varphi_\varepsilon,t)\|^3_{[0,T]} \leq I_0(p_1,p_2,p_3,p_4)_\varepsilon \|R_x(\varphi_\varepsilon,t)\|^3_{[0,T]} + \overline{\eta}(\varphi_\varepsilon), \tag{4.33}$$

where

$$\overline{\eta}(\varphi) \in \mathcal{N}. \tag{4.34}$$

By (4.33)–(4.34) we deduce that (for $q \geq N_1$, $\varphi \in \mathcal{A}_q$ and $0 < \varepsilon < \varepsilon_0$)

$$\|R_x(\varphi_\varepsilon,t)\|^3_{[0,T]} \leq c_3 \varepsilon^{\alpha(q)-N_1}. \tag{4.35}$$

If $t_0 \in (0,T)$, then (4.35) implies

$$R_{x^{(j)}}(\varphi,t_0) \in \mathcal{N} \qquad \text{for} \quad j = 0,1,2,3. \tag{4.36}$$

On the other hand $R_x(\varphi_\varepsilon,t)$ is a solution of the problem

$$L_{\varphi_\varepsilon}(x) = \eta(\varphi_\varepsilon,t) \tag{4.37}$$

$$x^{(j)}(t_0) = R_{x^{(j)}}(\varphi_\varepsilon,t_0). \tag{4.38}$$

Applying Remark 3.1 and (4.37)–(4.38) we conclude that

$$R_x(\varphi,t) \in \mathcal{N}[\mathbb{R}] \tag{4.39}$$

which completes the proof.

The proofs of Theorem 3.4–3.7 are similar to that of Theorem 3.3.

Proof of Thorem 3.4. We start with the equalities

$$R_{x''''}(\varphi,t) = -R_{p_4}(\varphi,t)R_x(\varphi,t) + \eta(\varphi,t), \tag{4.40}$$

$$\begin{aligned} R_x(\varphi,0) &= \eta_1(\varphi), \quad R_x(\varphi,T) = \eta_2(\varphi), \\ R_{x'}(\varphi,0) &= \eta_3(\varphi), \quad R_{x'}(\varphi,T) = \eta_4(\varphi), \end{aligned} \tag{4.41}$$

where $\eta_i(\varphi) \in \mathcal{N}$ (for $i = 1,2,3,4$), $\varphi \in \mathcal{A}_1$ and x is a solution of the problem (3.18). Applying (4.25) we get

$$\|R_x(\varphi_\varepsilon, t)\|^0_{[0,T]} \le I(p_4)_\varepsilon \|R_x(\varphi_\varepsilon, t)\|^0_{[0,T]} + \eta(\varphi_\varepsilon), \tag{4.42}$$

where

$$\eta(\varphi) \in \mathcal{N} \qquad (0 < \varepsilon < \varepsilon_0, \quad \varphi \in \mathcal{A}_N). \tag{4.43}$$

As in the proof of Theorem 3.3, we conclude that $R_x(\varphi, t)$ has property (4.39), which completes the proof of Theorem 3.4.

PROOF OF THEOREM 3.5. Let $x = [R_x(\varphi, t)]$ be a solution of the problem (3.19). Then

$$R_{x''''}(\varphi_\varepsilon, t) = -R_{p_3}(\varphi_\varepsilon, t) R_{x'}(\varphi_\varepsilon, t) + \eta(\varphi_\varepsilon, t) \tag{4.44}$$

and

$$\begin{aligned} R_x(\varphi_\varepsilon, 0) &= \eta_1(\varphi_\varepsilon), \quad & R_x(\varphi_\varepsilon, T) &= \eta_2(\varphi_\varepsilon), \\ R_{x'}(\varphi_\varepsilon, 0) &= \eta_3(\varphi_\varepsilon), \quad & R_{x'}(\varphi_\varepsilon, T) &= \eta_4(\varphi_\varepsilon), \end{aligned} \tag{4.45}$$

where

$$\eta_i(\varphi) \in \mathcal{N} \qquad (i = 1,2,3,4). \tag{4.46}$$

Hence

$$\begin{aligned} R_x(\varphi_\varepsilon, t) = &- \int_0^T G(t,s)(R_{p_3}(\varphi_\varepsilon, s) R_{x'}(\varphi_\varepsilon . s) - \eta(\varphi_\varepsilon, s)) ds \\ &+ \sum_{j=0}^{n} A_j(\varphi_\varepsilon) t^j, \end{aligned} \tag{4.47}$$

where

$$A_j(\varphi) \in \mathcal{N} \qquad (j = 0,1,2,3). \tag{4.48}$$

According to (4.14)–(4.18) we have

$$\|R_{x'}(\varphi_\varepsilon, t)\|^0_{[0,T]} \le I_3(p_3)_\varepsilon \|R_{x'}(\varphi_\varepsilon, t)\|^0_{[0,T]} + \eta_4(\varphi_\varepsilon), \tag{4.49}$$

where

$$\eta_4(\varphi) \in \mathcal{N}. \tag{4.50}$$

By (3.4) and (4.49), for $q \geq N_1$, $\varphi \in \mathcal{A}_q$ and $\varepsilon \in (0, \varepsilon_0)$ we get

$$\|R_{x'}(\varphi_\varepsilon, t)\|^0_{[0,T]} \leq c_1 \varepsilon^{\alpha(q)-N_1}. \tag{4.51}$$

Since

$$R_x(\varphi, t) = \int_0^t R_{x'}(\varphi_\varepsilon, s) ds + R_x(\varphi_\varepsilon, 0), \tag{4.52}$$

therefore (by the Schwartz inequality)

$$\|R_x(\varphi_\varepsilon, t)\|^0_{[0,T]} \leq c_0 \varphi^{\alpha(q)-N_1'} \tag{4.53}$$

(for $q \geq N_1'$, $\varphi \in \mathcal{A}_q$ and $0 < \varepsilon_0 < \varepsilon_0'$).
Taking into account relations (4.47)–(4.53) we infer that

$$\|R_{x''}(\varphi_\varepsilon, t)\|^0_{[0,T]} \leq c_2 \varepsilon^{\alpha(q)-N_2} \tag{4.54}$$

(for $q \geq N_2$, $\varphi \in \mathcal{A}_q$, $0 < \varepsilon < \varepsilon_2$).

We see that $R_x(\varphi, t)$ has property (4.35). Consequently $R_x(\varphi, t) \in \mathcal{N}[\mathbb{R}]$ which completes the proof of Theorem 3.5.

Proof of Theorem 3.6. We consider the following equality

$$R_x(\varphi_\varepsilon, t) = -\int_0^T G(t, s)(R_{p_2}(\varphi_\varepsilon, s) R_{x''}(\varphi_\varepsilon, s) - \eta(\varphi_\varepsilon, s) ds + \sum_{j=0}^{3} A_j(\varphi_\varepsilon) t^j, \tag{4.55}$$

where $\eta(\varphi, t) \in \mathcal{N}[\mathbb{R}]$ and $A_j(\varphi) \in \mathcal{N}$. Hence we have

$$\|R_{x''}(\varphi_\varepsilon, t)\|^0_{[0,T]} \leq c_2 \varepsilon^{\alpha(q)-N_2} \tag{4.56}$$

(for $\varphi \in \mathcal{A}_q$, $q \geq N_2$, $0 < \varepsilon < \varepsilon_2'$). On the other hand

$$R_{x'}(\varphi_\varepsilon, t) = \int_0^t R_{x''}(\varphi_\varepsilon, s) ds + R_{x'}(\varphi_\varepsilon, 0) \tag{4.57}$$

and

$$R_x(\varphi_\varepsilon, t) = \int_0^t R_{x'}(\varphi_\varepsilon, s) ds + R_x(\varphi_\varepsilon, 0), \tag{4.58}$$

$$R_x(\varphi,0) \in \mathcal{N}, \quad R_{x'}(\varphi,0) \in \mathcal{N}. \tag{4.59}$$

Therefore by the Schwartz inequality $R_x(\varphi,t)$ has properties (4.35) and (4.39), which completes the proof of the theorem.

PROOF OF THEOREM 3.7. We examine the equality

(4.60)
$$R_x(\varphi_\varepsilon,t) = -\int_0^T G(t,s)(R_{p_1}(\varphi_\varepsilon,s)R_{x'''}(\varphi_\varepsilon,s) - \eta(\varphi_\varepsilon,s)) + \sum_{j=0}^{3} A_j(\varphi_\varepsilon)t^j,$$

where η and A_j $(j=0,1,2,3)$ have properties (4.46) and (4.48). Obviously

$$\|R_{x'''}(\varphi_\varepsilon,t)\|^0_{[0,T]} \le c_3\varepsilon^{\alpha(q)-N_3} \tag{4.61}$$

(for $\varphi \in \mathcal{A}_q$, $q \ge N_3$, $0 < \varepsilon < \varepsilon_3'$). Relations (4.60)–(4.61) lead to inequalities (4.28) and (4.35) which completes the proof of Theorem 3.7.

PROOF OF THEOREM 3.8. Let $x = [R_x(\varphi,t)]$ be a solution of the problem (3.18). Then $R_x(\varphi,t)$ has properties (4.40)–(4.41). Hence we have (having integrated by parts)

$$\int_0^T R_{x''''}(\varphi_\varepsilon,t)R_x(\varphi_\varepsilon,t)dt = R_{x'''}(\varphi_\varepsilon,T)R_x(\varphi_\varepsilon,T) - \tag{4.62}$$

$$R_{x'''}(\varphi_\varepsilon,0)R_x(\varphi_\varepsilon,0) - R_{x''}(\varphi_\varepsilon,T)R_{x'}(\varphi_\varepsilon,T) + R_{x''}(\varphi_\varepsilon,0)R_x(\varphi_\varepsilon,0)$$
$$+\int_0^T R^2_{x''}(\varphi_\varepsilon,t)dt = -\int_0^T R_{p_4}(\varphi_\varepsilon,t)R^2_x(\varphi_\varepsilon,t)dt + \eta_5(\varphi_\varepsilon),$$

where

$$\eta_5(\varphi) \in \mathcal{N}. \tag{4.63}$$

Relations (4.62)–(4.63) lead to

$$\int_0^T R^2_{x''}(\varphi_\varepsilon,t)dt + \int_0^T R_{p_4}(\varphi_\varepsilon,t)R^2_x(\varphi_\varepsilon,t)dt = \eta_6(\varphi_\varepsilon), \tag{4.64}$$

where

$$\eta_6(\varphi) \in \mathcal{N}. \tag{4.65}$$

Therefore

$$\int_0^T R_{x''}^2(\varphi, t)dt \in \mathcal{N} \tag{4.66}$$

and

$$\int_0^T R_{p_4}(\varphi, t)R_x^2(\varphi, t)dt \in \mathcal{N}. \tag{4.67}$$

Applying the Schwarz inequality to the equalities

$$R_{x'}(\varphi_\varepsilon, t) = \int_0^t R_{x''}(\varphi_\varepsilon, s)ds + R_{x'}(\varphi_\varepsilon, 0), \tag{4.68}$$

$$R_x(\varphi_\varepsilon, t) = \int_0^t R_{x'}(\varphi_\varepsilon, s)ds + R_x(\varphi_\varepsilon, 0) \tag{4.69}$$

we obtain

$$\|R_{x'}(\varphi_\varepsilon, t)\|_{[0,T]}^0 \le c_1\varepsilon^{\alpha(q)-N_1} \tag{4.70}$$

and

$$\|R_x(\varphi_\varepsilon, t)\|_{[0,T]}^0 \le c_0\varepsilon^{\alpha(q)-N_1} \tag{4.71}$$

(for $\varphi \in \mathcal{A}_q,\ q \ge N_1,\ 0 < \varepsilon < \varepsilon_0'$).

On the other hand $R_x(\varphi_\varepsilon, t)$ satisfies the following equality

(4.72)

$$R_x(\varphi_\varepsilon, t) = -\int_0^t \frac{(t-s)^3}{3!}(R_{p_4}(\varphi_\varepsilon, s)R_x(\varphi_\varepsilon, s) - \eta(\varphi_\varepsilon, s))ds$$
$$+ R_x(\varphi_\varepsilon, 0) + R_{x'}(\varphi_\varepsilon, 0)t + \frac{R_{x''}(\varphi_\varepsilon, 0)}{2!}t^2 + \frac{R_{x'''}(\varphi_\varepsilon, 0)}{3!}t^3.$$

Hence

$$\begin{aligned} R_{x'}(\varphi_\varepsilon, t) &= -\int_0^t \frac{(t-s)^2}{2!}(R_{p_4}(\varphi_\varepsilon, s)R_x(\varphi_\varepsilon, s) - \eta(\varphi_\varepsilon, s))ds \\ &\quad + R_{x'}(\varphi_\varepsilon, 0) + R_{x''}(\varphi_\varepsilon, 0)t + \frac{R_{x'''}(\varphi_\varepsilon, 0)}{2!}t^2. \end{aligned} \tag{4.72'}$$

Taking into account (3.1), (4.41) and (4.70)–(4.72) we get a system of equations (putting $t = T$)
(4.73)
$$\begin{cases} \frac{R_{x''}(\varphi_\varepsilon,0)T^2}{2} + \frac{R_{x'''}(\varphi_\varepsilon,0)T^3}{6} = \eta_7(\varphi_\varepsilon) R_{x''}(\varphi_\varepsilon,0)T + \frac{R_{x'''}(\varphi_\varepsilon,0)T^2}{2} = \eta_8(\varphi_\varepsilon), \end{cases}$$

where $\eta_7(\varphi) \in \mathcal{N}$ and $\eta_8(\varphi) \in \mathcal{N}$. Relation (4.73) yields

$$R_{x''}(\varphi_\varepsilon, 0) = \frac{12}{T^4}\left(\frac{T^2}{2}\eta_7(\varphi_\varepsilon) - \eta_8(\varphi_\varepsilon)\frac{T^3}{6}\right) \tag{4.74}$$

and

$$R_{x'''}(\varphi_\varepsilon, 0) = \frac{12}{T^4}\left(\frac{T^2}{2}\eta_8(\varphi_\varepsilon) - T\eta_7(\varphi_\varepsilon)\right). \tag{4.75}$$

Consequently

$$R_{x''}(\varphi, 0) \in \mathcal{N} \qquad \text{and} \qquad R_{x'''}(\varphi, 0) \in \mathcal{N}. \tag{4.76}$$

By (4.41), (4.76) and Remark 3.1 we deduce that $R_x(\varphi, t) \in \mathcal{E}_M[\mathbb{R}]$, which ~ompletes the proof of Thorem 3.8.

5. Remarks on Caratheodory's and Colombeau's solutions of ordinary differential equations

REMARK 5.1. *If $g_1, g_2 \in C^\infty$, then the classical product $g_1 \cdot g_2$ and the product $g_1 \circ g_2$ in $\mathcal{G}(\mathbb{R})$ give rise to the same element of $\mathcal{G}(\mathbb{R})$.*
Hence we obtain

THEOREM 5.1. *We assume that*

$$p_v \in C^\infty, \quad d_i \in \mathbb{R} \qquad \text{for} \quad v = 1,2,3,4,5; \ \ i = 1,2,3,4, \tag{5.1}$$

(5.2) *the zero function is the unique solution of the problem (3.8) in the classical sense,*
(5.3) *x_1 is the solution of the problem (1.0)–(1.1) in the classical sense, $x_2 \in \mathcal{G}(\mathbb{R})$ is the solution of the problem*

$$\begin{cases} \tilde{L}(x) \equiv x''''(t) + p_1(t) \circ x'''(t) + p_2(t) \circ x''(t) + p_3(t) \circ x'(t) + \\ \quad p_4(t) \circ x(t) = p_5(t) \\ L_i(x) = d_i, \quad i = 1,2,3,4. \end{cases} \tag{5.4}$$

Then x_1 and x_2 give rise to the same element of $\mathcal{G}(\mathbb{R})$.

PROOF OF THEOREM 5.1. Let $x_2 = [R_{x_2}(\varphi, t)]$ be a solution of the problem (5.4) and let x_1 be a solution of problem (1.0)–(1.1). Then

$$L(x_1) = p_5(t), \quad L_i(x) = d_i \tag{5.5}$$

and

$$L(R_{x_2}(\varphi_\varepsilon, t)) = p_5(t) + \eta(\varphi_\varepsilon, t), \quad L_i(R_x(\varphi_\varepsilon, t)) = d_i + \eta_i(\varphi_\varepsilon), \tag{5.6}$$

where $\eta(\varphi, t) \in \mathcal{N}[\mathbb{R}]$, $\eta_i(\varphi) \in \mathcal{N}$ and $i = 1, 2, 3, 4$. Thus

$$L(R_x(\varphi_\varepsilon, t)) = \eta(\varphi_\varepsilon, t), \quad L_i(R_x(\varphi_\varepsilon, t)) = -\eta_i(\varphi_\varepsilon), \tag{5.7}$$

where

$$R_x(\varphi_\varepsilon, t) = x_1(t) - R_{x_2}(\varphi_\varepsilon, t) \quad \text{and} \quad i = 1, 2, 3, 4. \tag{5.8}$$

On the other hand $R_x(\varphi_\varepsilon, t)$ has a representation (4.3), where $Q(\varphi_\varepsilon, t)$ is defined by (4.4) (putting $R_{p_4}(\varphi_\varepsilon, s) = \eta(\varphi_\varepsilon, s)$). Relations (4.8)–(4.11), (4.3) and (5.7) imply

$$c_j(\varphi) \in \mathcal{N}, \qquad j = 0, 1, 2, 3 \tag{5.9}$$

and consequently

$$x_1(t) - R_{x_2}(\varphi, t) \in \mathcal{N}[\mathbb{R}]. \tag{5.10}$$

This proves of Theorem 5.1.

REMARK 5.2. *If $p_r \in L^1_{loc}(\mathbb{R})$, then*

$$R_{p_r}(\varphi, t) = \int_{-\infty}^{\infty} p_r(t + \varepsilon u)\varphi(u)du = (p_r * \varphi) \in \mathcal{E}_M[\mathbb{R}]$$

and p_r have property (3.1) for $r = 1, 2, 3, 4, 5$. It is known that every distribution is moderate (see [1]). Multiplication in $\mathcal{G}(\mathbb{R})$ does not coincide with usual multiplication of continuous functions in general (see [1]). As consequence solutions of differential equations in the Caratheodory sense and in the Colombeau sense are different (in general). To repair to consistency problem for multiplication we give the difinition introduced by J. F. Colombeau in [1].

An element u of $\mathcal{G}(\mathbb{R})$ is said to admit a member $W \in \mathcal{D}'(\mathbb{R})$ as the associated distribution, if it has a representative $R_u(\varphi, t)$ with the following property: for every $\psi \in \mathcal{D}(\mathbb{R})$ there is $N \in \mathbb{N}$ such that for every $\varphi \in \mathcal{A}_N$ we have

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} R_u(\varphi_\varepsilon, t)\psi(t)dt = W(\psi). \tag{5.11}$$

THEOREM 5.2. *We assume that*

$$p_v \in L^1_{loc}(\mathbb{R}) \quad \text{for} \quad v = 1,2,3,4,5; \tag{5.12}$$

the zero function is the unique solution of the problem

$$L(x) = 0, \quad L_i(x) = 0, \quad i = 1,2,3,4 \tag{5.13}$$

in the Caratheodory sense,
x is the solution of the problem

$$L(x) = p_5(t), \quad L_i(x) = d_i, \quad d_i \in \mathbb{R} \ \ (i = 1,2,3,4); \tag{5.14}$$

in the Caratheodory sense,
$\tilde{x} \in \mathcal{G}(\mathbb{R})$ is the solution of the problem

$$\tilde{L}(x) = p_5(t), \quad L_i(x) = d_i \quad (i = 1,2,3,4). \tag{5.15}$$

Then $\tilde{x}$ admits an associated distribution which equals x.

PROOF OF THEOREM 5.2. Proof of Theorem 5.2 follows from the fact that

$$R_{p_v}(\varphi_\varepsilon, t) = (p_v * \varphi_\varepsilon)(t) \to p_v(t)$$

in $L^1_{loc}(\mathbb{R})$ and the continuous dependence of x on coefficients p_v for $v = 1,2,3,4,5$. Indeed, let $R_{\psi_j}(\varphi_\varepsilon, t)$ $(j = 0,1,2,3)$ be the solution of the problems $(3.12)'$. Then we conclude that

$$\lim_{\varepsilon \to 0} \psi_{i-1}^{(r-1)}(\varphi_\varepsilon, t) = \psi_{i-1}^{(r-1)}(t) \tag{5.16}$$

(almost uniformly) for $i, r = 1,2,3,4$ and every fixed $\varphi \in \mathcal{A}_N$. This yields

$$\lim_{\varepsilon \to 0} |\det H_\varepsilon| = g \neq 0, \quad g \in \mathbb{R} \tag{5.17}$$

for every $\varphi \in \mathcal{A}_1$ (det H_ε is defined by (4.10)). Let $R_x(\varphi_\varepsilon, t)$ be a solution of equation (3.16) satisfying the conditions

$$L_i(R_x(\varphi_\varepsilon, t)) = d_i \quad \text{for} \quad \varepsilon \in (0, \varepsilon_1), \varphi \in \mathcal{A}_N \tag{5.18}$$

(N is sufficiently large) and $i = 1, 2, 3, 4$.

By virtue of relations (4.4)–(4.6), (4.11) and (5.16)–(5.18) we get

$$\lim_{\varepsilon \to 0} R_{x^{(r-1)}}(\varphi_\varepsilon, t) = x^{(r-1)}(t) \tag{5.19}$$

(almost uniformly for every fixed $\varphi \in \mathcal{A}_N$ and $r = 1, 2, 3, 4$) and x is a solution of the problem (5.14) in the Caratheodory sense. On the other hand $\overline{x} = [R_x(\varphi, t)]$ is the solution of the problem (5.15). (We put $R_x(\varphi, t) = 0$ if det $H_\varepsilon = 0$). This proves of Theorem 5.3.

Corollary 5.1. *We assume that*

$$p_v \in L^1_{loc}(\mathbb{R}), \quad v = 1, 2, 3, 4. \tag{5.20}$$

and

$$I(p_1, p_2, p_3, p_4) = b \left(\sum_{v=1}^{4} \int_0^T |p_v(t)| dt \right) < 1. \tag{5.21}$$

Then the problem (3.8) has only the trivial solution in the Caratheodory sense.

Corollary 5.2. *We assume conditions (5.20) and*

$$I_4(p_4) = a_4 \int_0^T |p_4(t)| dt < 1. \tag{5.22}$$

Then the problem (3.18) has only the trivial solution in the Caratheodory sense.

Corollary 5.3. *We assume conditions (5.20) and*

$$I_3(p_3) = a_3 \int_0^T |p_3(t)| dt < 1. \tag{5.23}$$

Then the problem (3.19) has only the trivial solution in the Caratheodory sense.

COROLLARY 5.4. *We assume conditions (5.20) and*

$$I_2(p_2) = a_2 \int_0^T |p_2(t)|dt < 1. \tag{5.24}$$

Then the problem (3.20) has only the trivial solution in the Caratheodory sense.

COROLLARY 5.5. *We assume conditions (5.20) and*

$$I_1(p_1) = \int_0^T |p_3(t)|dt < 1. \tag{5.25}$$

Then the problem (3.21) has only the trivial solution in the Caratheodory sense.

COROLLARY 5.6. *We assume conditions (5.20) and*

$$p_4(t) \geq 0 \qquad \text{for almost all} \quad t \text{ in } [0, T]. \tag{5.26}$$

Then the problem (3.18) has only the trivial solution in the Caratheodory sense.

REMARK 5.3. *The boundary value problems for generalized differential equations can be considered on the other way (for example: [2]–[4], [6]–[8], [11]–[14]).*

REMARK 5.4. *The definition of generalized function on an open interval $(a, b) \subset \mathbb{R}$ is almost the same as the definition in the whole $\mathbb{R}$ (see [1]). It is not difficult to observe that the proved theorems are also true in the case when generalized functions p_i and x are considered on an interval $(a, b) \supset [0, T]$. For this purpose it is necessary to formulate properties (3.0)–(3.7) on the iterval (a, b).*

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Instytut Matematyki,
Uniwersytet Śląski,
ul. Bankowa 14,
40–007 Katowice,
Poland