Annales Mathematicae Silesianae **15** (2001)

Prace Naukowe Uniwersytetu Śląskiego nr 1999, Katowice

ON FUNCTIONS OF BOUNDED N–TH VARIATION

Małgorzata Wróbel

1. Introduction

The class of functions of bounded n–th variation, denoted by $BV_n[a,b]$, was introduced by M. T. Popoviciu in 1933. In 1979 A. M. Russell in [6] proved the Jordan–type decomposition theorem for functions from this class, then, applying this result, showed that each $BV_n[a,b]$, with a suitable norm, is a Banach space. For $n = 0$ and $n = 1$ the above facts are well known classical results (cf., for instance, [5], [7]).

However, the proofs given by A. M. Russell for $n \geq 2$, based on some properties of divided differences, are rather complicated. The aim of this note, is to give an essentially simpler arguments both, for the Jordan–type decomposition theorem as well as for the completness of the space $BV_n[a,b]$. In our proofs we apply the Popoviciu theorem and the fact that for every positive integer $n \geq 2$, a function f is n–convex iff the derivative f' is $(n-1)$–convex.

Let us mention that the completness of $BV_n[a,b]$ can be applied in the theory of iterative functional equations (cf. [2] where the solutions of the class $BV_1[a,b]$ are investigated).

2. Preliminaries

We begin with the following definitions:

Definition 1. (cf. [8], p. 237). Let $f : [a,b] \to \mathbb{R}$ and let $x_1, \ldots, x_{n+1}$ be distinct points in $[a,b]$. The divided difference of order n of f at points

Received: 22.12.1997. Revised: 13.06.2001.

$x_1, ..., x_{n+1}$ we define by reccurence:

$$(1) \qquad \begin{aligned} &[x_1; f] = f(x_1), \\ &[x_1, ..., x_{n+1}; f] = \frac{[x_2, ..., x_{n+1}; f] - [x_1, ..., x_n; f]}{x_{n+1} - x_1}. \end{aligned}$$

DEFINITION 2. (cf. [8], p. 239). A function $f : [a, b] \to \mathbb{R}$ is said to be n–convex iff for all choices of $x_1 < x_2 < ... < x_{n+2}$ in $[a, b]$:

$$[x_1, ..., x_{n+2}; f] \geq 0.$$

It is easily seen that 0–convex function is increasing and 1–convex function is convex in the classical sense.

DEFINITION 3. (cf. [8], p. 239). Let $f : [a, b] \to \mathbb{R}$ and let

$$(2) \qquad V_n(f) := \sup_P \sum_{i=1}^{m-n-1} \mid [x_{i+1}, ..., x_{i+n+1}; f] - [x_i, ..., x_{i+n}; f] \mid,$$

where the supremum is taken over all the partitions

$$P = \{(x_1, \ldots, x_m) : \; a = x_1 < ... < x_m = b; \;\; m \geq n + 2\}.$$

We say that f is of bounded n–th variation on $[a, b]$ if and only if $V_n(f) < \infty$, and denote by $BV_n[a, b]$ the class of all such functions.

The following result by Popoviciu [4] plays a crucial role in our paper.

LEMMA 1. *If $f \in BV_n[a, b]$, $n > 1$, then f' exists, $f' \in BV_{n-1}[a, b]$, and*

$$(3) \qquad nV_n(f) = V_{n-1}(f'), \;\; n > 1.$$

If $f \in BV_1[a, b]$ then f'_- exists on $(a, b]$, f'_+ exists on $[a, b)$ and $V_1(f) = V_0(f'_+)$. (By f'_- and f'_+ we denote the right and left hand derivatives of f, respectively).

Lemma 1 implies that $f^{(k)}$, $k = 1, ..., n-1$, exist and they are finite in (a, b). Moreover (cf. [8], p. 27), if $f \in BV_1[a, b]$ then f is absolutely continuous on $[a, b]$ and it has finite derivatives exept for at most countably many points.

Now let us quote

LEMMA 2. *(cf. [3], p. 392). For every positive integer $n > 1$, a function $f : (a, b) \to \mathbb{R}$ is n–convex, iff f is of class C^{n-1} in (a, b) and the derivative f' is $(n-1)$–convex.*

REMARK 1. *Let $n \geq 1$ be a natural number and suppose that $f : [a,b] \to \mathbb{R}$ is n-convex function. Then $V_n(f) < \infty$ if and only if both $f_+^{(n)}(a)$ and $f_-^{(n)}(b)$ exist and are finite.*

PROOF. Notice that if f is n-convex function on $[a,b]$ then, by Definition 3, we have

$$V_n(f) = \sup_P ([x_{m-n}, \ldots, x_{m-1}, b; f] - [a, x_2, \ldots, x_{n+1}; f])$$

where the supremum is taken over all the partitions

$$P = \{(x_1, \ldots, x_m) : \; a = x_1 < \ldots < x_m = b; \;\; m \geq n+2\}.$$

Since for $n = 1$ it is obvious (cf. [5], p. 569), assume that $n > 1$.

According to [1] (cf. [1], Theorem 6) we have that the functions

$$(x_{m-n}, \ldots, x_{m-1}, b; f) \to [x_{m-n}, \ldots, x_{m-1}, b; f]$$

$$(a, x_2, \ldots, x_{n+1}; f) \to [a, x_2, \ldots, x_{n+1}; f]$$

are monotonic increasing with respect to all the variables, and, consequently, we have

$$V_n(f) = \lim_{x_{m-n} \to b^-} [x_{m-n}, \ldots, x_{m-1}, b; f] - \lim_{x_{n+1} \to a^+} [a, x_2, \ldots, x_{n+1}; f].$$

Thus (cf. [1], Theorem 7), if $f_+^{(n)}(a)$ and $f_-^{(n)}(b)$ are finite then

$$V_n(f) = f_-^{(n)}(b) - f_+^{(n)}(a) < \infty.$$

If $f \in BV_n[a,b]$ then, by Lemma 1, we have $f^{(n-1)} \in BV_1[a,b]$ and, therefore $f_+^{(n)}(a)$ and $f_-^{(n)}(b)$ exist and are finite.

According to this remark, not every n-convex function belongs to $BV_n[a,b]$.

Note that, in view of Lemma 1, Lemma 2 and Remark 1 we immediately get

LEMMA 3. *Let $n > 1$ be a natural number. Suppose that a function $f \in BV_{n-1}[a,b]$ is $(n-1)$-convex and put $F(x) := \int_a^x f(t)dt$. Then $V_n(F) < \infty$.*

3. A decomposition of functions of $BV_n[a,b]$

We are going to prove the following

THEOREM 1. *Every function of bounded n-th variation in a closed interval is a difference of two functions which are n-convex and of n-th bounded variation.*

PROOF. Suppose that $f = g - h$ where g and h are n-convex functions on $[a,b]$ which are of bounded n-th variation. Let us remark here (cf. Definition 2) that $V_n(f) \leq V_n(g) + V_n(h)$, therefore $f \in BV_n[a,b]$.

The proof of the converse implication is by induction on n. It is true for $n = 0$ and $n = 1$ (cf. [8], Theorem 14D). Now assume that $n > 1$ and that every function $f \in BV_k[a,b]$, $k = 0, 1, \ldots, n-1$, may be represented as a difference $g - h$ of two k-convex functions such that g and h are of k-th bounded variation, and take an arbitrary $f \in BV_n[a,b]$. From Lemma 1 it follows that f is a differentiable in $[a,b]$ and $f' \in BV_{n-1}[a,b]$. By the induction hypothesis, there exist $(n-1)$-convex functions g_1 and h_1 such that $f' = g_1 - h_1$ and $g_1, h_1 \in BV_{n-1}[a,b]$. Now put

$$g(x) := \int_a^x g_1(t)dt + \frac{1}{2}f(a), \quad x \in [a,b];$$

$$h(x) := \int_a^x h_1(t)dt - \frac{1}{2}f(a), \quad x \in [a,b].$$

According to Lemma 2 and Lemma 3, g and h are n-convex and of bounded n-th variation. Since

$$f(x) = f(a) + \int_a^x f'(t)dt = g(x) - h(x), \quad x \in [a,b],$$

the induction completes the proof.

REMARK 2.. *It is evident that the decomposition of a $BV_n[a,b]$-function in Theorem 1 is not uniquelly determined as we may add an arbitrary n-convex function to each component.*

4. $BV_n[a,b]$ is a Banach space

Define $\|\cdot\|_n$: $BV_n[a,b] \to \mathbb{R}_+$ by the formula

$$\|f\|_n := V_n(f) + |f(a)| + |f'_+(a)| + \ldots + |f_+^{(n)}(a)|. \tag{4}$$

It is easy to see that $f \in BV_n[a,b]$ iff $\| f \|_n < \infty$ and that $BV_n[a,b]$ is a real linear space with the usual addition of functions and multiplication by scalars.

LEMMA 4. *For every n, $n = 0, 1, \ldots$, the function $\| \cdot \|_n$ defined by (4) is a norm.*

PROOF. If $\| f \|_n = 0$ then for every $x_1, \ldots, x_{n+2} \in [a,b]$ such that $x_1 < \ldots < x_{n+2}$ we have that $| [x_2, \ldots, x_{n+2}; f] - [x_1, \ldots, x_{n+1} : f] | = 0$ which means that the divided difference $[x_1, \ldots, x_{n+2}; f] = 0$. Thus (cf. [3], p. 398) f is a polynomial of degree at most n. Since

$$| f(a) | = | f'_+(a) | = \ldots = | f_+^{(n)}(a) | = 0,$$

we have $f \equiv 0$. It is obvious $f \equiv 0$ implies $\| f \|_n = 0$.

Since

$$[x_1, \ldots, x_{n+1}; f + g] \leq [x_1, \ldots, x_{n+1}; f] + [x_1, \ldots, x_{n+1}; g]$$

and

$$[x_1, \ldots, x_{n+1}; \alpha f] = \alpha [x_1, \ldots, x_{n+1}; f]$$

for f, $g \in BV_n[a,b]$, $\alpha \in \mathbb{R}$ and $x_1, \ldots, x_{n+1} \in [a,b]$, it follows that $\| \cdot \|_n$ defined by (4) satisfies all axioms of the norm. This completes the proof.

Put

$$BVN_n[a,b] := \{ f \in BV_n[a,b] : \; f(a) = f'_+(a) = \ldots = f_+^{(n)}(a) = 0 \}$$

and note that $(BV_nN[a,b], \mathbb{R}, +, \cdot, \| \cdot \|_n)$ is a normed linear subspace of $(BV_n[a,b], \mathbb{R}, +, \cdot, \| \cdot \|_n)$ such that for $f \in BV_n[a,b]$ we have $\| f \|_n = V_n(f)$.

In the sequel we write $BV_n[a,b]$ and $BV_nN[a,b]$ to denote the spaces $(BV_n[a,b], \mathbb{R}, +, \cdot, \| \cdot \|_n)$ and $(BV_nN[a,b], \mathbb{R}, +, \cdot, \| \cdot \|_n)$, respectively.

LEMMA 5. *For every $n \in \{0, 1, \ldots\}$, $BV_nN[a,b]$ is a Banach space.*

PROOF. Since the space $BV_0[a,b]$ coincides with the Banach space $BV[a,b]$ of functions of bounded variation, our lemma is true for $n = 0$. In the paper [5] there is the proof that it is also true for $n = 1$. Assume now that $n > 1$ and that $BV_iN[a,b]$, $i = 0, 1, \ldots, n-1$, are Banach spaces. In order to prove that $BV_nN[a,b]$ is a Banach space take a Cauchy sequence $(f_s)_{s \in \mathbb{N}}$ of elements of the space $BV_nN[a,b]$. Thus, given $\varepsilon > 0$, there exists a number $s_0 \in \mathbb{N}$ such that

$$\| f_k - f_s \|_n < \varepsilon, \quad k, s \geq s_0. \tag{5}$$

In view of (3) we have

$$\| f_k - f_s \|_n = \frac{1}{n} \| f_k' - f_s' \|_{n-1}, \quad k, s \in \mathbb{N}.$$

Therefore from (5) it follows that $(f_s')_{s\in\mathbb{N}}$ is a Cauchy sequence in $BV_{n-1}N[a,b]$. By the induction hypothesis, $BV_{n-1}N[a,b]$ is a complete space, thus $(f_s')_{s\in\mathbb{N}}$ converges to an element F in $BV_{n-1}N[a,b]$. Define $f : [a,b] \to \mathbb{R}$ by the formula

$$f(x) = \int_a^x F(t)dt, \quad x \in [a,b].$$

Obviously $f(a) = 0$, and since $f_+^{(r)}(a) = F_+^{(r-1)}(a)$ for $r = 1, ..., n$, we also have $f_+'(a) = f_+''(a) = ... = f_+^{(n)}(a) = 0$. According to Theorem 1, there are $(n-1)$–convex and of $(n-1)$–bounded variation G and H such that $F = G - H$. Hence $f = g - h$ where

$$g(x) = \int_a^x G(t)dt, \quad h(x) = \int_a^x H(t)dt, \quad x \in [a,b].$$

In view of Lemma 2 the functions g and h are n–convex. Using Lemma 3 we obtain that g and h are of bounded n-th variation, therefore $f \in BV_nN[a,b]$. Because $f \in BV_nN[a,b]$ and $f_s \in BV_nN[a,b]$, $s \in \mathbb{N}$, we have $f - f_s \in BV_nN[a,b]$ and, by Lemma 1, we obtain

$$\| f_s - f \|_n = \frac{1}{n} \| f_s' - F \|_{n-1},$$

and, consequently $\| f_s - f \|_n \to 0$ as $s \to \infty$. This implies that $BV_nN[a,b]$ is a complete space, which ends the proof.

LEMMA 6. *A function $f \in BV_n[a,b]$ if and only if there exist $g \in BV_nN[a,b]$ and $A_0, ..., A_n \in \mathbb{R}$ such that*

$$f(x) = g(x) + \sum_{k=0}^{n} A_k x^k.$$

The function g and numbers $A_0, ..., A_n$ are uniquelly determined.

PROOF. Let us consider the system of $n+1$ linear equations given by the formula

$$\sum_{i=0}^{n-k} \frac{(n-i)!}{(n-k-i)!} A_{n-i} a^{n-k-i} = f^{(k)}(a)$$

for $k = 0, 1, ..., n$, with the unknown $A_0, \ldots, A_n$. Because the main matric of this system is triangle, it is obious that the determinal reduces to the product of the elements of diagonal and it is easy to check that it is equal $\prod_{k=1}^{n} k!$, so it is a Cramer system, and, consequently $A_0, \ldots, A_n$ exist and are uniquelly determined. Put

$$g(x) := f(x) - \sum_{k=0}^{n} A_k x^k.$$

From the definition of $A_0, ..., A_n$ it is easily seen that $g(a) = g'_+(a) = \ldots = g_+^{(n)}(a) = 0$ and, since $V_n(f) = V_n(g)$, we have $g \in BV_nN[a, b]$. The proof of the converse implication follows immediately from the equality $V_n(f) = V_n(g)$. The last equation holds true by the property that n–th variation of polynomials of degree at most n is equal zero (cf. [1], p.82).

Now we are in a position to prove

THEOREM 2. *For every $n \in \{0, 1, ..., \}$, $BV_n[a, b]$ is a Banach space.*

PROOF. Let n be arbitrarily fixed and let $(f_s)_{s \in \mathbb{N}}$ be a Cauchy sequence in $BV_n[a, b]$. Thus, for every $\varepsilon > 0$, we can find $s_0 \in \mathbb{N}$ such that

$$\| f_s - f_l \|_n < \varepsilon, \quad s, l \geq s_0. \tag{6}$$

From Lemma 6, for every $s \in \mathbb{N}$, there exist a function $g_s \in BV_nN[a, b]$ and $A_n^s, ..., A_0^s$ such that

$$f_s(x) = g_s(x) + P_s(x),$$

where $P_s(x) := A_n^s x^n + ... + A_0^s$. By (6) we hence get

$$\| (g_s - g_l) + (P_s - P_l) \|_n < \varepsilon, \tag{7}$$

for s, $l \geq s_0$.

Therefore, from the Definition 4 of the norm, we obtain

$$V_n(g_s - g_l) < \varepsilon, \quad s, l \geq s_0, \tag{8}$$

and

$$\left| \frac{n!}{(n-k)!}(A_{n-k}^s - A_{n-k}^l)a^{n-k} + ... + k!(A_k^s - A_k^l) \right| < \varepsilon, \tag{9}$$

$k = 0, ..., n$.

According to (8), $(g_s)_{s\in\mathbb{N}}$ is a Cauchy sequence in $BV_nN[a,b]$. Consequently, there exists a function $g \in BV_nN[a,b]$ such that

$$\| g_s - g \|_n \to 0 \qquad as \;\; s \to \infty,$$

whereas (9) implies the convergence of the sequences $(A_k^s)_{s\in\mathbb{N}}$, $k = 0, ..., n$ in $\mathbb{R}$. Now we define

$$A_k := \lim_{s\to\infty} A_k^s, \qquad k = 0, ..., n,$$

and put

$$f(x) = g(x) + A_n x^n + ... + A_0, \qquad x \in [a,b].$$

Since $g \in BV_nN[a,b]$ so $f \in BV_n[a,b]$. Letting $l \to \infty$ in (7), we obtain

$$\| f_s - f \|_n < \varepsilon, \quad s \geq s_0,$$

which means that $f_s \to f$ as $s \to \infty$. Thus we have proved that every Cauchy sequence in $BV_n[a,b]$ converges, and the proof is completed.

References

[1] P. S. Bullen, *A criterion for n-convexity*, Pac. J. Math. **36**, no 1 (1971), 81–98.

[2] T. Kostrzewski, *Existence and uniqueness of $BC(a,b)$ solutions of nonlinear functional equation*, Demonstr. Math. **26**,1 (1993), 61–74.

[3] M. Kuczma, *An introduction to the theory of functional equations and inequalities*, PWN, Warszawa–Kraków–Katowice 1985.

[4] M. T. Popoviciu, *Sur quelques proprietes des functions d'une variable reele convexes d'orde superieur*, Mathematica, Cluj **8**.

[5] A. W. Roberts, D. E. Varberg, *Functions of bounded convexity*, Bulletin of AMS **75.3**, (1969), 568–572.

[6] A. M. Russell, *A commutative Banach algebra of functions of generalized variation*, Pac. J. Math. **84**, no 2 (1979), 455–463.

[7] A. E. Taylor, *Introduction to functional analysis*, Wiley, New York 1967.

[8] D. E. Varberg, *Convex functions*, Academic Press 1973.

INSTYTUT MATEMATYKI I INFORMATYKI
WSP W CZĘSTOCHOWIE
UL. ARMII KRAJOWEJ 13/15
42–200 CZĘSTOCHOWA