THE EXPANSION OF SOME DISTRIBUTIONS INTO THE WIENER SERIES

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1. Introduction

The aim of this paper is to investigate a discrete integral transform on the real line, which seems to be better adapted for some applications then the Hermite transform (see for example [6]). Another complete orthonormal system (CON) of functions on the real line, which was introduced by Wiener is more appropriate for nonlinear differential equations of mathematical physics. The reasons are that there exist linearization formulas with respect to the argument as well as with respect to the index and that the functions tend to zero as $|x|$ tends to infinity as quickly as $|x|^{-1}$.

Notations: Let $\mathbb{Z}$ be the set of integers, $\mathbb{N}$ the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}$–the field of real numbers, $\mathbb{C}$–the field of complex numbers. Instead of $\sum\limits_{n=-\infty}^{+\infty} a_n$, we will write $\sum a_n$. For abbreviation we denote $L_2(R) = L_2$, $C^\infty(R) = C^\infty$. By $\| \; \|$ we will denote the norm and by $(\cdot, \cdot)$ the inner product of L_2.

2. The spaces A and A_k and their characterizations

The set of functions $\{\varrho_n\}_{n=-\infty}^{+\infty}$, where

$$(1) \qquad \varrho_n(x) = \frac{1}{\sqrt{2\pi}} \frac{(-ix - \frac{1}{2})^n}{(-ix + \frac{1}{2})^{n+1}} \qquad \text{for} \quad n \in \mathbb{Z}, \; x \in \mathbb{R}$$

forms CON in L_2. These functions were introduced by N. Wiener in 1949. For details we refer to [6], [9].

Obviously $\varrho_n \in C^\infty$. Let S be the differential operator defined by

$$(2) \qquad (Su)(x) = i(x - \frac{1}{2}i)\frac{d}{dx}[(x + \frac{1}{2}i)u(x)].$$

We can easily check that, the functions ϱ_n are eigenfunctions of the differential operator S belonging to the eigenvalues $\lambda_n = -n$;

$$(3) \qquad S\varrho_n = -n\varrho_n.$$

DEFINITION 1. We consider the space A defined by:

$$(4) \qquad A = \{\varphi \in C^\infty : \|S^k\varphi\| < \infty, \ k \in \mathbb{N}_0\}.$$

By integration by parts it is easy to verify that $(S\varphi, \varrho_n) = (\varphi, S\varrho_n)$ for $\varphi \in A$, $n \in \mathbb{Z}$. Notice that from this definition $A \subset L_2$. A. H. Zemanian [10] proves that A is a complete countable multinormed space with the system of semi–norms $\{\alpha_k\}_{k=0}^\infty$, where $\alpha_k(\varphi) := \|S^k\varphi\|$. Notice also that functions ϱ_n, $n \in \mathbb{Z}$ belong to A. S is a continuous linear operator of A into itself.

THEOREM 1. *For each element $\varphi \in A$ we have the representation*

$$(5) \qquad \varphi = \sum(\varphi, \varrho_n)\varrho_n.$$

We have the following characterization of the space A:

THEOREM 2. *If φ is in A, then $\varphi = \sum a_n\varrho_n$ and $\sum |n|^{2k}|a_n|^2$ is convergent for each $k \in \mathbb{N}_0$, where $a_n = (\varphi, \varrho_n)$ Conversely if $\sum |n|^{2k}|a_n|^2$ is convergent, then the series $\sum a_n\varrho_n$ converges to some φ in A.*

For the proof see [10], p. 312–313 or [11], p. 268.

DEFINITION 2. For each $k \in \mathbb{N}$ and $\varphi \in A$ we define

$$(6) \qquad |\varphi|_k^2 := ((S^2 + I)^k\varphi, \varphi).$$

Immediatly we see that the following theorem holds.

THEOREM 3. *We have following properties of $|\ |_k$:*

$$(7) \qquad |\varphi|_k^2 = \sum_{l=0}^k \binom{k}{l}\|S^l\varphi\|^2 \qquad \text{for each} \quad k \in \mathbb{N} \quad \text{and} \quad \varphi \in A,$$

(8) $\qquad$ $|\varphi|_{k+1} \geq |\varphi|_k \geq \|\varphi\|$ $\qquad$ *for each* $k \in \mathbb{N}$ *and* $\varphi \in A$.

Notice, that $|\ |_k$ is the norm in A for each $k \in \mathbb{N}$.

DEFINITION 3. By A_k we denote the completion of A in the norm $|\ |_k$, for each $k \in \mathbb{N}$.

THEOREM 4. *The norms* $|\ |_k$ *are compatible in the following sense: If a sequence* $\{\varphi_n\}$ *of elements of* A *converges to zero with respect to the norm* $|\ |_m$ *and is a Cauchy sequence in the norm* $|\ |_k$ *($m \leq k$), then it also converges to zero in the norm* $|\ |_k$. *(Compare [7], p. 14.)*
$\quad$ *Since the norms* $|\ |_k$ *are compatible,* A *is complete and the relation (8) holds, so we have*

(9) $\qquad$ $A \subset \ldots \subset A_{k+1} \subset A_k \subset \ldots \subset A_2 \subset A_1 \subset L_2,$

(10) $$A = \bigcap_{k=1}^{\infty} A_k.$$

For the proof see [4] or [7].
We have also a characterization of A_k:

THEOREM 5. *The function* φ *belonging to* L_2 *is in* A_k *if and only if the series* $\sum n^{2k}|a_n|^2$ *is convergent, where* $a_n = (\varphi, \varrho_n)$.

For the proof see [7], p. 34.

3. The dual spaces A' and A'_k and their characterizations

DEFINITION 4. By A'_k and A' we denote the dual space of A_k and A, respectively.

It means that $A'(A'_k)$ is the space of all continuous linear functionals on $A(A_k)$.
$\quad$ For $f \in A'$ and $\varphi \in A$, we use the following notation:

$$(f, \varphi) = f(\overline{\varphi}).$$

THEOREM 6. *If* $f \in A'$ *then* $f = \sum(f, \varrho_n)\varrho_n$, *where the series is convergent in* A'.

For the proof see [10] or [11].

THEOREM 7. *The following inclusions take place:*

(11) $L_2 \subset A'_1 \subset A'_2 \subset \ldots \subset A'_k \subset A'_{k+1} \subset \ldots \subset A'$ *and* $A' = \bigcup_{k=1}^{\infty} A'_k.$

For the proof see [4].

We have also characterizations of A'_k and A'.

THEOREM 8. *If f is in A', then the series $\sum a_n \varrho_n$, $a_n = f(\overline{\varrho_n})$ converges to f in A' and there exists an integer $k \in \mathbb{N}_0$ such that the series $\sum_{n \neq 0} |n|^{-2k} |a_n|^2$ is convergent.*

Conversely if the series $\sum_{n \neq 0} |n|^{-2k} |a_n|^2$ is convergent, then there exists a continuous linear functional f in A' such that $f(\overline{\varrho_n}) = a_n$.

For the proof see [10], p. 323–324 or [11], p. 271–272.

THEOREM 9. *If f is in A', then f belongs to A'_k if and only if the series $\sum_{n \neq 0} |n|^{-2k} |a_n|^2$ is convergent, where $a_n = f(\overline{\varrho_n})$.*

For the proof see [7], p. 36, theorem 4.9.

4. Connection between A_k and the Sobolev space $W_{k,2}$

DEFINITION 5. We shall denote by $W_{m,2}$ for $m \in \mathbb{N}$, the subspaces of L_2 defined by

$$\{f \in L_2 : \ f^{(\nu)} \in L_2 \quad \text{for} \quad \nu \in \{0, 1, \ldots, m\}, \ \text{where}$$
$$f^{(\nu)} \ \text{is the weak derivative of} \ f\}.$$

The vector space $W_{m,2}$ equipped with the norm:

$$\|f\|_{m,2} = \left(\sum_{\nu=0}^{m} \|f^{(\nu)}\|^2 \right)^{1/2}$$

is called Sobolev space and it is Banach space.

DEFINITION 6. By D_{L_2} we shall denote the space of all smooth functions φ such that $\varphi^{(\nu)} \in L_2$ for $\nu \in \mathbb{N}$.

The convergence in D_{L_2} may be defined by the family of norms $\|\cdot\|_{m,2}$ for $m = 0, 1, 2, \ldots$.

Let Λ be a linear functional on D_{L_2} continuous with respect to the norm $\|\cdot\|_{m,2}$. It is known that such functional takes the following form (see [8], p. 201)

$$(12) \qquad \Lambda(\varphi) = \sum_{\nu=0}^{m} \int_{R} f_{\nu}(x)\varphi^{(\nu)}(x)dx \qquad \text{for} \quad \varphi \in D_{L_2},$$

where f_{ν} for $\nu \in \{0, 1, \ldots, m\}$ are fixed elements of L_2. We shall now show that the functional Λ can be extended on $W_{m,2}$ by continuity and formula (12) also holds.

Let be $\varphi \in W_{m,2}$. Then by virtue of density of D_{L_2} in $W_{m,2}$ (see for example [1]), there exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}}$, $\varphi_n \in D_{L_2}$ for $n \in \mathbb{N}$, such that:

$$(13) \qquad \left\|\varphi_n^{(\nu)} - \varphi^{(\nu)}\right\| \underset{n\to\infty}{\longrightarrow} 0 \qquad \text{for} \quad \nu = 0, 1, \ldots, m.$$

From Hölder's inequality we have:

$$(14) \qquad \left|\int_{R}[f_{\nu}(x)\varphi_n^{(\nu)}(x) - f_{\nu}(x)\varphi^{(\nu)}(x)]dx\right| \leq \|f_{\nu}\| \, \|\varphi_n^{(\nu)} - \varphi^{(\nu)}\|.$$

Since (13) and (14), therefore

$$\Lambda(\varphi) = \lim_{n\to\infty} \Lambda(\varphi_n) = \sum_{\nu=0}^{m} \int_{R} f_{\nu}(x)\varphi^{(\nu)}(x)dx \qquad \text{for} \quad \varphi \in W_{m,2}.$$

THEOREM 10. *Let $\Lambda : D_{L_2} \to \mathbb{C}$ be a linear functional continuous with respect to the norm $\|\cdot\|_{m,2}$ and $g \in D_{L_1}, \int_{R} g(x)dx = 1$ and $g_{\varepsilon}(x) = \varepsilon^{-1}g\left(\frac{x}{\varepsilon}\right)$ and $\Lambda_{g_{\varepsilon}}(x) = \Lambda(g_{\varepsilon}(x - \cdot))$.*

Then

$$\int_{R} \Lambda_{g_{\varepsilon}}(x)\varphi(x)dx \qquad \text{tends to} \quad \Lambda(\varphi) \text{ as } \varepsilon \to 0 \text{ for } \varphi \in W_{m,2}.$$

PROOF. An easy computation shows that:

$$(15)$$
$$\int_{R} \Lambda_{g_{\varepsilon}}(x)\varphi_n(x)dx - \Lambda(\varphi_n) =$$

$$= \sum_{\nu=0}^{m} \int_{R} \int_{R} [f_{\nu}(x - \varepsilon y) - f_{\nu}(x)]g(y)dy\varphi_n^{(\nu)}(x)dx \quad \text{for} \quad \varphi_n \in D_{L_2}.$$

Assume that a sequence $\{\varphi_n\}_{n \in \mathbb{N}}$, $\varphi_n \in D_{L_2}$ for $n \in \mathbb{N}$, converges to φ in $W_{m,2}$, then

a)
$$\int_R \Lambda_{g_\epsilon}(x)\varphi_n(x)dx \overset{n \to \infty}{\longrightarrow} \int_R \Lambda_{g_\epsilon}(x)\varphi(x)dx$$

b)
$$\Lambda(\varphi_n) \overset{n \to \infty}{\longrightarrow} \Lambda(\varphi)$$

c)
$$\sum_{\nu=0}^m \int_R \int_R [f_\nu(x - \varepsilon y) - f_\nu(x)]g(y)dy\varphi_n^{(\nu)}(x)dx \to$$
$$\sum_{\nu=0}^m \int_R \int_R [f_\nu(x - \varepsilon y) - f_\nu(x)]g(y)dy\varphi^{(\nu)}(x)dx.$$

From a) b) c) it follows:

(15′)
$$\int_R \Lambda_{g_\epsilon}(x)\varphi(x)dx - \Lambda(\varphi) = \sum_{\nu=0}^m \int_R \int_R [f_\nu(x - \varepsilon y) - f_\nu(x)]$$
$$g(y)dy\varphi^{(\nu)}(x)dx \qquad \text{for} \quad \varphi \in W_{m,2}.$$

The rest part of the proof is as in the proof of Lemma 1 in [2].

THEOREM 11.

(16)
$$A_k \subset W_{k,2} \qquad \text{for} \quad k = 1, 2, \ldots$$

PROOF. (Compare Th. 6 in [2]). Let φ be in A_k. In accordance with Definition 3 φ is in L_2.

It is easy to see that

(17)
$$\varrho'_n = -ni\varrho_{n-1} + (2n + 1)i\varrho_n - (n + 1)i\varrho_{n+1}.$$

According to Theorem 5 the function φ can be represented as follows

$$\varphi = \sum a_n \varrho_n, \qquad \text{where} \quad a_n = (\varphi, \varrho_n).$$

Formally differentiating this series we obtain

$$\sum a_n \varrho'_n = -i \sum na_n \varrho_{n-1} + i \sum (2n + 1)a_n \varrho_n$$
$$- i \sum (n + 1)a_n \varrho_{n+1}.$$

By virtue of Theorem 5 it follows that the series $\sum |n|^2 |a_n|^2$, $\sum |2n+1|^2 |a_n|^2$ and $\sum |n + 1|^2 |a_n|^2$ are convergent so $\sum n a_n \varrho_{n-1}, \sum(2n + 1)a_n \varrho_n$ and $\sum(n + 1)a_n \varrho_{n+1}$ are convergent in L_2. Hence φ' is in L_2.

Analogously we will prove theorem 14. First using the formules (26) and (27) it can be shown that $\varphi^{(\nu)} \in L_2$ for $\nu \in \{0, 1, \ldots, k\}$ in the case $k > 1$.

REMARK 1. *Remark 1 If $\Lambda \in D'_{L_2}$ is functional continuous with respect to the norm $\| \cdot \|_{m,2}$, then we will writte that*

$$(18) \qquad \Lambda \in W_{-m,2} := (W_{m,2})'.$$

THEOREM 12. *If Λ belongs to $W_{-k,2}$, then the restriction $\tilde{\Lambda}$ of the functional Λ to A_k belongs to A'_k.*

PROOF. By virtue of Theorem 10

$$(19) \qquad \int\limits_R \Lambda_{g_\varepsilon}(x)\varphi(x)dx$$

tends to $\Lambda(\varphi)$ as $\varepsilon \to 0$ for $\varphi \in W_{k,2}$.

Therefore, in view of Theorem 11 we infer that the expression (19) converges to $\Lambda(\varphi)$ when $\varepsilon \to 0$ for $\varphi \in A_k$. Taking into account that A_k is complete we conclude, by the Banach–Steinhaus theorem that $\tilde{\Lambda}$ belongs to A'_k.

COROLLARY 1. *If Λ is in $W_{-k,2}$, then $\tilde{\Lambda} = \sum a_n \varrho_n$ in A'_k, where $a_n = \tilde{\Lambda}(\bar{\varrho}_n)$.*

5. Some functionals from the space A'_k

Now we are going to show the examples of functionals in A'_k.

DEFINITION 7. For $k \in \mathbb{N}$ and $\varphi \in A$ we define the functionals:

$$(20) \qquad \frac{1}{(\cdot + 0i)^k}(\varphi) = \lim_{\varepsilon \to 0+} \int\limits_{-\infty}^{+\infty} \frac{1}{(x + \varepsilon i)^k} \cdot \varphi(x)dx$$

$$(21) \qquad \frac{1}{(\cdot - 0i)^k}(\varphi) = \lim_{\varepsilon \to 0+} \int\limits_{-\infty}^{+\infty} \frac{1}{(x - \varepsilon i)^k} \cdot \varphi(x)dx$$

DEFINITION 8. For $k = 1$ we define functional

$$(22) \qquad \frac{1}{(\cdot)}(\varphi) := \int\limits_{-\infty}^{+\infty} (\varphi(x) - \varphi(-x)) \frac{1}{2x} dx \qquad \text{for} \quad \varphi \in A.$$

For $k > 1$ we define functionals depending on evenness and oddness of k:
for even $k = 2m$

$$(23)$$
$$\frac{1}{(\cdot)^{2m}}(\varphi) := \int\limits_{-\infty}^{+\infty} x^{-2m} \left[(\varphi(x) + \varphi(-x)) \frac{1}{2} - \left(\varphi(0) + \varphi^{(2)}(0) \frac{1}{2!} x^2 + \ldots + \right. \right.$$
$$\left. \left. + \varphi^{(2m-2)}(0) \frac{1}{(2m-2)!} x^{2m-2} \right) \right] dx \qquad \text{for} \quad m = 1, 2, \ldots$$

for odd $k = 2m + 1$:

$$(24)$$
$$\frac{1}{(\cdot)^{2m+1}}(\varphi) := \int\limits_{-\infty}^{+\infty} x^{-(2m+1)} \left[(\varphi(x) - \varphi(-x)) \frac{1}{2} \right.$$
$$\left. - \left(\varphi'(0) \frac{1}{1!} x + \ldots + \varphi^{(2m-1)}(0) \frac{1}{(2m-1)!} x^{2m-1} \right) \right] dx$$
$$\text{for} \quad m = 1, 2, \ldots \text{ and } \varphi \in A.$$

Compare the definition in [5], p. 199.

Notice that these functionals belongs to $W_{-k,2}$ for appropriate k.

THEOREM 13. *Functionals* $\frac{1}{(\cdot)}$, $\frac{1}{\cdot+0i}$, $\frac{1}{\cdot-0i}$, $\delta \in A'_1$.

THEOREM 14. *Functionals* $\frac{1}{(\cdot)^k}$, $\frac{1}{(\cdot+0i)^k}$, $\frac{1}{(\cdot-0i)^k}$, $\delta^{(k-1)} \in A'_k \setminus A'_{k-1}$, *for* $k > 1$.

We base on the following result (see for example [3]):

Cauchy's representation theorem. *If a simple closed curve C, positively oriented and lying in the region R, contains only points c of R in its interior, then a function $f(c)$ analytic in R can be represented for points c interior to C by*

$$(25) \qquad f(c) = \frac{1}{2\pi i} \int\limits_C \frac{f(z)}{z - c} dz,$$

and it's n-th derivative

$$(26) \qquad f^{(n)}(c) = \frac{n!}{2\pi i} \int\limits_C \frac{f(z)}{(z - c)^{n+1}} dz,$$

Moreover,

$$(27) \qquad \int_C f(z)dz = 0.$$

(A region is defined as an open set which is arcwise connected.)
We denote by Γ_R^+ and Γ_R^- the parts of circle

$$(28) \qquad \Gamma_R^+ = \{Re^{it} : 0 \le t \le \pi\} \text{ and } \Gamma_R^- = \{Re^{-it} : 0 \le t \le \pi\}$$
$$\text{for constant } R > 0$$

and we use the following property of rational functions:

REMARK 2. *Remark 2 If*

a) $W(z)$ is rational function,
b) W has no poles on the real line,
c) $\lim\limits_{|z| \to +\infty} zW(z) = 0$

then:

$$(29) \qquad \int_{-\infty}^{+\infty} W(x)dx = \lim_{R \to +\infty} \int_{[-R,R] \cup \Gamma_R^+} W(z)dz,$$

and

$$(30) \qquad \int_{-\infty}^{+\infty} W(x)dx = \lim_{R \to +\infty} \int_{[-R,R] \cup \Gamma_R^-} W(z)dz.$$

Now using the Cauchy's representation theorem and remark 2 we can calculate the integrals:

LEMMA 1. *For each $\varepsilon > 0$ we have*

$$(31) \qquad \int_{-\infty}^{+\infty} \frac{1}{x + \varepsilon i} \overline{\varrho}_n(x)dx = \begin{cases} 0, & \text{for } n < 0 \\ i\sqrt{2\pi}(-1)^{n+1} \frac{\left(\frac{1}{2} - \varepsilon\right)^n}{\left(\frac{1}{2} + \varepsilon\right)^{n+1}}, & \text{for } n \ge 0 \end{cases}$$

$$(32) \qquad \int_{-\infty}^{+\infty} \frac{1}{x - \varepsilon i} \overline{\varrho}_n(x)dx = \begin{cases} i\sqrt{2\pi}(-1)^n \frac{\left(\frac{1}{2} - \varepsilon\right)^{-n-1}}{\left(\frac{1}{2} + \varepsilon\right)^{-n}}, & \text{for } n < 0 \\ 0, & \text{for } n \ge 0 \end{cases}$$

PROOF. We want to calculate the integral

(33)
$$\int_{-\infty}^{+\infty} \frac{1}{x+\varepsilon i} \overline{\varrho}_n(x) dx = \frac{1}{i\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{1}{x+\varepsilon i} \frac{\left(x+\frac{1}{2}i\right)^n}{\left(x-\frac{1}{2}i\right)^{n+1}} dx.$$

For $n \geq 0$ the function $f(z) = \frac{\left(z+\frac{1}{2}i\right)^n}{\left(z-\frac{1}{2}i\right)^{n+1}}$ is analytic in the half–plane $\operatorname{Im} z < \frac{1}{2}$, so by the Cauchy's representation theorem we have

(34)
$$\int_{[-R,R]\cup\Gamma_R^-} \frac{1}{z+\varepsilon i} f(z) dz = -2\pi i f(-\varepsilon i) \quad \text{for} \quad R > \varepsilon.$$

Using the remark 2 we receive
(35)
$$\int_{-\infty}^{+\infty} \frac{1}{x+\varepsilon i} \frac{\left(x+\frac{1}{2}i\right)^n}{\left(x-\frac{1}{2}i\right)^{n+1}} dx = -2\pi i \frac{\left(-\varepsilon i+\frac{1}{2}i\right)^n}{\left(-\varepsilon i-\frac{1}{2}i\right)^{n+1}} = 2\pi(-1)^n \frac{\left(\frac{1}{2}-\varepsilon\right)^n}{\left(\frac{1}{2}+\varepsilon\right)^{n+1}}.$$

So from (33) and (35) we have (31) for $n \geq 0$.
  For $n < 0$
(36)
$$\int_{-\infty}^{+\infty} \frac{1}{x+\varepsilon i} \overline{\varrho}_n(x) dx = \frac{1}{i\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{1}{x+\varepsilon i} \frac{\left(x-\frac{1}{2}i\right)^{p-1}}{\left(x+\frac{1}{2}i\right)^p} dx \quad \text{where} \quad p = -n.$$

Since the function $f(z) = \frac{1}{z+\varepsilon i} \frac{\left(z-\frac{1}{2}i\right)^{p-1}}{\left(z+\frac{1}{2}i\right)^p}$ is analytic in the half–plane $\operatorname{Im} z \geq 0$ so $\int_{[-R,R]\cup\Gamma_R^+} f(z) dz = 0$ for each $R > 0$, and by the remark 2 we have (31) for $n < 0$.
  It means that:

(37)
$$\frac{1}{x+\varepsilon i} = i\sqrt{2\pi} \sum_{n=0}^{\infty} (-1)^{n+1} \frac{\left(\frac{1}{2}-\varepsilon\right)^n}{\left(\frac{1}{2}+\varepsilon\right)^{n+1}} \varrho_n(x) \quad \text{in} \quad L_2$$

Similary we have

(38)
$$\frac{1}{x-\varepsilon i} = -i\sqrt{2\pi} \sum_{n=-\infty}^{-1} (-1)^n \frac{\left(\frac{1}{2}-\varepsilon\right)^{-n-1}}{\left(\frac{1}{2}+\varepsilon\right)^{-n}} \varrho_n(x) \quad \text{in} \quad L_2.$$

By definition 7 we have Fourier representation:

$$(39) \qquad \frac{1}{\cdot + 0i}(\overline{\varphi}) = i2\sqrt{2\pi} \sum_{n=0}^{+\infty} (-1)^{n+1} \varrho_n$$

and from theorem 9 $\frac{1}{\cdot + 0i} \in A_1'$.

Analogously we can receive

$$(40) \qquad \frac{1}{\cdot - 0i} = i2\sqrt{2\pi} \sum_{n=-\infty}^{-1} (-1)^n \varrho_n \quad \text{in} \quad A_1'.$$

We can also easily calculate that:

$$(41) \qquad \delta = \frac{2}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} (-1)^n \cdot \varrho_n.$$

Moreover, from the Sochotzki formulas:

$$(42) \qquad (-1)\pi i \delta + \frac{1}{(\cdot)} = \frac{1}{\cdot + 0i}$$

or

$$(43) \qquad \pi i \delta + \frac{1}{(\cdot)} = \frac{1}{\cdot - 0i},$$

we have:

$$(44) \qquad \frac{1}{(\cdot)} = i\sqrt{2\pi} \sum_{n=-\infty}^{-1} (-1)^n \cdot \varrho_n + i\sqrt{2\pi} \sum_{n=0}^{\infty} (-1)^{n+1} \cdot \varrho_n.$$

From the above statement and theorem 9, theorem 13 follows.

Analogously we will prove theorem 14. First using the formulas (26) and (27) of the Cauchy's representation theorem and remark 2, we calculate the integrals.

LEMMA 2. *For* $k > 1, k \in \mathbb{N}$ *and* $\varepsilon > 0$ *we have:*

(45)

$$\int_{-\infty}^{+\infty} \frac{1}{(x + \varepsilon i)^k} \overline{\varrho}_n(x)dx =$$

$$= \begin{cases} 0, & \text{for } n < 0 \\ \dfrac{\sqrt{2\pi}}{i^k}(-1)^n \\ \quad \left\{ \displaystyle\sum_{l=0}^{\min\{k-1,n\}} \dfrac{1}{l!(k-1-l)!}(n-l+1)\dots(n+k-1-l) \right. \\ \quad \left. \left(\dfrac{1}{2}-\varepsilon\right)^{n-l}\left(\dfrac{1}{2}+\varepsilon\right)^{-n-k+l} \right\} & \text{for } n \geq 0 \end{cases}$$

(46)

$$\int_{-\infty}^{+\infty} \frac{1}{(x - \varepsilon i)^k} \overline{\varrho}_n(x)dx =$$

$$= \begin{cases} \dfrac{\sqrt{2\pi}}{i^k}(-1)^{n+1} \\ \quad \left\{ \displaystyle\sum_{l=0}^{\min\{k-1,-n-1\}} \dfrac{1}{l!(k-1-l)!}(n+l)\dots(n+l-k+2) \right. \\ \quad \left. \left(\dfrac{1}{2}-\varepsilon\right)^{-n-1-l}\left(\dfrac{1}{2}+\varepsilon\right)^{n-k+1+l} \right\} & \text{for } n < 0 \\ 0, & \text{for } n \geq 0 \end{cases}$$

This implies from definition 7

(47)

$$\frac{1}{(\cdot + 0i)^k}(\overline{\varrho}_n) =$$

$$= \begin{cases} 0, \quad \text{for } n < 0 \\[2ex] \dfrac{\sqrt{2\pi} \cdot 2^k}{i^k}(-1)^n \\[2ex] \quad \left\{ \displaystyle\sum_{l=0}^{\min\{k-1,n\}} \dfrac{1}{l!(k-1-l)!}(n+1-l)\ldots(n+k-1-l) \right\} \\[2ex] \text{for } n \geq 0 \end{cases}$$

(48)

$$\frac{1}{(\cdot - 0i)^k}(\overline{\varrho}_n) =$$

$$= \begin{cases} \dfrac{\sqrt{2\pi}}{i^k}(-1)^{n+1} \\[2ex] \quad \left\{ \displaystyle\sum_{l=0}^{\min\{k-1,-n-1\}} \dfrac{1}{l!(k-1-l)!}(n+l)\ldots(n+l-k+2) \right\} \\[2ex] \text{for } n < 0 \\[2ex] 0, \quad \text{for } n \geq 0 \end{cases}$$

Notice that, since $\overline{\varrho}_n(x) = \varrho_n(-x)$, for $k > 1, k \in \mathbb{N}$ we have:

(49) $$\delta^{(k-1)}(\overline{\varrho}_n) = (-1)^{k-1}\delta\left(\overline{\varrho}_n^{(k-1)}\right) = \varrho_n^{(k-1)}(0)$$

and

(50)

$$
\varrho_n^{(k-1)}(0) = \begin{cases} \dfrac{i}{\sqrt{2\pi}}(-1)^n \dfrac{2^k}{i^k} \\ \left\{ \displaystyle\sum_{l=0}^{\min\{k-1,-n-1\}} \binom{k-1}{l}(-n-l)\ldots(-n+k-l-2) \right\} \\ \qquad \text{for } n < 0 \\ \dfrac{i}{\sqrt{2\pi}}(-1)^{n+k-1}\dfrac{2^k}{i^k} \\ \left\{ \displaystyle\sum_{l=0}^{\min\{k-1,n\}} \binom{k-1}{l}(n-l+1)\ldots(n+k-1-l) \right\} \\ \qquad \text{for } n \geq 0 \end{cases}
$$

From Sochotzki formulas:

(51)
$$
\frac{1}{(\ )^k} = \frac{1}{(\cdot+0i)^k} - \frac{\pi i(-1)^k}{(k-1)!}\delta^{(k-1)}
$$

or

(52)
$$
\frac{1}{(\ )^k} = \frac{1}{(\cdot-0i)^k} + \frac{\pi i(-1)^k}{(k-1)!}\delta^{(k-1)}
$$

we can receive for $k \in \mathbb{N}, k > 1$:

(53)

$$
\frac{1}{(\ )^k}(\overline{\varrho}_n) = \begin{cases} \dfrac{\sqrt{2\pi}2^{k-1}}{i^k}(-1)^{n+k} \\ \left\{ \displaystyle\sum_{l=0}^{\min\{k-1,-n-1\}} \dfrac{1}{l!(k-1-l)!}(-n-l)\ldots(-n+k-l-2) \right\} \\ \qquad \text{for } n < 0 \\ \dfrac{\sqrt{2\pi}2^{k-1}}{i^k}(-1)^{n} \\ \left\{ \displaystyle\sum_{l=0}^{\min\{k-1,n\}} \dfrac{1}{l!(k-1-l)!}(n-l+1)\ldots(n+k-1-l) \right\} \\ \qquad \text{for } n \geq 0 \end{cases}
$$

From the previous statement and theorem 9, theorem 14 follows.

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