A NOTE ON SEPARATION BY SUBADDITIVE AND SUBLINEAR FUNCTIONS

Kazimierz Nikodem, Elżbieta Sadowska and Szymon Wąsowicz

Abstract. The necessary and sufficient conditions for separation by subadditive and sublinear functions are given.

1. Introduction

In this note we establish the necessary and sufficient conditions under which two functions can be separated by subadditive and sublinear functions. These results are related to the theorem on separation by convex functions presented in [1] as well as the known theorems on separation by additive functions due to R. Kaufman [3] and P. Kranz [4].

2. Separation by subadditive functions

Let $(S,+)$ be a semigroup. Recall that a function $f : S \to \mathbb{R}$ is called *subadditive* if $f(x+y) \le f(x)+f(y)$ for any $x,y \in S$; f is *superadditive* if $-f$ is subadditive. It is known that if $\emptyset \ne A \subset S$, then a subsemigroup of S generated by A (i. e. an intersection of all subsemigroups of S containing A) has the form $\{x_1+\ldots+x_n : n \in \mathbb{N},\ x_1,\ldots,x_n \in A\}$. Notice that the epigraph of a subadditive function f (the set $\operatorname{epi} f = \big\{(x,y) \in S \times \mathbb{R} : y \ge f(x)\big\}$) is a subsemigroup of $S \times \mathbb{R}$.

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THEOREM 1. *Let $f, g : S \to \mathbb{R}$. The following conditions are equivalent:*
(i) there exists a subadditive function $h : S \to \mathbb{R}$ such that $f \le h \le g$;
(ii) the inequality

$$f\Big(\sum_{i=1}^{n} x_i\Big) \le \sum_{i=1}^{n} g(x_i)$$

holds true for any $n \in \mathbb{N}$, $x_1, \ldots, x_n \in S$.

PROOF. An implication (i)⇒(ii) is obvious. Assume the condition (i), (ii). Let W be a subsemigroup of $S \times \mathbb{R}$ generated by $\operatorname{epi} g$. For any $x \in S$, $(x, g(x)) \in \operatorname{epi} g \subset W$, so $\{y \in \mathbb{R} : (x, y) \in W\} \neq \emptyset$. Moreover, if $(x, y) \in W$, then there exist $n \in \mathbb{N}$, $x_1, \ldots, x_n \in S$, $y_1, \ldots, y_n \in \mathbb{R}$ such that $(x_i, y_i) \in \operatorname{epi} g$, $i = 1, \ldots, n$, and $(x, y) = \sum_{i=1}^{n} (x_i, y_i)$. From (ii) we obtain

$$f(x) = f\Big(\sum_{i=1}^{n} x_i\Big) \le \sum_{i=1}^{n} g(x_i) \le \sum_{i=1}^{n} y_i = y. \tag{1}$$

Put

$$h(x) = \inf\{y \in \mathbb{R} : (x, y) \in W\}, \quad x \in S.$$

By (1) this definition is correct and $f \le h$ on S. Since $(x, g(x)) \in W$ for any $x \in S$, we get $h \le g$ on S. We will check that $h : S \to \mathbb{R}$ is subadditive. Fix $x_1, x_2 \in S$ and $y_1, y_2 \in \mathbb{R}$ such that $(x_i, y_i) \in W$, $i = 1, 2$. As $(x_1 + x_2, y_1 + y_2) \in W$, $h(x_1 + x_2) \le y_1 + y_2$. Passing to the infimum by y_1 and then by y_2, we infer that $h(x_1 + x_2) \le h(x_1) + h(x_2)$, which finishes the proof.

REMARK 1. *The following question connected with the above theorem was posed by Prof. R. Ger during a discussion with the first author: assume that functions $f, g : [0, \infty) \to \mathbb{R}$ satisfy the condition*

$$f(x_1 + x_2) \le g(x_1) + g(x_2), \qquad x_1, x_2 \ge 0.$$

Does there exist a subadditive function $h : [0, \infty) \to \mathbb{R}$ such that $f \le h \le g$? It appears that the answer is negative. The example below shows that it is not true even if we assume that analogous inequalities with k components hold for all $k \le n$, where $n \in \mathbb{N}$ is arbitrarily fixed.

EXAMPLE 1. Fix an $n \in \mathbb{N}$ and consider the functions

$$f(x) = \begin{cases} 0, & \text{if } x \neq n+1, \\ 1, & \text{if } x = n+1, \end{cases} \qquad g(x) = \begin{cases} 0, & \text{if } x \le 1, \\ 1, & \text{if } x > 1. \end{cases}$$

Then f and g satisfy

$$f(x_1 + \ldots + x_k) \leq g(x_1) + \ldots + g(x_k)$$

for all $k \leq n$ and $x_1, \ldots, x_k \in [0, \infty)$. Suppose that there exists a subadditive function $h : [0, \infty) \to \mathbb{R}$ such that $f \leq h \leq g$. Then

$$1 = f(n+1) \leq h(n+1) \leq (n+1)h(1) \leq (n+1)g(1) = 0,$$

which is impossible.

REMARK 2. *The previous theorem as well as the results of [1] show that if f and g fulfil an inequality of "convex" type, then they can be separated by a "convex" (in the same sense) function. It is proved in [5] that two functions f and g mapping a real interval into $\mathbb{R}$ can be separated by an affine function if and only if they fulfil a system of two inequalities: one of the convex (in the usual sense) type and the second one — of the concave type. In connection with this the natural question arises: if $f, g : S \to \mathbb{R}$ fulfil a system of inequalities*

$$
\begin{aligned}
f\Big(\sum_{i=1}^{n} x_i\Big) &\leq \sum_{i=1}^{n} g(x_i), \\
g\Big(\sum_{i=1}^{n} x_i\Big) &\geq \sum_{i=1}^{n} f(x_i)
\end{aligned}
\tag{2}
$$

for any $n \in \mathbb{N}$, $x_1, \ldots, x_n \in S$, is there an additive function separating f and g? Note that the inequalities (2) are satisfied if, in particular, g is subadditive, f is superadditive and $f \leq g$ or g is subadditive and the second inequality of (2) holds. In these two cases f and g can be separated by an additive function in view on Kranz's [4] and Kaufman's [3] theorems, respectively. The following counterexample shows that in the general case it is not true.

EXAMPLE 2. Consider two functions $f, g : [0, \infty) \to \mathbb{R}$ given by the formulas

$$f(x) = \begin{cases} 0, & \text{if } x \neq 1, \\ 1, & \text{if } x = 1, \end{cases} \qquad g(x) = \begin{cases} \sqrt{x}, & \text{if } 0 \leq x < 2, \\ x, & \text{if } x \geq 2. \end{cases}$$

Fix any $n \in \mathbb{N}$ and $x_1, \ldots, x_n \geq 0$. If $\sum_{i=1}^{n} x_i = 1$, then $0 \leq x_i \leq 1$, $i = 1, \ldots, n$, which yields

$$f\Big(\sum_{i=1}^{n} x_i\Big) = \sqrt{\sum_{i=1}^{n} x_i} \leq \sum_{i=1}^{n} \sqrt{x_i} = \sum_{i=1}^{n} g(x_i).$$

If $\sum_{i=1}^{n} x_i \neq 1$, then

$$f\Big(\sum_{i=1}^{n} x_i\Big) = 0 \leq \sum_{i=1}^{n} g(x_i).$$

So, f and g fulfil the first inequality of (2). If $x_i \neq 1$ for all $i \in \{1,\dots,n\}$, then $g\Big(\sum_{i=1}^{n} x_i\Big) \geq 0 = \sum_{i=1}^{n} f(x_i)$. Now assume that $x_1,\dots,x_k = 1$ and $x_{k+1},\dots,x_n \neq 1$ for some $k \in \{1,\dots,n\}$. If $\sum_{i=1}^{n} x_i \in [0,2)$, then $k = 1$. Therefore

$$g\Big(\sum_{i=1}^{n} x_i\Big) = \sqrt{\sum_{i=1}^{n} x_i} \geq 1 = \sum_{i=1}^{n} f(x_i).$$

On the other hand, if $\sum_{i=1}^{n} x_i \geq 2$, then

$$g\Big(\sum_{i=1}^{n} x_i\Big) = \sum_{i=1}^{n} x_i \geq k = \sum_{i=1}^{n} f(x_i).$$

We have shown that f and g also fulfil the second inequality of (2). Suppose that there is an additive function $a : [0,\infty) \to \mathbb{R}$ such that $f \leq a \leq g$. Since $a(1) = 1$, we have $a(x) = x$ for $x \in \mathbb{Q}$. So, $a(x) > g(x)$ for $x \in (1,2) \cap \mathbb{Q}$, which is a contradiction.

3. Separation by sublinear functions

Let X be a real vector space. Recall that a functional $p : X \to \mathbb{R}$ is called *sublinear* if it is subadditive and the equality $p(tx) = tp(x)$ holds for any $x \in X$ and $t \geq 0$; p is *superlinear* if $-p$ is sublinear. A nonempty set $K \subset X$ is a *cone* if $tK \subset K$ for all $t \geq 0$; if moreover $K + K \subset K$ then K is a *convex cone*. If $\emptyset \neq A \subset X$, then the cone generated by A is the set

$$\operatorname{cone} A = \{ta : a \in A,\ t \geq 0\};$$

the convex cone generated by A is the set of all linear combinations of elements of A with nonnegative coefficients. Note that the epigraph of a sublinear functional defined in X is a convex cone in $X \times \mathbb{R}$.

THEOREM 2. *Let $f, g : X \to \mathbb{R}$. The following conditions are equivalent:*
(i) there exists a sublinear functional $p : S \to \mathbb{R}$ such that $f \le p \le g$;
(ii) the inequality

$$f\Big(\sum_{i=1}^{n} t_i x_i\Big) \le \sum_{i=1}^{n} t_i g(x_i) \tag{3}$$

holds true for any $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$ and $t_1, \dots, t_n \ge 0$.

PROOF. The implication (i)$\Rightarrow$(ii) is clear. We will prove its converse. Consider a convex cone $K \subset X \times \mathbb{R}$ generated by $\operatorname{epi} g$, and for any $x \in X$ put

$$p(x) = \inf\{y \in \mathbb{R} : (x, y) \in K\}.$$

Similarly to the proof of the previous theorem we check that $p : X \to \mathbb{R}$ is well defined subadditive functional such that $f \le p \le g$. Fix an $x \in X$ and $t \ge 0$. Then for every $y \in \mathbb{R}$ such that $(x, y) \in K$, we have $(tx, ty) \in K$. Hence $p(tx) \le ty$, and in consequence we obtain $p(tx) \le tp(x)$. A standard argument yields $p(tx) = tp(x)$. So, p is sublinear and the proof is complete.

REMARK 3. *In the case $X = \mathbb{R}$ we can replace condition (3) in the above theorem by the simpler one:*

$$f(t_1x_1 + t_2x_2) \le t_1 g(x_1) + t_2 g(x_2), \ t_1, t_2 \ge 0, \ x_1, x_2 \in \mathbb{R}.$$

So, this time we have a different situation than described previously in Remark 1. Analysing carefully the structure of the convex cone generated by epi g and using the Carathéodory's theorem we obtain the following result.

THEOREM 3. *If $f, g : \mathbb{R}^k \to \mathbb{R}$ satisfy the condition*

$$f\Big(\sum_{i=1}^{k+1} t_i x_i\Big) \le \sum_{i=1}^{k+1} t_i g(x_i) \tag{4}$$

for all $x_1, \dots, x_{k+1} \in \mathbb{R}^k$ and $t_1, \dots, t_{k+1} \ge 0$, then there exists a sublinear function $p : \mathbb{R}^k \to \mathbb{R}$ such that $f(x) \le p(x) \le g(x)$, $x \in \mathbb{R}^k$.

PROOF. Consider the convex cone $K \subset \mathbb{R}^{k+1}$ generated by $\operatorname{epi} g$ and note that $K = \operatorname{conv}(\operatorname{cone} \operatorname{epi} g)$, i. e. K is the convex hull of the cone generated by $\operatorname{epi} g$. Fix an $x \in \mathbb{R}^k$ and take a point $(x, z) \in K$. By the Carathéodory's theorem there exists a $(k+1)$–dimensional simplex S with the vertices in $\operatorname{cone} \operatorname{epi} g$ such that $(x, z) \in S$. Let

$$y_0 = \inf\{y \in \mathbb{R} : (x, y) \in S\}.$$

Then (x, y_0) belongs to the boundary of S and hence

$$(x, y_0) = t_1(x_1, y_1) + \ldots + t_{k+1}(x_{k+1}, y_{k+1})$$

with some $(x_i, y_i) \in \operatorname{cone}\operatorname{epi} g$ and $t_i \geq 0$ summing up to 1. For $i = 1, \ldots, k+1$ there exist $(\overline{x}_i, \overline{y}_i) \in \operatorname{epi} g$ and $s_i \geq 0$ such that $(x_i, y_i) = s_i(\overline{x}_i, \overline{y}_i)$. Hence, using (4) we get

$$\begin{aligned} z \geq y_0 = \sum_{i=1}^{k+1} t_i y_i = \sum_{i=1}^{k+1} t_i s_i \overline{y}_i \geq \sum_{i=1}^{k+1} t_i s_i g(\overline{x}_i) \\ \geq f\left(\sum_{i=1}^{k+1} t_i s_i \overline{x}_i\right) = f\left(\sum_{i=1}^{k+1} t_i x_i\right) = f(x). \end{aligned}$$

Define

$$p(x) = \inf\{z \in \mathbb{R} : (x, z) \in K\}.$$

Then $f(x) \leq p(x)$ and $p(x) \leq g(x)$, because $(x, g(x)) \in K$. In the same way as in the proof of Theorem 2 we show that p is sublinear.

REMARK 4. *Similarly as in Remark 2 we can consider the following problem: assume that $f, g : X \to \mathbb{R}$ satisfy the system of inequalities*

$$\begin{aligned} f\left(\sum_{i=1}^{n} t_i x_i\right) \leq \sum_{i=1}^{n} t_i g(x_i), \\ g\left(\sum_{i=1}^{n} t_i x_i\right) \geq \sum_{i=1}^{n} t_i f(x_i) \end{aligned}$$

for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$, $t_1, \ldots, t_n \geq 0$. Does there exist a linear functional separating f and g?

We know only partial answers of this question. For instance, as a consequence of Theorem we get the following result.

COROLLARY 1. *If $f, g : \mathbb{R} \to \mathbb{R}$ satisfy the inequalities*

$$\begin{aligned} f(t_1 x_1 + t_2 x_2) \leq t_1 g(x_1) + t_2 g(x_2), \\ g(t_1 x_1 + t_2 x_2) \geq t_1 f(x_1) + t_2 f(x_2) \end{aligned}$$

for all $x_1, x_2 \in \mathbb{R}$ and $t_1, t_2 \geq 0$, then there exists a linear function $l : \mathbb{R} \to \mathbb{R}$ separating f and g.

PROOF. By Theorem there exist a sublinear function p and a superlinear function q which separate f and g. The function $\bar{l} = \max\{p, q\}$ is sublinear (because it has a linear support at every point), and the function $\underline{l} = \min\{p, q\}$ is superlinear. Moreover, $\underline{l} \le \bar{l}$.

By the classical Hahn–Banach sandwich theorem (cf. [2, Theorem 5]) there exists a linear function l separating $\underline{l}$ and $\bar{l}$. It also separates f and g.

REMARK 5. *We know that in the case $X = \mathbb{R}^2$ the above problem has also a positive solution. However, we obtained it using a more complicated method, different than in the proof of Corollary 1. Since that method can not be applied in the general case (even for $X = \mathbb{R}^n$ with $n \ge 3$), we do not present it.*

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DEPARTMENT OF MATHEMATICS
TECHNICAL UNIVERSITY OF ŁÓDŹ
BRANCH IN BIELSKO-BIAŁA,
WILLOWA 2,
PL 43-309 BIELSKO-BIAŁA
POLAND

e-mail: knik@pb.bielsko.pl
lk@pb.bielsko.pl
szw@pb.bielsko.pl