ON INVOLUTIONS SATISFYING A SYSTEM OF FUNCTIONAL EQUATIONS

MARTA DOBOSZ-SMELA AND MAREK CEZARY ZDUN

Abstract. In this paper we investigate a system of functional equations

$$\begin{cases} N \circ N = \mathrm{id} \\ N \circ f_k = f_{p-1-k} \circ N \quad k = 0, \ldots, p-1 \end{cases}$$

in finite and infinite interval, where $f_0, \ldots, f_{p-1}$ are given real functions. Under suitable assumptions on f_i we prove that the system has a unique solution and this solution is continuous and decreasing.

Let us assume the following hypothesis

(H_1) $f_0, f_1, \ldots, f_{p-1} : [0,1] \to [0,1]$ are strictly increasing and continuous functions with $f_0(0) = 0$, $f_{k-1}(1) = f_k(0)$, $k = 1, \ldots, p-1$ and $f_{p-1}(1) = 1$, such that

$$(1) \quad |f_k(x) - f_k(y)| < |x - y|, \quad \textit{for } x, y \in (0,1),\ x \neq y,\ k = 0, \ldots, p-1.$$

The starting point of our considerations is the following result on generalized de Rham system.

PROPOSITION 1. *(see [4]) Let hypothesis (H_1) be fulfilled. Then the system*

$$(2) \qquad R\left(\frac{x+k}{p}\right) = f_k\left(R(x)\right), \quad \textit{for } x \in [0,1],\ k = 0, \ldots, p-1$$

has exactly one solution $R : [0,1] \to [0,1]$. This solution is strictly increasing and continuous.

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LEMMA 1. *Let γ be an arbitrary homeomorphism of $[0,1]$ onto $[0,1]$. Then the formula*

$$N(x) := \gamma\left(1-\gamma^{-1}(x)\right) \tag{3}$$

for $x \in [0,1]$, defines a strictly decreasing involution i.e. $N^2(x) = x$ for all $x \in [0,1]$. Conversely, each decreasing involution on $[0,1]$ admits a representation of form (1).

PROOF. Obviously, only the latter assertion requires an argument. Let $N : [0,1] \to [0,1]$ be a decreasing solution of

$$N^2(x) = x. \tag{4}$$

Then N is a surjection and consequently N is continuous. Put $\sigma(x) := \frac{1}{2}\left(1+x-N(x)\right)$, $x \in [0,1]$. Hence

$$\sigma\left(N(x)\right) = \frac{1}{2}\left(1+N(x)-x\right) = 1-\frac{1}{2}(1-N(x)+x) = 1-\sigma(x), \tag{5}$$

for $x \in [0,1]$. Clearly, σ is a strictly increasing function of $[0,1]$ onto $[0,1]$ and continuous since $N(0)=1$ and $N(1)=0$. Therefore according to (5) we get

$$N(x) = \sigma^{-1}\left(1-\sigma(x)\right)$$

for $x \in [0,1]$. The function $\gamma(x) := \sigma^{-1}(x)$ is the desired homeomorphism.

LEMMA 2. *Let hypothesis (H_1) be fulfilled and R be a solution of (2). Then the function defined by formula*

$$N(x) = R\left(1-R^{-1}(x)\right) \tag{6}$$

satisfies simultaneously equations (4) and

$$N(f_k(x)) = f_{p-1-k}(N(x)) \quad k=0,\dots,p-1 \tag{7}$$

for $x \in [0,1]$.

PROOF. First, by Lemma 1 we obtain, that N is an involution. By (2) we

$$\begin{aligned}
N(f_k(x)) &= R\left(1-R^{-1}(f_k(x))\right) = R\left(1-\frac{R^{-1}(x)+k}{p}\right)\\
&= R\left(\frac{p-k-1+(1-R^{-1}(x))}{p}\right) = f_{p-1-k}(R(1-R^{-1}(x)))\\
&= f_{p-1-k}(N(x)),
\end{aligned}$$

have for $x \in [0,1]$, $k=0,\dots,p-1$.

THEOREM 1. *Let hypothesis* (H_1) *be fulfilled and* R *be a solution of (2). The only solution of the system of functional equations*

$$(8) \qquad \begin{cases} N^2(x) = x \\ N(f_k(x)) = f_{p-1-k}(N(x)) \end{cases} \quad \text{for } x \in [0,1],\ k = 0, \ldots, p-1$$

is given by (6). *This function is strictly decreasing and continuous.*

PROOF. By Lemma 2 the function N given by (6) satisfies (8). Moreover N is strictly decreasing and continuous. To prove the uniqueness, let N' be a solution of (8). Note that $r(x) := N'(R(1-x))$, $x \in [0,1]$ satisfies (2). In fact

$$\begin{aligned} r\left(\frac{x+k}{p}\right) &= N'\left(R\left(1 - \frac{x+k}{p}\right)\right) \\ &= N'\left(R\left(\frac{p-k-1+(1-x)}{p}\right)\right) = N'(f_{p-1-k}(R(1-x))) \\ &= f_k(N'(R(1-x))) = f_k(r(x)) \end{aligned}$$

for $x \in [0,1]$ and $k = 0, \ldots, p-1$. By the uniqueness of solution of system (2) $r = R$ and consequently $N'(x) = R(1 - R^{-1}(x))$ for all $x \in [0,1]$.

Theorem 1 generalizes result of Mayor and Torrens in paper [2].

If there exist limit $\lim_{x\to\infty} h(x) = a$ then we shall use the notation $h(\infty) := a$.

REMARK 1. *Let* $h_0, h_1, \ldots, h_{p-1} : [0,\infty) \to [0,\infty)$ *be strictly increasing and continuous functions with* $h_0(0) = 0$, $h_{k-1}(\infty) = h_k(0)$, $k = 1, \ldots, p-1$ *and* $h_{p-1}(\infty) = \infty$. *Then for every strictly increasing homeomorphism* $\alpha : [0,\infty) \to [0,1)$ *and*

$$f_k(x) := \begin{cases} \alpha \circ h_k \circ \alpha^{-1} & \text{if } x \in [0,1) \\ \lim_{x\to 1^-} \alpha \circ h_k \circ \alpha^{-1}(x) & \text{if } x = 1 \end{cases} \quad k = 0, \ldots, p-1$$

we have $f_0(0) = 0$, $f_{k-1}(1) = f_k(0)$, $k = 1, \ldots, p-1$ *and* $f_{p-1}(1) = 1$. *Moreover relations (1) hold iff the functions* $\alpha \circ h_k - \alpha$, $k = 0, \ldots, p-1$ *are strictly decreasing.*

Assume now the following hypothesis:

(H_2) $h_0, h_1, \ldots, h_{p-1} : [0,\infty) \to [0,\infty)$ are strictly increasing and continuous functions with $h_0(0) = 0$, $h_{k-1}(\infty) = h_k(0)$, $k = 1, \ldots, p-1$, $h_{p-1}(\infty) = \infty$ and there exists a strictly increasing homeomorphism $\alpha : [0,\infty) \to [0,1)$ such that functions $\alpha \circ h_k - \alpha$, $k = 0, \ldots, p-1$ are strictly decreasing.

THEOREM 2. *Let hypothesis* (H_2) *be fulfilled. Then the system of functional equations*

$$(9) \qquad \begin{cases} N^2(x) = x \\ N(h_k(x)) = h_{p-1-k}(N(x)) \end{cases} \text{for } x \in (0,\infty),\ k = 0,\dots,p-1$$

with the initial condition

$$(10) \qquad N(h_k(0)) = h_{p-k}(0), \quad k = 1,\dots,p-1$$

has a unique solution $N : (0,\infty) \to (0,\infty)$. *This solution is strictly decreasing and continuous. Every continuous solution of* (9) *satisfies condition* (10).

PROOF. To prove the existence put

$$f_k(x) := \begin{cases} \alpha \circ h_k \circ \alpha^{-1}(x) & \text{if } x \in [0,1) \\ \lim_{x\to 1^-} \alpha \circ h_k \circ \alpha^{-1}(x) & \text{if } x = 1 \end{cases} \quad k = 0,\dots,p-1.$$

By Remark 1 the function $f_k, k = 0,\dots,p-1$ fulfill (H_1). Hence by Theorem 1 there exists exactly one solution M of (8). This function is strictly decreasing, continuous and $M(0) = 1, M(1) = 0$.

Let $N : (0,\infty) \to (0,\infty)$ be defined by

$$N(x) := \alpha^{-1} \circ M \circ \alpha(x).$$

We shall show that N satisfies (9). It is easy to check, that $N^2(x) = x$, x in $(0,\infty)$. Moreover we have

$$\begin{aligned} N \circ h_k(x) &= \alpha^{-1} \circ M \circ \alpha \circ \alpha^{-1} \circ f_k \circ \alpha(x) = \alpha^{-1} \circ M \circ f_k \circ \alpha(x) \\ &= \alpha^{-1} \circ f_{p-1-k} \circ M \circ \alpha(x) = \alpha^{-1} \circ f_{p-1-k} \circ \alpha \circ \alpha^{-1} \circ M \circ \alpha(x) \\ &= h_{p-1-k} \circ N(x), \end{aligned}$$

for $x \in (0,\infty)$, $k = 0,\dots,p-1$. For $1 \le k \le p-1$ we have

$$\begin{aligned} N \circ h_k(0) &= \alpha^{-1} \circ M \circ \alpha \circ h_k(0) = \alpha^{-1} \circ M \circ \alpha \circ \alpha^{-1} \circ f_k \circ \alpha(0) \\ &= \alpha^{-1} \circ M \circ f_k(0) = \alpha^{-1} \circ f_{p-1-k} \circ M(0) = \alpha^{-1} \circ f_{p-1-k}(1) \\ &= \alpha^{-1} \circ f_{p-k}(0) = \alpha^{-1} \circ f_{p-k} \circ \alpha(0) = h_{p-k}(0). \end{aligned}$$

It remains to prove that this solution is unique. Let $\overline{N} : (0,\infty) \to (0,\infty)$ be a solution of (9) satisfying condition (10). Put

$$\overline{M}(x) := \begin{cases} 1 & \text{if } x = 0 \\ \alpha \circ \overline{N} \circ \alpha^{-1}(x) & \text{if } x \in (0,1) \\ 0 & \text{if } x = 1. \end{cases}$$

We shall show that $\overline{M}$ verifies (8). It is easily seen that $\overline{M}^2(x) = x$, $x \in [0,1]$. Evidently $\overline{M}$ satisfies (7) in $(0,1)$. At the point $x = 0$ we have
1) for $k = 0$:

$$\overline{M} \circ f_0(0) = \overline{M}(0) = 1 = f_{p-1}(1) = f_{p-1} \circ \overline{M}(0),$$

2) for $0 < k \le p-1$:

$$\begin{aligned}\overline{M} \circ f_k(0) &= \alpha \circ \overline{N} \circ \alpha^{-1} \circ f_k(0) = \alpha \circ \overline{N} \circ \alpha^{-1} \circ f_{k-1}(1)\\ &= \alpha \circ \overline{N} \circ \alpha^{-1} \circ \alpha \circ h_{k-1}(\infty)\\ &= \alpha \circ \overline{N} \circ h_{k-1}(\infty) = \alpha \circ \overline{N} \circ h_k(0) = \alpha \circ h_{p-k}(0)\\ &= \alpha \circ h_{p-k} \circ \alpha^{-1}(0) = f_{p-k}(0) = f_{p-1-k}(1) = f_{p-1-k} \circ \overline{M}(0).\end{aligned}$$

At the point $x = 1$ we have
1) for $k = p-1$:

$$\overline{M} \circ f_{p-1}(1) = \overline{M}(1) = 0 = f_0(0) = f_0 \circ \overline{M}(1),$$

2) for $0 \le k < p-1$:

$$\begin{aligned}\overline{M} \circ f_k(1) &= \alpha \circ \overline{N} \circ \alpha^{-1} \circ f_k(1) = \alpha \circ \overline{N} \circ \alpha^{-1} \circ \alpha \circ h_k(\infty)\\ &= \alpha \circ \overline{N} \circ h_{k+1}(0) = \alpha \circ h_{p-1-k}(0) = f_{p-1-k}(0)\\ &= f_{p-1-k} \circ \overline{M}(1).\end{aligned}$$

Thus $\overline{M}$ satisfies (8) in $[0,1]$ and consequently by the uniqueness of solution of (8) we have $\overline{M}(x) = M(x)$, $x \in [0,1]$. Hence $\alpha \circ \overline{N} \circ \alpha^{-1}(x) = \alpha \circ N \circ \alpha^{-1}(x)$, $x \in (0,1)$ and finally $N(x) = \overline{N}(x)$ for $x \in (0,\infty)$.

To prove the last thesis suppose N is continuous solution of (9). The equation $N^2(x) = x$ implies that N is strictly monotonic surjection of $(0,\infty)$ onto itself. By (9) we have

$$\begin{aligned}N[h_0[(0,\infty)]] &= h_{p-1}[(0,\infty)]\\ N[h_{p-1}[(0,\infty)]] &= h_0[(0,\infty)].\end{aligned} \tag{11}$$

Let $x \in h_0[(0,\infty)]$ and $y \in h_{p-1}[(0,\infty)]$. Since $h_0(\infty) \le h_{p-1}(0)$ we infer that $x < y$ and by (11) $N(x) > N(y)$. Thus N is strictly decreasing and consequently $N(0+) = \infty$ and $N(\infty) = 0$. Hence by (9)
$N(h_k(0)) = \lim_{x\to 0^+} N(h_k(x)) = \lim_{x \to 0^+} h_{p-1-k}(N(x)) = h_{p-1-k}(\infty) = h_{p-k}(0)$.
This ends the proof.

Further we shall deal with particular case of system (9). Given k, $k \geq 1$, consider the system

$$(12) \qquad \begin{cases} N^2(x) = x \\ N\left(\frac{x}{kx+1}\right) = N(x) + k \end{cases} \text{for } x \in (0, \infty).$$

As an application of Theorem 2 we shall prove the following result

THEOREM 3. *If $k = 1$ then the only solution of system (12) is the function $N(x) = 1/x$ (see [3]). If $k > 1$ then for every increasing bijection $f : [0, \infty) \to [1/k, k)$ such that*

$$(13) \qquad \frac{f(x) - f(y)}{1 + f(x)f(y)} < \frac{x - y}{1 + xy} \quad \textit{for } x > y$$

there exists exactly one solution of system (12) such that $N \circ f = f \circ N$ and $N(k) = \frac{1}{k}$. This solution is strictly decreasing and continuous.

PROOF. The first assertion where $k = 1$ is the Volkmann's theorem (see [3]) but we give a new proof of this theorem. In this case system (12) has the form

$$(14) \qquad \begin{cases} N^2(x) = x \\ N\left(\frac{x}{x+1}\right) = N(x) + 1 \end{cases}$$

for $x \in (0, \infty)$. The thesis results directly from Theorem 2 for $p = 2$ with $h_0(x) = \frac{x}{x+1}$, $h_1(x) = x + 1$, $x \in (0, \infty)$. Observe that these functions fulfill hypothesis (H_2) with $\alpha(x) = \frac{2}{\pi}\arctan x$. In fact, h_0, h_1 are strictly increasing, continuous and $h_0(0) = 0$, $h_0(\infty) = h_1(0)$, $h_1(\infty) = \infty$. Moreover it is easy to check that functions

$$(\alpha \circ h_0 - \alpha)(x) = \frac{2}{\pi}\arctan\frac{x}{x+1} - \frac{2}{\pi}\arctan x$$
$$(\alpha \circ h_1 - \alpha)(x) = \frac{2}{\pi}\arctan(x+1) - \frac{2}{\pi}\arctan x$$

are strictly decreasing in $[0, \infty)$.

We shall show that for every solution N of system (14) $N(1) = 1$. By the second equation of system (14) we get that for $x < 1$ $N(x) > 1$. Moreover $N(1) \geq 1$ since otherwise $1 = N(N(1)) > 1$ is contradiction. We shall show that $N(1) = 1$. Let us note that by (14) we get

$$N\left(\frac{N(x)}{N(x)+1}\right) = x + 1$$

for $x > 0$. Suppose $N(1) > 1$. Then there exists an $x_0 > 0$ such that

$$N\left(\frac{N(x_0)}{N(x_0)+1}\right) = N(1).$$

Hence $\frac{N(x_0)}{N(x_0)+1} = 1$, a contradiction. Thus $N(1) = 1$.

By Theorem 2 there is a unique function N satisfying system (14) in $(0, \infty)$. The involution $N(x) = 1/x$, $x \in (0, \infty)$ is a solution of system (14). Consequently it is the only solution of this system. This ends the proof in case $k = 1$.

Let $k > 1$. Consider the system

$$\begin{cases} N^2(x) = x \\ f(N(x)) = N(f(x)) \\ N\left(\frac{x}{kx+1}\right) = N(x) + k \end{cases} \tag{15}$$

for $x \in (0, \infty)$. The proof results directly from Theorem 2 for $p = 3$ with $h_0(x) = \frac{x}{kx+1}$, $h_1(x) = f(x)$, $h_2(x) = x + k$, $x \in (0, \infty)$. Observe that these functions fulfill hypothesis (H_2) with $\alpha(x) = \frac{2}{\pi}\arctan(x)$. Evidently h_0, h_1, h_2 are strictly increasing, continuous and

$$h_0(0) = 0,\ h_0(\infty) = h_1(0) = \frac{1}{k},\ h_1(\infty) = h_2(0) = k,\ h_2(\infty) = \infty.$$

Let us note, that inequality (13) is equivalent to the fact that function $(\alpha \circ h_1 - \alpha)(x) = \frac{2}{\pi}\arctan f(x) - \frac{2}{\pi}\arctan x$ is strictly decreasing. In fact, for $x > y$, $x, y \in (0, \infty)$ we get

$$\begin{aligned} &(\alpha \circ f - \alpha)(x) - (\alpha \circ f - \alpha)(y) \\ &= \frac{2}{\pi}[(\arctan f(x) - \arctan x) - (\arctan f(y) - \arctan y)] \\ &= \frac{2}{\pi}[(\arctan f(x) - \arctan f(y)) - (\arctan x - \arctan y)] \\ &= \frac{2}{\pi}\left[\arctan \frac{f(x) - f(y)}{1 + f(x)f(y)} - \arctan \frac{x - y}{1 + xy}\right]. \end{aligned}$$

Thus $(\alpha \circ f - \alpha)(x) - (\alpha \circ f - \alpha)(y) < 0$ iff

$$\frac{f(x) - f(y)}{1 + f(x)f(y)} < \frac{x - y}{1 + xy}.$$

Moreover it is easy to check that functions

$$(\alpha \circ h_0 - \alpha)(x) = \frac{2}{\pi}\arctan\frac{x}{kx+1} - \frac{2}{\pi}\arctan x$$

$$(\alpha \circ h_2 - \alpha)(x) = \frac{2}{\pi}\arctan(x+k) - \frac{2}{\pi}\arctan x$$

are strictly decreasing in $(0,\infty)$. Since $h_1(0) = \frac{1}{k}$ and $h_2(0) = k$, the condition (10) is equivalent to the equality $N(k) = \frac{1}{k}$. By Theorem 2 there is a unique function N satisfying system (15) in $(0,\infty)$. This ends the proof.

REMARK 2. *If N satisfies system* (15) *then $N(k) \in \{k, \frac{1}{k}\}$. If moreover N is continuous, then $N(k) = \frac{1}{k}$. In fact, by* (4) *N is a bijection of $(0,\infty)$ onto itself. By the third equation of system* (15) *we get that $N((0,\frac{1}{k})) \subset (k,\infty)$ and further by* (4), *$(0,\frac{1}{k}) \subset N((k,\infty))$. Let us note that by* (15) *we get*

$$\frac{N(x)}{kN(x)+1} = N(x+k),$$

whence we infer that $N((k,\infty)) \subset (0,\frac{1}{k})$ and by (4) *$(k,\infty) \subset N((0,\frac{1}{k}))$. Thus $N((0,\frac{1}{k})) = (k,\infty)$ and $N((k,\infty)) = (0,\frac{1}{k})$. Similary by equation $f \circ N = N \circ f$ we obtain that $N((\frac{1}{k},k)) = (\frac{1}{k},k)$. Hence by bijectivity of N we have that $N(k) \in \{k,\frac{1}{k}\}$. If N is continuous then by Theorem 2 $N(k) = \frac{1}{k}$.*

EXAMPLE 1. Given $k > 1$, consider the system

$$\text{(16)} \qquad \begin{cases} N^2(x) = x \\ N\left(\frac{kx+1}{x+k}\right) = \frac{kN(x)+1}{N(x)+k} \\ N\left(\frac{x}{kx+1}\right) = N(x)+k \end{cases}$$

for $x \in (0,\infty)$. We apply Theorem 3 with $f(x) = \frac{kx+1}{x+k}$, $x \in [0,\infty)$. The function $f(x)$ is strictly increasing, continuous and $f(0) = \frac{1}{k}$, $f(\infty) = k$. Moreover

$$\frac{f(x)-f(y)}{1+f(x)f(y)} = \frac{(k^2-1)(x-y)}{2kx+2ky+(k^2+1)xy+k^2+1} < \frac{(k^2-1)(x-y)}{(k^2+1)(1+xy)} < \frac{x-y}{1+xy},$$

for $x > y$, $x, y \in (0,\infty)$. Thus by Theorem 3 there exists a unique solution N of system (16) such that $N(k) = \frac{1}{k}$. Let us note that function $\frac{1}{x}$ satisfies

(16). Consequently the only solution of system (16) such that $N(k) = \frac{1}{k}$ is given by $N(x) = \frac{1}{x}$, $x \in (0, \infty)$.

EXAMPLE 2. Consider the system

$$(17) \qquad \begin{cases} N^2(x) = x \\ N\left(\frac{\frac{3}{2}x+\frac{2}{3}}{x+1}\right) = \frac{\frac{3}{2}N(x)+\frac{2}{3}}{N(x)+1} \\ N\left(\frac{x}{\frac{3}{2}x+1}\right) = N(x) + \frac{3}{2} \end{cases}$$

for $x \in (0, \infty)$. We apply Theorem 3 with $k = \frac{3}{2}$, $f(x) = \frac{\frac{3}{2}x+\frac{2}{3}}{x+1}$, $x \in [0, \infty)$. The function $f(x)$ is strictly increasing, continuous and $f(0) = \frac{2}{3}$, $f(\infty) = \frac{3}{2}$. Moreover

$$\begin{aligned} \frac{f(x) - f(y)}{1 + f(x)f(y)} &= \frac{(\frac{3}{2} - \frac{2}{3})(x - y)}{2x + 2y + ((\frac{3}{2})^2 + 1)xy + (\frac{2}{3})^2 + 1} \\ &< \frac{(\frac{3}{2} - \frac{2}{3})(x - y)}{((\frac{2}{3})^2 + 1)(xy + 1)} < \frac{x - y}{1 + xy}, \end{aligned}$$

for $x > y$, $x, y \in (0, \infty)$. Thus by Theorem 3 there exists a unique solution N of system (17) such that $N(\frac{3}{2}) = \frac{2}{3}$. But in this case the function $\frac{1}{x}$ does not commute with f. Consequently we get a solution, which is different from $\frac{1}{x}$.

REFERENCES

[1] M. Kuczma, B. Choczewski, R. Ger, *Iterative functional equations*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, New York, Port Chester, Melbourne, Sydney 1990.

[2] G. Mayor, J. Torrens, *De Rham systems and the solution of a class of functional equations*, Aequat. Math. **47**(1994), 43–49.

[3] P. Volkmann, *Charakterisierung der Function $\frac{1}{x}$ durch Functionalgleichungen*, Ann. Polonici Math. **48**(1988), 91–94.

[4] M. C. Zdun, *On conjugacy of some systems of functions*, in Aequat. Math.

INSTITUTE OF MATHEMATICS
PEDAGOGICAL UNIVERSITY
REJTANA 16 A
PL-35-310 RZESZÓW

INSTITUTE OF MATHEMATICS
PEDAGOGICAL UNIVERSITY
PODCHORĄŻYCH 2
PL-30-084 KRAKÓW