REMARK ON HOMOMORPHISMS OF GROUPS

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Assume that $(X,+)$ are commutative groups, and C is a subset of X fulfilling the conditions $C+C \subseteq C$ and $C-C=X$. A function $f:X \to Y$ is called C–additive function if it satisfies functional equation $f(x+y) = f(x)+f(y)$, for all $x,y \in X$ such that $y-x \in C \cup (-C) \cup \{0\}$. In [1, Theorem 8.4] has been proved that every C–additive function $f:X\to Y$ is additive. In the proof the comutativity has been essentially used. Here we present a simple proof of an analogous statement in the case of arbitrary groups.

THEOREM 1. *Let X,Y be groups, let C be a subset of X fulfilling the conditions $C\cdot C \subseteq C$ and $C\cdot C^{-1} = X$ and let $f:X\to Y$ be a function satisfying $f(xy)=f(x)f(y)$, for all $x,y\in X$ such that $x\cdot y^{-1} \in C\cup C^{-1}$. Then f is a homomorphism of groups.*

PROOF. The proof is carried out through five steps.

First step: For $c\in C$ we have $f(c)=f(ce)=f(c)f(e)$, so $f(e)=e$.

Second step: We are going to prove that $f(x^{-1})=f(x)^{-1}$, for every $x\in X$. At first suppose that $x\in C$. Then $e=f(e)=f(xx^{-1})=f(x)f(x^{-1})$. Let $x\in X$ be arbitrary. There exist $u,v\in C$ such that $x=uv^{-1}$. Therefore $f(x^{-1})=f(vu^{-1})=f(v)f(u^{-1})=f(v^{-1})^{-1}f(u)^{-1}=(f(u)f(v^{-1}))^{-1}=f(uv^{-1})^{-1}=f(x)^{-1}$.

Third step: Suppose that $x,y\in C$. Then from the second step we get

$$f(xy)=f(xy)f(y)^{-1}f(y)=f(xy)f(y^{-1})f(y)=f(xyy^{-1})f(y)=f(x)f(y).$$

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Fourth step: Suppose that $c \in C$ and $x \in X$. Then by the third step

$$f(cx) = f(cuv^{-1}) = f((cu)v^{-1}) = f(cu)f(v^{-1}) = f(c)f(u)f(v^{-1}) = f(c)f(x).$$

From this and from the second step, we get $f(xc^{-1}) = f(x)f(c^{-1})$, for every $c \in C$ and every $x \in X$.

Fifth step: Suppose that $x, y \in X$ are arbitrary. Then there exist $u, v, z, w \in C$ such that $x = uv^{-1}$ and $y = zw^{-1}$. By the fourth step we have

$$\begin{aligned} f(xy) &= f(uv^{-1}zw^{-1}) = f(u)f(v^{-1}zw^{-1}) = f(u)f(v^{-1}z)f(w^{-1}) \\ &= f(u)f(v^{-1})f(z)f(w^{-1}) = f(uv^{-1})f(zw^{-1}) = f(x)f(y). \end{aligned}$$

The theorem is proved.

In the sequel we will show that the condition $C \cdot C \subseteq C$ in the theorem 1 can be replaced by a weaker one.

THEOREM 2. *Let X, Y be groups, let C be a subset of X fulfilling the conditions $C \cdot C \cdot C \subseteq C$ and $C \cdot C^{-1} = X$ and let $f : X \to Y$ be a function satisfying $f(xy) = f(x)f(y)$, for all $x, y \in X$ such that $x \cdot y^{-1} \in C \cup C^{-1}$.*
Then f is a homomorphism of groups.

PROOF. **1. step:** $f(e) = e$, as in the first step in the proof of Theorem 1.

2. step: If $c \in C$ then $f(c^{-1}) = f(c)^{-1}$.

a) There exist $u, v \in C$ such that $c = uv^{-1}$. Therefore, $e = f(uu^{-1}) = f(cvu^{-1}) = f(cv)f(u^{-1}) = f(u)f(u^{-1})$, hence $f(u^{-1}) = f(u)^{-1}$. Since $uv^{-1} \in C$, we have $f(uv) = f(u)f(v)$. Therefore, $f(u) = f(uvv^{-1}) = f(uv)f(v^{-1}) = f(u)f(v)f(v^{-1})$, so $f(v^{-1}) = f(v)^{-1}$.

b) $f(u)f(u^{-2}) = f(uu^{-2}) = f(u^{-1}) = f(u)^{-1}$, so $f(u^{-2}) = f(u)^{-2}$. Similarly we can prove that $f(v^{-2}) = f(v)^{-2}$.

c) $f(uv^{-1}) = f(uv^{-1})f(v^{-1})f(v) = f(uv^{-2})f(v) = f(u)f(v^{-2})f(v) = f(u)f(v)^{-2}f(v) = f(u)f(v)^{-1}$.

d) From a), b) and c) we have $f(c^{-1}) = f(vu^{-1}) = f(vu^{-1})f(u^{-1})f(u) = f(vu^{-2})f(u) = f(v)f(u)^{-2}f(u) = f(v)f(u)^{-1} = (f(u)f(v)^{-1})^{-1} = (f(uv^{-1}))^{-1} = f(c)^{-1}$.

3. step: Let $a, b \in C$. Then, by the 2. step we have:

$$f(ab) = f(ab)f(b)^{-1}f(b) = f(ab)f(b^{-1})f(b) = f(abb^{-1})f(b) = f(a)f(b).$$

Especially, $f(c^2) = f(c)^2$, for every $c \in C$.

4. step: Similarly as in the 2. step c), using the 2. step d) we get $f(ab^{-1}) = f(a)f(b)^{-1}$, for all $a, b \in C$. Also, from the 3. step and the fact $C^{-1} \cdot C^{-1} \cdot C^{-1} \subseteq C^{-1}$ we obtain

$$\begin{aligned} f(a^{-1}b) &= f(a^{-1}b)f(b)f(b)^{-1} = f(a^{-1}b^2)f(b)^{-1} = f(a^{-1})f(b^2)f(b)^{-1} \\ &= f(a^{-1})f(b)^2 f(b)^{-1} = f(a^{-1})f(b), \qquad \text{for all} \quad a, b \in C. \end{aligned}$$

5. step: Let $x \in X$. Then $x = ab^{-1}$, for $a, b \in C$. From the 4. step, similarly as in the proof of the 2. step d), we get $f(x^{-1}) = f(x)^{-1}$.

The rest of the proof is as the proofs of the fourth step and the fifth step of Theorem 1.

References

[1] Z. Kominek, *Convex functions in linear spaces*, Katowice 1989.

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