DISTRIBUTIONAL CHAOS FOR TRIANGULAR MAPS

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To the memory of Professor György Targonski

Abstract. In this paper we show that triangular maps of the unit square can have properties that are impossible in the one-dimensional case. In particular, we find a map with infinite spectrum; a distributionally chaotic map whose principal measure of chaos is not generated by a pair of points and which has the empty spectrum; a distributionally chaotic map that is not chaotic in the sense of Li and Yorke.

1. Introduction

Triangular maps have been recently considered by many authors, since the dynamical systems generated by them exhibit phenomena impossible in the one-dimensional case, regardless that in some properties they are surprisingly regular, cf., e. g., [2], [4], [1]. We give here further examples.

Let $I = [0,1]$ be the unit interval. By a *triangular map* we mean a continuous map $F : I^2 \to I^2$ of the form $F(x,y) = (f(x), g_x(y))$. The map f is called the *base* for F, g_x is a map from the *layer* $I_x = I \times \{x\}$ to I.

Let f be a map from a compact metric space (M,d) into itself. For any integer $i \geq 0$, let f^i denote the i-th iterate of f. For any x in M, the sequence of iterates $\{f^i(x)\}_{i=0}^{\infty}$, where $f^0(x) = x$, is the *trajectory* of x; and the set $\omega_f(x)$ of all limit points of this trajectory is the ω*-limit set* of x. An ω-limit set is *maximal* if it is not properly contained in any other ω-limit set. If

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for any nonvoid subsets U and V of M, both relatively open in M, there exists an $n \in N$ such that $f^n(U) \cap V \neq \emptyset$, then we say that f is *topologically transitive* on M.

For any pair (x, y) of points of M and any positive integer n, we define a distribution function $\Phi_{xy}^{(n)} : R \to [0, 1]$ by

$$\Phi_{xy}^{(n)}(t) = \frac{1}{n}\#\{i, 0 \leq i \leq n-1 : \quad d(f^i(x), f^i(y)) < t\}.$$

Obviously, $\Phi_{xy}^{(n)}$ is a left-continous non-decreasing function, $\Phi_{xy}^{(n)}(0) = 0$ and $\Phi_{xy}^{(n)}(t) = 1$ for all t greater than the maximum of the numbers $d(f^i(x), f^i(y))$, $0 \leq i \leq n-1$. Put $\Phi_{xy}(t) = \liminf_{n\to\infty} \Phi_{xy}^{(n)}(t)$, $\Phi_{xy}^*(t) = \limsup_{n\to\infty} \Phi_{xy}^{(n)}(t)$. The function Φ_{xy} is called the *lower distribution*, and Φ_{xy}^* the *upper distribution* of x and y. If there is a pair of points (x, y) in M such that $\Phi_{xy}(t) < \Phi_{xy}^*(t)$ for all t in some nondegenerate interval, then we say that f is *distributionally chaotic* (briefly, *d-chaotic*). The (*principal*) *measure of chaos* of f is the number

$$\mu_p(f) = \sup_{x,y\in M} \frac{1}{d_M} \int_0^\infty (\Phi_{xy}^*(t) - \Phi_{xy}(t))dt,$$

where d_M is the (finite) diameter of the metric space (M, d). It follows at once that $\mu_p(f) \neq 0$ if and only if f is d-chaotic. Using results from [7] it can be proved that for $f \in C(I, I)$, $\mu_p(f)$ is always generated by a pair of points (cf. also [3]).

A pair (x, y), $x, y \in M$, is called *isotectic* (with respect to f) if, for every positive integer n, the ω-limit sets $\omega_{f^n}(x)$ and $\omega_{f^n}(y)$ are subsets of the same maximal ω-limit set of f^n. The *spectrum* of f, denoted by $\Sigma(f)$, is the set of minimal elements of $D(f)$, where $D(f) = \{\Phi_{xy}; (x, y) \textit{ is isotectic}\}$. For $f \in C(I, I)$ the spectrum is always nonempty and finite (see [7]).

A map $f : M \to M$ is called *chaotic in the sense of Li and Yorke* if there exist distinct points $x, y \in M$ such that

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0, \qquad \limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0.$$

It is easy to see that any d-chaotic function $f \in C(I, I)$ is chaotic in the sense of Li and Yorke (see also [7]).

2. A triangular map with infinite spectrum

In [6], there was given an instruction how to construct a function $f \in C(I,I)$ such that its spectrum has exactly n elements. Using this example we construct a triangular map F of the unit square with infinite spectrum.

Let C be the middle-third Cantor set, and let $J = [a,b]$ denote a complementary interval to the Cantor set. Define the base for F, $f : I \to I$, by

$$f(x) = \begin{cases} x, & \text{if } x \in C, \\ a, & \text{if } x \in [a, b - \frac{b-a}{4}], \\ 4x - 3b, & \text{if } x \in [b - \frac{b-a}{4}, b]. \end{cases} \tag{1}$$

This map is obviously continuous.

Now, define maps $g_x : I_x \to I$. For $a \in C$ denote $p_a = (1+a)/5$, $q_a = (4-a)/5$, so that $0 < p_a < q_a < 1$ for each $a \in C$. Define g_a as a piecewise linear map given by $g_a(0) = g_a(q_a) = 0$, $g_a(p_a/2) = q_a$, $g_a(p_a) = p_a$, $g_a(1) = 1$. For $t \in (a,b)$ (where $J = [a,b]$ is a complementary interval) put

$$g_t(x) = \frac{b-t}{b-a}\, g_a(x) + \frac{t-a}{b-a}\, g_b(x). \tag{2}$$

LEMMA 1. *The triangular map $F = (f(x), g_x(y))$ has infinite spectrum.*

PROOF. For $0 < p_a < q_a < 1$ the restriction of g_a to the interval $[0, q_a]$ is topologically conjugate to the tent map $\tau(x) = 1 - |2x-1|$. Exploiting this, it follows that $\Sigma(g_a) = \{\Psi_{p_a q_a}\}$, where $\Psi_{p_a q_a}(x) = 0$ if $x \in [0, p_a]$, $\Psi_{p_a q_a}(x) = 1$ if $x \in [q_a, 1]$, and $0 < \Psi_{p_a q_a}(x) < 1$ if $x \in (p_a, q_a)$. So, if $p < p' < q' < q$, then Ψ_{pq} and $\Psi_{p'q'}$ are incomparable and therefore $\Sigma(F) = \bigcup_{a \in C} \Sigma(g_a)$.

3. A triangular map with empty spectrum

Now, we construct a triangular map F that is d-chaotic but its principal measure of chaos is not generated by a pair of points and, moreover, it has the empty spectrum.

REMARK. In the sequel we denote by ϵ_a, for $a \in R$, the distribution function such that $\epsilon_a(t) = 0$ if $t \le a$, $\epsilon_a(t) = 1$ if $t > a$.

LEMMA 2. *Let $f \in C(I,I)$ be a piecewise monotone map (with finite number of pieces of monotonicity), topologically transitive on an interval*

$J = [a, b] \subset I$ such that $f(a) = a$, $f(b) = b$. Then there exist $u, v \in J$ such that $\Phi_{uv} \leq \epsilon_{b-a}$, $\Phi^*_{uv} \equiv 1$. Consequently, $\mu_p(f) \geq b - a$.

PROOF. Since f is transitive and piecewise monotone on J,

$$f^n(V) \supset [a, b] \tag{3}$$

for any interval $V \subset J$ and for any sufficiently large n (depending on V) (cf. [5]). If $U_n \subset [a, b]$, $n = 0, 1, \dots$, are compact intervals such that $a \in U_{2n}$ and $b \in U_{2n+1}$ for each $n = 0, 1, \dots$, and diam $U_n \to 0$ for $n \to \infty$ then, by (3), there exist nonnegative integers r_n such that $f^r(U_{2n}) \supset U_{2n+1}$ for each $r \geq r_{2n}$, and $f^r(U_{2n+1}) \supset U_{2(n+1)}$ for each $r \geq r_{2n+1}$. Moreover, integers r_n can be chosen so that $\lim_{n\to\infty}(r_0 + \ldots + r_n)/r_{n+1} = 0$. By the itinerary lemma and transitivity of f, there exists a $u \in U_0$ such that for any n, $f^{2(r_0+\ldots+r_n)+k}(u) \in U_{n+1}$, whenever $1 \leq k \leq r_{n+1}$ (to see this, note that any U_n contains a fixed point).

Let $\delta > 0$. Find n such that diam $U_{2n+1} < \delta$. Then $\Phi_{ub}^{(2(r_1+\ldots+r_{2n})+r_{2n+1})}(\delta) \geq \frac{r_{2n+1}}{2(r_1+\ldots+r_{2n})+r_{2n+1}}$, hence $\limsup_{n\to\infty} \Phi_{ub}^{(n)}(\delta) = 1$, and therefore $\Phi^*_{ub} \equiv 1$. Similarly, find n such that diam $U_{2n} < \delta$. Then $\Phi_{ub}^{(2(r_1+\ldots+r_{2n-1})+r_{2n})}(b - a - \delta) \leq \frac{2(r_1+\ldots+r_{2n-1})}{2(r_1+\ldots+r_{2n-1})+r_{2n}}$, hence $\liminf_{n\to\infty} \Phi_{ub}^{(n)}(b - a - \delta) = 0$, and therefore $\Phi_{ub} \leq \epsilon_{b-a}$.

To define F, let the base f for F be the same as in the previous example, cf. (1). Now, define the maps $g_x : I_x \to I$. For $a \in C$, $a \neq 0$, let g_a be a map such that:

(i) $|g_a(x) - x| \leq a/10$,

(ii) $g_a(x) = x$ for $x \in [0, a/5] \cup [1 - a/5, 1]$,

(iii) g_a is piecewise monotone and topologically transitive on $[a/5, 1 - a/5]$.

Such a map always exists (see, e. g., [8]).

Put $g_0(x) = x$. For $t \in (a, b)$ (where $[a, b]$ is a complementary interval to C), let g_t be given by (2).

LEMMA 3. *Let $F = (f(x), g_x(y))$, where f and g_x are as above. Then*

(i) $\Sigma(F) = \emptyset$.

(ii) *The triangular map F is d-chaotic but the principal measure of chaos of F is generated by no pair of points.*

PROOF. For each $a \in C$, $a \neq 0$, there exist u_y and v_y such that, for $u = (a, u_y)$ and $v = (a, v_y)$, $\Phi^*_{uv} \equiv 1$ and $\Phi_{uv} \equiv \epsilon_{1-2a/5}$ (cf. Lemma 2). This implies that F is d-chaotic with $\mu_p(F) = 1$. If $\Sigma(F) \neq \emptyset$, then $\epsilon_1 \in \Sigma(F)$, which is impossible. Indeed, assume $\epsilon_1 = \Phi_{wz}$ for some w, z such that $\Phi^*_{wz} \equiv 1$. By the definition of the base f it is easy to see that for each

$x \in I$ there exists an $n \geq 0$ such that $f^n(x) \in C$. Hence, for some $n \geq 0$, the first coordinate both of $F^n(w)$ and $F^n(z)$ is a fixed point $b \in C$. We may assume without loss of generality, that $n = 0$. If $b > 0$, then, as above, $\Phi_{wz} = \epsilon_{1-2b/5} > \epsilon_1$. So there must be $b = 0$. But in this case $\Phi_{wz} \equiv 1$ which is a contradiction.

The property (ii) is obvious.

4. A distributionally chaotic map not chaotic in the sense of Li and Yorke

Define the base f for F. Let $y_0 = 0$, $y_i = \sum_{k=1}^{i} 2^{-k}$ be endpoints of the intervals $J_i = [y_{i-1}, y_i]$, for $i = 1, 2, \ldots$. Divide each interval J_i to $n_i = 2^{2^i}$ parts of the same length; then

$$\lim_{k\to\infty} \frac{n_1 + \ldots + n_k}{n_{k+1}} = 0. \tag{4}$$

In this way we obtain an increasing sequence of points $\{x_i\}_{i=0}^{\infty}$ such that $x_0 = y_0$, $x_{n_1} = y_1$, $\ldots$, $x_{n_k} = y_k$, $\ldots$, $\lim_{n\to\infty} x_n = 1$. Define $f : I \to I$ as a piecewise linear map (with infinitely many pieces) given by $f(x_k) = x_{k+1}$ for $k = 0, 1, 2, \ldots$, $f(1) = 1$.

Now, define maps $g_x : I_x \to I$. Let ε_k be such that $(1 - \varepsilon_k)^{n_k} = 1/3$. For $t \in J_{2n}$, $n = 1, 2, \ldots$, set

$$g_t(x) = x.$$

For $t \in J_k \setminus [x_{n_k-1}, y_k]$, where $k = 4n + 1$, $n = 0, 1, 2, \ldots$, put

$$g_t(x) = (1 - \varepsilon_k)x,$$

and for $t \in J_l \setminus [y_{l-1}, x_{n_{l-1}+1}]$, where $l = 4n - 1$, $n = 1, 2, \ldots$, set

$$g_t(x) = \begin{cases} \frac{1}{1-\varepsilon_l}x & \text{for } x \in [0, 1 - \varepsilon_l], \\ 1 & \text{for } x \in [1 - \varepsilon_l, 1]. \end{cases}$$

Let $[a, b]$ be an interval such that either $a = x_{n_k-1}$ and $b = y_k$ $(k = 4n + 1,\ n = 0, 1, \ldots)$ or $a = y_{l-1}$ and $b = x_{n_{l-1}+1}$ $(l = 4n - 1,\ n = 1, 2, \ldots)$. For $t \in [a, b]$ define g_t by (2).

LEMMA 4. *The triangular map $F = (f(x), g_x(y))$ is d-chaotic but not chaotic in the sense of Li and Yorke.*

PROOF. The map F is d-chaotic, since for $u = (0,0)$ and $v = (0,1)$ we have $\liminf_{n\to\infty} |F^n(u) - F^n(v)| = 1/3$ and $\limsup_{n\to\infty} |F^n(u) - F^n(v)| = 1$, so that $0 \le \Phi_{uv} \le \Phi^*_{uv} \le \epsilon_{1/3}$. Moreover, for each $\delta > 0$, $k = 1, 2, \ldots$,

$$\Phi_{uv}^{(n_1+n_2+\ldots+n_{4k})}(1-\delta) < \frac{n_1+n_2+n_3+\ldots+n_{4k-3}+n_{4k-2}+n_{4k-1}}{n_1+n_2+\ldots+n_{4k}},$$

hence $\liminf_{n\to\infty} \Phi_{uv}^{(n)}(1-\delta) = 0$, which gives $\Phi_{uv} \equiv 0$. On the other hand,

$$\Phi_{uv}^{(n_1+n_2+\ldots+n_{4k-2})}(1/3+\delta) > \frac{n_{4k-2}}{n_1+n_2+\ldots+n_{4k-2}},$$

so that $\limsup_{n\to\infty} \Phi_{uv}^{(n)}(1/3+\delta) = 1$, and therefore $\Phi^*_{uv} = \epsilon_{1/3}$.

Let us show that F is not chaotic in the sense of Li and Yorke. By means of the map F we can define a relation of equivalence $\sim$ in I^2 such that $u \sim v$ if and only if $\liminf_{n\to\infty} |F^n(u) - F^n(v)| = 0$. Denote $u = (u_x, u_y)$, $v = (v_x, v_y)$ and suppose $u \sim v$. By (4) we may assume without loss of generality, that $u_x = v_x$. From the construction of g_x it is easy to see that if $u_y \ne v_y$ then $u \not\sim v$ which is a contradiction.

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