SOME INVARIANTS FOR σ–PERMUTATION MAPS

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To the memory of Professor György Targonski

Abstract. In this paper we obtain some formulas which allow us to compute topological and metric entropy and topological pressure for a new class of maps. It is also shown that similar formulas do not hold for metric and topological sequence entropy and a new commutativity problem is posed.

Introduction

Let $X_0, X_1, \ldots, X_{n-1}$ be compact metric spaces. Maps $F : X_0 \times X_1 \times \ldots \times X_{n-1} \to X_0 \times X_1 \times \ldots \times X_{n-1}$ of the type

$$F(x_0, x_1, \ldots, x_{n-1}) = (f_0(x_1), f_1(x_2), \ldots, f_{n-1}(x_0)),$$

where $f_i : X_{i+1} \to X_i$ are continuous maps (throughout the paper, subindexes for spaces "X" and maps "f" must be always taken mod n) were extensive studied by Linero in [8] and could be useful to describe the behaviour of populations of several non-competitive species living in the same habitat. Moreover, when $n = 2$ and $X = [0, 1]$ its periodic structure has been described in [3]. In this case these maps are used to study some models of duopoly games (see [4]).

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Our aim is to obtain formulas to compute the topological and metric entropies and the topological pressure of these maps. We will also show that similar formulas do not hold for topological and metric sequence entropies.

Indeed we will work with a slightly larger class of maps. Namely, if $\mathcal{S}_n$ is the set of permutations of $\{0, 1, \ldots, n-1\}$ we will say that $F : X_0 \times X_1 \times \ldots \times X_{n-1} \to X_0 \times X_1 \times \ldots \times X_{n-1}$ is a *σ-permutation map* (or shortly, a *p-map*) if there are $\sigma \in \mathcal{S}_n$ and continuous maps $f_i : X_{\sigma(i)} \to X_i$ such that

$$F(x_0, x_1, \ldots, x_{n-1}) = (f_0(x_{\sigma(0)}), f_1(x_{\sigma(1)}), \ldots, f_{n-1}(x_{\sigma(n-1)})).$$

We will also say that σ is the *associated permutation* to F. Notice that if σ is the *shift permutation* $s(i) = i + 1$ (i is again taken mod n) then we receive Linero's maps. In the particular case when σ is a cyclic permutation we will call F a *cyclic σ-permutation map* (or just a *cp-map*).

It is worth emphasizing that if (with the notation above) F is a p-map associated to a permutation σ, τ is another permutation and $G : X_{\tau(0)} \times X_{\tau(1)} \times \ldots . \times X_{\tau(n-1)} \to X_{\tau(0)} \times X_{\tau(1)} \times \ldots \times X_{\tau(n-1)}$ is defined by

$$G(x_{\tau(0)}, \ldots, x_{\tau(n-1)}) = (f_{\tau(0)}(x_{\sigma(\tau(0))}), \ldots, f_{\tau(n-1)}(x_{\sigma(\tau(n-1))}))$$

then F and G are topologically conjugated via the map $\pi : X_0 \times X_1 \times \ldots \times X_{n-1} \to X_{\tau(0)} \times X_{\tau(1)} \times \ldots \times X_{\tau(n-1)}$ given by

$$\pi(x_0, x_1, \ldots, x_{n-1}) = (x_{\tau(0)}, x_{\tau(1)}, \ldots, x_{\tau(n-1)}).$$

Further, if μ is an invariant measure of F then the measure $\widetilde{\pi_\tau}\mu$ is an invariant measure of G and, respect to these measures, these maps are metrically isomorphic. [Here we are using the following definition. Let X, Y be compact metric spaces, let $f : X \to Y$ be continuous, denote the set of probabilistic measures on the Borel σ-algebra $\beta(X)$ by $\mathcal{M}(X)$ and let $\mu \in \mathcal{M}(X)$. Then $\widetilde{f}\mu \in \mathcal{M}(Y)$ is the measure defined by $\widetilde{f}\mu(B) = \mu(f^{-1}(B))$ for any $B \in \beta(Y)$.] In particular, when working with cp-maps and after taking τ defined by $\tau(i) = \sigma^i(0)$, this will allow us later to assume that they are associated to the shift permutation s.

Let us complete this introductory section recalling the definitions of topological and metric entropies, topological and metric sequence entropies and topological pressure. Until the end of the section X, $F : X \to X$ and $A = \{a_i\}_{i=1}^{\infty}$ will respectively denote a compact metric space, a continuous map and an increasing sequence of non negative integers.

Let $\mathcal{A}$ be a finite open cover of X and denote by $\mathcal{N}(\mathcal{A})$ the smallest cardinality of any subcover chosen from $\mathcal{A}$. Let $\bigvee_{i=1}^{n} F^{-a_i}\mathcal{A}$ be the finite open cover of X given by $\{\bigcap_{i=1}^{n} F^{-a_i} A_i : A_i \in \mathcal{A}\}$. Define the *topological sequence entropy of F relative to $\mathcal{A}$* and A as

$$h_A(F,\mathcal{A}) = \limsup_{n\to\infty} \frac{1}{n} \log \mathcal{N}\left(\bigvee_{i=1}^{n} F^{-a_i}\mathcal{A}\right),$$

and define the *topological sequence entropy of F relative to A* ([5]) as

$$h_A(F) = \sup{}_{\mathcal{A}} h_A(F,\mathcal{A}).$$

When $A = \{i\}_{i=0}^{\infty}$ we get standard *topological entropy* $h(F)$.

Let μ be an invariant measure of F, that is, a probabilistic measure on $\beta(X)$ such that $\mu(F^{-1}B) = \mu(B)$ for all $B \in \beta(X)$. Denote by $\mathcal{M}(X,F)$ the set of invariants measures of F. Given a finite partition $\mathcal{B} = \{B_1,\ldots,B_k\}$ of Borel sets of X define

$$\mathcal{H}_\mu(\mathcal{B}) = -\sum_{i=1}^{k} \mu(B_i)\log\mu(B_i).$$

The *metric sequence entropy of F relative to $\mathcal{B}$, A and μ* is

$$h_{\mu,A}(F,\mathcal{B}) = \limsup_{n\to\infty} \frac{1}{n}\mathcal{H}_\mu\left(\bigvee_{i=1}^{n} F^{-a_i}\mathcal{B}\right),$$

and the *metric sequence entropy of F relative to A and μ* ([7]) is

$$h_{\mu,A}(F) = \sup{}_{\mathcal{B}} h_{\mu,A}(F,\mathcal{B}).$$

When $A = \{i\}_{i=0}^{\infty}$ we have the standard *metric entropy* $h_\mu(F)$.

Finally, if $\phi : X \to \mathbb{R}$ is a continuous map then define the *topological pressure of F relative to the function ϕ* (see page 207 from [9]) as

$$P(F,\phi) = \sup{}_{\mu\in\mathcal{M}(X,F)} \left\{h_\mu(F) + \int_X \phi\, d\mu\right\}.$$

1. Results

The computation of the above topological and metric invariants for p-maps requires some commutativity formulas. We will introduce some useful notation in order to write these formulas in a short way.

As usual, let $X_0, X_1, \ldots, X_{n-1}$ be compact metric spaces and let $f_i : X_{i+1} \to X_i$ for $i = 0, 1, \ldots, n-1$ be continuous maps. For any i and j define $f_i^j : X_{i+j} \to X_i$ by $f_i^j = f_i \circ f_{i+1} \circ \ldots \circ f_{i+j-1}$ (here f_i^0 will denote the identity map on X_i). Also, for any i let $\widetilde{f_i} : \mathcal{M}(X_{i+1}) \to \mathcal{M}(X_i)$ be given by $\widetilde{f_i}(\mu) := \widetilde{f_i}\mu$ and define analogously the maps $\widetilde{f_i^j} : \mathcal{M}(X_{i+j}) \to \mathcal{M}(X_i)$ by $\widetilde{f_i^j}(\mu) := \widetilde{f_i^j}\mu$. Notice that $\widetilde{f_i^j} = \widetilde{f_i} \circ \widetilde{f_{i+1}} \circ \ldots \circ \widetilde{f_{i+j-1}}$, as it is easy to check.

THEOREM 1. *Under the above conditions, for any $k = 0, 1, \ldots, n-1$:*

(a) $h(f_0^n) = h(f_k^n)$.

(b) *If $\mu_0 \in \mathcal{M}(X_0, f_0^n)$ then $\widetilde{f_k^{n-k}}\mu_0 \in \mathcal{M}(X_k, f_k^n)$.*

(c) *If $\phi : X_0 \to \mathbb{R}$ is a continuous function then $P(f_0^n, \phi) = P(f_k^n, \phi \circ f_0^k)$.*

(d) *If $\mu_0 \in \mathcal{M}(X_0, f_0^n)$ then $h_{\mu_0,A}(f_0^n) = h_{\widetilde{f_k^{n-k}}\mu_0,A}(f_k^n)$ for every increasing sequence A.*

(e) *If f_i are surjective maps for $i = 0, 1, \ldots, n-1$, then $h_A(f_0^n) = h_A(f_k^n)$ for every increasing sequence A.*

The formula $h(f_0 \circ f_1) = h(f_1 \circ f_0)$ (with $n = 2$) was previously proved by Kolyada and Snoha in [6]. Its proof, jointly with that of Theorem 1, is also contained in [1] when $X_0 = X_1 = X$. The argument in this general case is similar so we omit it. On the other hand it was shown in [2] that Theorem 1(e) does not necessarily hold if the maps f_i are not surjective.

We are ready now to speak about cp-maps F. Still a preparatory lemma. The used notation has the obvious meaning: for instance $f_0^n = f_0 \circ f_{\sigma(0)} \circ \ldots \circ f_{\sigma^{n-1}(0)}$.

LEMMA 1. *Let F be a cp-map and let σ be its associated permutation. Let $\mu_0 \in \mathcal{M}(X_0, f_0^n)$ and define $\mu_{\sigma^i(0)} = \widetilde{f_{\sigma^i(0)}^{n-i}}\mu_0$ for $i = 1, 2, \ldots, n-1$. Then the product measure $\mu = \mu_0 \times \mu_1 \times \ldots \times \mu_{n-1}$ is an invariant measure for F.*

PROOF. According to a previous comment we can assume that $\sigma = s$.

By Theorem 1.1 from [9] it suffices to prove that for any $A_i \in \beta(X_i)$ with $i = 0, 1, \ldots, n-1$ we have

$$\mu(F^{-1}(A_0 \times \ldots \times A_{n-1})) = \mu(A_0 \times \ldots . \times A_{n-1}).$$

Since $F^{-1}(A_0 \times \ldots \times A_{n-1}) = f_{n-1}^{-1}(A_{n-1}) \times f_0^{-1}(A_0) \times \ldots . \times f_{n-2}^{-1}(A_{n-2})$, using Theorem 1 we have that

$$\begin{aligned}
\mu(F^{-1}(A_0 \times \ldots \times A_{n-1})) &= \mu(f_{n-1}^{-1}(A_{n-1}) \times f_0^{-1}(A_0) \times \ldots \times f_{n-2}^{-1}(A_{n-2})) \\
&= \mu_0(f_{n-1}^{-1}(A_{n-1}))\mu_1(f_0^{-1}(A_0))\cdot \\
&\qquad \cdots \mu_{n-1}(f_{n-2}^{-1}(A_{n-2})) \\
&= \widetilde{f_{n-1}}\mu_0(A_{n-1})\widetilde{f_0}\mu_1(A_0)\ldots\widetilde{f_{n-2}}\mu_{n-1}(A_{n-2}) \\
&= \mu_{n-1}(A_{n-1})\mu_0(A_0)\ldots\mu_{n-2}(A_{n-2}) \\
&= (\mu_0 \times \ldots \times \mu_{n-1})(A_0 \times \ldots \times A_{n-1}) \\
&= \mu(A_0 \times \ldots \times A_{n-1}),
\end{aligned}$$

which concludes the proof. □

Let F be a cp-map having $\sigma_2 \in \mathcal{S}_2$ defined by $\sigma_2(0) = 1$ and $\sigma_2(1) = 0$ as its associated permutation. In general, given two invariant measures $\mu \in \mathcal{M}(X_0, f_0 \circ f_1)$ and $\nu \in \mathcal{M}(X_1, f_1 \circ f_0)$ it is not true that the product measure $\mu \times \nu$ is an invariant measure of F. For example consider the tent map $f_0 : [0,1] \to [0,1]$ given by $f_0(x) = 1 - |2x - 1|$ and the map $f_1 : [0,1] \to [0,1]$ given by $f_1(x) = \frac{1}{2}x$ and construct the corresponding cp-map F. Consider the invariant measure $\mu \in \mathcal{M}([0,1], f_0 \circ f_1)$ supported on the fixed point 0, that is, $\mu(\{0\}) = 1$, and similarly an invariant measure $\nu \in \mathcal{M}([0,1], f_1 \circ f_0)$ supported on the fixed point $\frac{1}{2}$. Then $\mu \times \nu(\{0\} \times \{\frac{1}{2}\}) = 1$ but $\mu \times \nu(f_1^{-1}(\{\frac{1}{2}\}) \times f_0^{-1}(\{0\}) = \mu(\{1\})\nu(\{0,1\}) = 0$.

Now we are ready to compute some topological and metric invariants of cp-maps. The formula of the topological entropy of a cp-map was proved at the same time by A. Linero and us when $n = 2$ and $X_0 = X_1$ and it previously appeared in [1].

THEOREM 2. *Let F be a cp-map and let σ be its associated permutation. Then for any $k = 0, 1, \ldots, n-1$ we have:*

(a) $h(F) = h(f_k^n)$.

(b) *With the notation of Lemma 1,* $h_\mu(F) = h_{\mu_k}(f_k^n)$.

(c) *Let $\phi_i : X_i \to \mathbb{R}$ be continuous functions for $i = 0, 1, \ldots, n-1$ and define $\phi : X_0 \times X_1 \times \ldots \times X_{n-1} \to \mathbb{R}$ by*

$$\phi(x_0, x_1, \ldots, x_{n-1}) = \sum_{i=0}^{n-1} \phi_i(x_i).$$

Then

$$P(F_\sigma, \phi) = P\left(f^n_{\sigma^k(0)}, \sum_{j=0}^{n-1} \phi_{\sigma^{k-j}(0)} \circ f^j_{\sigma^{k-j}(0)}\right).$$

PROOF. As before there is no loss of generality in assuming $\sigma = s$.

It is easy to check that $F^n = f_0^n \times f_1^n \times \ldots \times f_{n-1}^n$. Applying Theorem 7.10 from [9] and Theorem 1(a) we get that

$$h(F) = \frac{1}{n} h(F^n) = \frac{1}{n} \sum_{i=0}^{n-1} h(f_i^n) = h(f_k^n),$$

so (a) is proved. Similarly, apply Theorem 4.13 and 4.23 from [9] and Theorem 1(d) to get

$$h_\mu(F) = \frac{1}{n} h_\mu(F^n) = \frac{1}{n} \sum_{i=0}^{n-1} h_{\mu_i}(f_i^n) = h_{\mu_k}(f_k^n).$$

and obtain (b).

Finally we prove the formula for topological pressure. In what follows subindexes for maps "ϕ" must be taken mod n.

It is easy to check that

$$\sum_{i=0}^{n-1} \phi \circ F^i(x_0, \ldots, x_{n-1}) = \sum_{i=0}^{n-1} \left(\sum_{j=0}^{n-1} \phi_{i-j} \circ f^j_{i-j} \right)(x_i),$$

and then by Theorem 9.8(i) and (v) from [9] we have that

$$\begin{aligned} P(F, \phi) &= \frac{1}{n} P\left(F^n, \sum_{i=0}^{n-1} \phi \circ F^i\right) \\ &= \frac{1}{n} \sum_{i=0}^{n-1} P\left(f_i^n, \sum_{j=0}^{n-1} \phi_{i-j} \circ f^j_{i-j}\right). \end{aligned}$$

Let $i \in \{1, 2, \ldots, n-1\}$. Applying Theorem 1(c) and using that

$$\int_{X_i} (g \circ f_i^n) d\nu = \int_{X_i} g d\nu$$

for any continuous map $g : X_i \to \mathbb{R}$ and any $\nu \in \mathcal{M}(X_i, f_i^n)$, it follows that

$$\begin{aligned}
P\left(f_0^n, \sum_{j=0}^{n-1} \phi_{-j} \circ f_{-j}^j\right) &= P\left(f_i^n, \sum_{j=0}^{n-1} \phi_{-j} \circ f_{-j}^j \circ f_0^i\right) \\
&= P\left(f_i^n, \sum_{j=0}^{n-i-1} \phi_{-j} \circ f_{-j}^j \circ f_0^i\right. \\
&\qquad \left. + \sum_{j=n-i}^{n-1} \phi_{-j} \circ f_{-j}^{j+i-n} \circ f_i^n\right) \\
&= \sup{}_\nu \left\{ h_\nu(f_i^n) + \int_{X_i} \left(\sum_{j=0}^{n-i-1} \phi_{-j} \circ f_{-j}^j \circ f_0^i\right) d\nu \right. \\
&\qquad \left. + \int_{X_i} \left(\sum_{j=n-i}^{n-1} \phi_{-j} \circ f_{-j}^{j+i-n} \circ f_i^n\right) d\nu \right\} \\
&= P\left(f_i^n, \sum_{j=0}^{n-1} \phi_{i-j} \circ f_{i-j}^j\right),
\end{aligned}$$

which concludes the proof. □

In the case of topological and metric sequence entropies Theorem 2 does not hold. In order to see this consider the sequence $A = \{2^i\}_{i=1}^{\infty}$ and the map $G : S^1 \times S^1 \to S^1 \times S^1$ defined by $G(x, y) = (x, x + y \bmod 1)$ for all $(x, y) \in S^1 \times S^1$. This map preserves the normalized Haar measure m and it can be seen in [7] and [5] that $h_{m,A}(G) = h_A(G) = \log 2$. Define $T((x_1, x_2), (y_1, y_2)) = (G(y_1, y_2), G(x_1, x_2))$. Since $h_A(T^2) = h_A(T)$ and $h_{m,A}(T^2) = h_{m,A}(T)$ (see [1]) and $h_A(G^2 \times G^2) = 2h_A(G^2)$ and $h_{m\times m,A}(G^2 \times G^2) = 2h_{m,A}(G^2)$ (see [5] and [7]) we have that $h_A(G^2) = \log 2$ and $h_A(T) = h_A(T^2) = 2h_A(G^2) = \log 4$ and similarly $h_{m,A}(G^2) < h_{m\times m,A}(T)$.

The case of arbitrary p-maps does not involve any significant difference. First recall a bit of standard notation. If $\{i_1, \ldots, i_l\} \subset \{0, 1, \ldots, n-1\}$ then the cycle $(i_1, \ldots, i_l)$ is the permutation $\tau \in \mathcal{S}_n$ defined by $\tau(i_j) = i_{(j+1)}$ for any j (mod l) and $\tau(i) = i$ elsewhere. Notice that if $l = 1$ then τ is just the identity. As is well known, for any $\sigma \in \mathcal{S}_n$ there are cycles $\sigma_1 = (i_1^1, \ldots, i_{l_1}^1)$, $\sigma_2 = (i_1^2, \ldots, i_{l_2}^2), \ldots, \sigma_k = (i_1^k, \ldots, i_{l_k}^k)$, such that $l_1 + l_2 + \cdots l_k = n$, their components are pairwise disjoint and $\sigma = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_k$.

Let F be a p-map having σ as its associated permutation. We can assume that there are numbers $0 = m_1 < m_2 < \cdots < m_k < m_{k+1} = n$ such that $\sigma = \sigma_1 \circ \cdots \circ \sigma_k$ with $\sigma_j = (m_j, m_j + 1, \ldots, m_{j+1} - 1)$ for any $j = 1, 2, \ldots, k$. Write $l_j = m_{j+1} - m_j$, fix measures $\mu_{m_j} \in \mathcal{M}(X_{m_j}, f_{m_j}^{l_j})$ for any $j = 1, \ldots, k$ and define for each j the measures $\mu_{m_j+i} = f^i_{\sigma^i(m_j)} \mu_{m_j}$, $i = 1, 2, \ldots, l_j - 1$. Put $\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_{n-1}$. Then $\mu \in \mathcal{M}(X_0 \times \ldots \times X_{n-1}, F)$ and we have:

THEOREM 3. *In the above conditions*
(a) $h(F) = \sum_{j=1}^k h\left(f_{m_j}^{l_j}\right)$.
(b) $h_\mu(F) = \sum_{j=1}^k h_{\mu_{m_j}}\left(f_{m_j}^{l_j}\right)$.
(c) *Let* ϕ_i, $i = 0, 1, \ldots, n-1$, *and* ϕ *defined as in Theorem 2. Then* $P(F, \phi) = \sum_{j=1}^k P\left(f_{m_j}^{l_j}, \sum_{s=0}^{l_j-1} \phi_{\sigma^{-s}(m_j)} \circ f^s_{\sigma^{-s}(m_j)}\right)$.

PROOF. Just use Theorems 4.13, 4.23, 7.10 and 9.8 from [9] and Theorem 2. □

An important particular case of p–maps is when $X_i = X$ for any i. Now, for any given family of continuous maps $f_i : X \to X$, $i = 0, 1, \ldots, n-1$, and any $\sigma \in S_n$ one can construct the p-map

$$F_\sigma(x_0, x_1, \ldots, x_{n-1}) := (f_0(x_{\sigma(0)}), f_1(x_{\sigma(1)}), \ldots, f_{n-1}(x_{\sigma(n-1)})).$$

Given two arbitary permutations σ_1 and σ_2 one would suppose that $F_{\sigma_1 \circ \sigma_2}$ and $F_{\sigma_2 \circ \sigma_1}$ share the same invariants as above, but unfortunately this is not the case. For instance, let $f_i : [0,1] \to [0,1]$, $i = 0, 1, 2$ be respectively defined by

$$f_0(x) = \begin{cases} \frac{1}{3}(f(3x) + 1) & \text{if } x \in [0, 1/3] \\ 1/3 & \text{if } x \in [1/3, 1], \end{cases}$$

$$f_1(x) = \begin{cases} \frac{1}{3}(f(3(x - 1/3)) + 2) & \text{if } x \in [1/3, 2/3] \\ 2/3 & \text{if } x \in [0,1] \setminus [1/3, 2/3] \end{cases}$$

and

$$f_2(x) = \begin{cases} \frac{1}{3}(1 - f(3(x - 2/3))) & \text{if } x \in [2/3, 1] \\ 1/3 & \text{if } x \in [0, 2/3] \end{cases}$$

(where $f(x) = 1 - |2x - 1|$ is the tent map). Then $h(f_0 \circ f_2 \circ f_1) = \log 2$ but $h(f_0 \circ f_1 \circ f_2) = 0$ (see [6]). Taking the cycles $\sigma_1 = (0,1)$ and $\sigma_2 = (1,2)$ we get $h(F_{\sigma_1 \circ \sigma_2}) = h(f_0 \circ f_2 \circ f_1) > h(f_0 \circ f_1 \circ f_2) = h(F_{\sigma_2 \circ \sigma_1})$ by Theorem 3.

Concerning metric entropy, assume that the maps $f_i : X \to X$ preserve the same measure μ and let $\sigma_1, \sigma_2 \in S_n$. We conjecture that the formula $h_{\underbrace{\mu \times \ldots \times \mu}_{n}}(F_{\sigma_1 \circ \sigma_2}) = h_{\underbrace{\mu \times \ldots \times \mu}_{n}}(F_{\sigma_2 \circ \sigma_1})$ does not hold and that, for instance, there are three maps preserving the same measure μ and satisfying

$$h_\mu (f_0 \circ f_2 \circ f_1) \neq h_\mu (f_0 \circ f_1 \circ f_2) .$$

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