THE "0 TO ∞–HOMOCLINIC BIFURCATION" OF A CLASS OF DISCRETE TWO-DIMENSIONAL MAPS

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To the memory of Professor György Targonski

Abstract. Based on a previous theoretical result of the same authors the present paper deals with discrete perturbed two-dimensional maps having a semi-hyperbolic fixed point. We give applicable sufficient conditions assuring a particular kind of bifurcation of homoclinic orbits when the perturbative parameter μ varies in a small neighborhood of zero: no homoclinic orbits when μ is on one side of zero, one homoclinic orbit when $\mu = 0$, and infinite homoclinics when μ is on the other side of zero.

1. Introduction

In [3, Theorem 2] we proved a general result which give sufficient conditions assuring that the following discrete perturbed map

$$(1)_\mu \qquad x_{n+1} = f(x_n) + \mu h(x_n, \mu), \qquad x_n \in \mathbb{R}^N,\ n \in \mathbb{Z},\ \mu \in \mathbb{R},\ |\mu| \ll 1,$$

where f and h are C^3–functions of their arguments, has a particular kind of bifurcation of homoclinic orbits when the perturbative parameter μ crosses zero, under the assumption that the unperturbed map $(1)_0$, that is $x_{n+1} = f(x_n)$, has a "critical" orbit $\{q_n\}_{n\in\mathbb{Z}}$ (by "critical" we mean that the jacobian matrices $A_n := f'(q_n)$ are invertible for any $n \neq 0$, but $A_0 := f'(q_0)$ is not invertible), and $(1)_\mu$ has a "semi-hyperbolic" fixed point $p \in \mathbb{R}^N$ (by "semi-hyperbolic" we mean that, for any (small) value of $|\mu|$, $f(p) +$

Received: April 6, 1999.

AMS (1991) subject classification: Primary 58F14, 58F30.

$\mu h(p, \mu) = p$ and $f'(p) + \mu h_x(p, \mu)$ has 1 as simple eigenvalue and all the other eigenvalues have modulus different from 1); in this case we obtain: zero homoclinic orbits "near" $\{q_n\}_{n\in\mathbb{Z}}$ when μ is on one side of $\mu = 0$; one homoclinic orbit, just the $\{q_n\}_{n\in\mathbb{Z}}$, when $\mu = 0$; an infinite number of homoclinic orbits "near" $\{q_n\}_{n\in\mathbb{Z}}$ when μ is on the other side of $\mu = 0$. We called this kind of bifurcation a "0 to ∞ − *bifurcation*". The present paper deals with the study of a particular but remarkable class of discrete two-dimensional maps [5,6] for which the general, but difficult, sufficient conditions become easier and applicable. For shortness reasons, concerning definitions, notations and preliminaries, we refer to [2,3].

2. The two-dimensional case

Consider the following two-dimensional discrete perturbed map

$$(2)_\mu \qquad \begin{cases} x_{n+1} = f_1(x_n, y_n) + \mu h_1(x_n, y_n, \mu) \\ y_{n+1} = f_2(x_n, y_n) + \mu h_2(x_n, y_n, \mu) \end{cases}$$

where $x_n, y_n \in \mathbb{R}$, f_1, f_2, h_1, h_2 are real-valued C^3−functions and assume:

A1) $f_1(0, y) = h_1(0, 0, \mu) = f_2(0, 0) = h_2(0, 0, \mu) = 0$;

A2) $f_{2x}(0, y) = h_{2x}(0, 0, \mu) = h_{2y}(0, 0, \mu) = 0, \quad |f_{1x}(0, 0)| > 1$, $|f_{2y}(0, 0)| = 1, \quad f_{2yy}(0, 0) \neq 0$;

A3) the unperturbed map $(2)_0$, that is

$$(2)_0 \qquad \begin{cases} x_{n+1} = f_1(x_n, y_n) \\ y_{n+1} = f_2(x_n, y_n) \end{cases}$$

has the (critical) orbit $\{\bar{q}_n = (0, \bar{y}_n)\}_{n\in\mathbb{Z}}$ homoclinic (snap-back [4]) to the fixed point $(0, 0)$, such that $\bar{q}_0 \neq (0, 0)$, $f_{1x}(0, \bar{y}_0) \neq 0$, $f_{2y}(0, \bar{y}_0) = 0$, $f_{2yy}(0, \bar{y}_0) \neq 0$, $\bar{q}_n = (0, 0)$ for any $n > \bar{n} > 0$.

From (A1) we get that $(0, 0)$ is a fixed point of $(2)_\mu$ for any μ; from (A1) and (A2), the jacobian matrix $A(\mu)$ of $(2)_\mu$ evaluated at $(0, 0)$ writes

$$A(\mu) = \begin{pmatrix} f_{1x}(0, 0) + \mu h_{1x}(0, 0, \mu) & \mu h_{1y}(0, 0, \mu) \\ 0 & 1 \end{pmatrix};$$

it has the eigenvalues $\lambda_1(\mu) = f_{1x}(0, 0) + \mu h_{1x}(0, 0, \mu)$ and $\lambda_2(\mu) = 1$ (for any μ). It is obvious, from (A2) and the smoothness of h_{1x}, that $|\lambda_1(\mu)| > 1$ for $|\mu|$ small. So, $(0, 0)$ is a "semi-expanding" fixed point of $(2)_\mu$ for $|\mu|$ small.

The variational system of $(2)_0$ evaluated at $\{\bar{q}_n = (0, \bar{y}_n)\}_{n\in\mathbb{Z}}$ is

$$(3)\quad \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = A_n \begin{pmatrix} x_n \\ y_n \end{pmatrix} \quad A_n := \begin{pmatrix} f_{1x}(0,\bar{y}_n) & 0 \\ 0 & f_{2y}(0,\bar{y}_n) \end{pmatrix}, \quad n \in \mathbb{Z}.$$

Hence we have $A_0 := \begin{pmatrix} f_{1x}(0,\bar{y}_0) & 0 \\ 0 & f_{2y}(0,\bar{y}_0) \end{pmatrix}$; the homoclinic orbit $\{\bar{q}_n = (0,\bar{y}_n)\}_{n\in\mathbb{Z}}$ is critical if $f_{1x}(0,\bar{y}_0) = 0$, or $f_{2y}(0,\bar{y}_0) = 0$, or $f_{1x}(0,\bar{y}_0) = f_{2y}(0,\bar{y}_0) = 0$; for reason that will be clear later on we restrict to the case $f_{1x}(0,\bar{y}_0) \neq 0$, $f_{2y}(0,\bar{y}_0) = 0$, as we assumed in (A3). Then $\mathcal{N}A_0$, the kernel of A_0, is exactly span $\{e_2\}$, $e_2 = (0,1)^*$. Observe that $A_0 e_2 = 0$ and $A_0^* = A_0$, so that $e_2^* A_0 = 0$. Since we are studying a two-dimensional semi-expanding case, it is not difficult to see, following the notations of [3], that the projections of the trichotomy (see also [1]) of the linear map (3) are

$$P_\pm = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_\pm = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad R_\pm = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

so that $\mathcal{R}P_+$, the range of P_+ is $\{0\}$, $\mathcal{N}P_- = \mathbb{R}^2$, and (see also [3], the Remark at the end of the proof of Theorem 2)

$$V := \{\eta \in \mathcal{N}P_- : \ A_0\eta \in \mathcal{R}P_+\} = \mathcal{N}A_0 = \mathcal{N}A_0^* = \operatorname{span}\{e_2\}.$$

Moreover, again from (A1) and (A2) we get that (H1) in [3] is satisfied. Then $(2)_\mu$ has, for any small fixed $|\mu|$, a (local) center manifold

$$C_\mu = \{t v_r(\mu) + H(t,\mu) : \ t \in \mathbb{R}, |t|\,\text{small}\} \subset \mathbb{R}^2,$$

where $v_r(\mu)$ is the normalized right eigenvector of $A(\mu)$ associated with the eigenvalue $\lambda_2(\mu) = 1$, and $H(\cdot,\mu) : \operatorname{span}\{v_r(\mu)\} \to W(\mu)$ is a C^3–function such that $H(0,\mu) = H_t(0,\mu) = 0$, $W(\mu)$ being the eigenspace of $\lambda_1(\mu)$ [3, Theorem 1]. From (A1)–(A3) we easily get that $(2)_\mu$ has the center manifold $C_\mu = C_0 = \{(0,y) \in \mathbb{R}^2 : y \in \mathbb{R}\}$. The jacobian matrix $A := A(0) = \begin{pmatrix} f_{1x}(0,0) & 0 \\ 0 & 1 \end{pmatrix}$ has $v_l(0) = v_r(0) = e_2$ as left and right eigenvectors associated with the eigenvalue 1. We get $v_l^*(0)f''(0,0)(v_r(0),v_r(0)) = e_2^* \begin{pmatrix} f_{1yy}(0,0) \\ f_{2yy}(0,0) \end{pmatrix} = f_{2yy}(0,0) \neq 0$; then, $(H2)$ and $(H3)$ in [3] are satisfied. Moreover, there exists $\beta_0 > 0$ such that, for any β with $0 < \beta < \beta_0$, $(2)_\mu$ has a smooth solution $\{q_n^+(\beta,\mu) = (0, y_n^+(\beta,\mu))\}_{n\in\mathbb{Z}} \subset C_\mu$ such that $q_n^+(0,\mu) = 0$ for any $n > \bar{n}$ with $\sup_{n\geq 1} |q_n^+(\beta,\mu) - \bar{q}_n| \to 0$ as $|\beta| + |\mu| \to 0$. We show now how Theorem 2 in [3] writes for the case here considered. Let

$$T_n = \begin{cases} \mathbb{I}, & n = 0, 1 \\ A_{n-1}T_{n-1}, & n > 1 \\ A_n^{-1}T_{n+1}, & n < 0 \end{cases}$$

be the pseudo-fundamental solution [2] of the linear variational system (3), and

$$\psi_l^{(k)} = \begin{cases} (T_{l+1}^{-1})^*\psi^{(k)}, & l \geq 0 \\ (T_{l+1}^{-1})^*A_0^*\psi^{(k)}, & l < 0. \end{cases}$$

In the present case we have : $k = 1$, $\psi^{(k)} = \phi_k = e_2$, $\psi_l^{(k)} = (T_{l+1}^{-1})^*A_0^*e_2 = 0$, $l < 0$, $\psi_l^{(k)} = (T_{l+1}^{-1})^*e_2$, $l \geq 0$; so, a, b, c of Theorem 2 in [3] take now the simple form (recall that $A_0e_2 = 0, e_2^*A_0 = 0$)

$$\begin{aligned} a &= -e_2^*[\frac{\partial q_1^+}{\partial \mu}(0,0) - h(0, \bar{y}_0, 0)], \\ b &= e_2^* f''(0, \bar{y}_0)(e_2, e_2), \\ c &= -e_2^* \frac{\partial q_1^+}{\partial \beta}(0,0). \end{aligned} \tag{4}$$

We have now the unique quadratic form, say $b\lambda_1^2 + c\lambda_2^2$, and the unique equation $b\lambda_1^2 + c\lambda_2^2 = \pm a$. Then we get the following simplified version of Theorem 2 in [3]:

PROPOSITION 1. *Let a, b, c be as in* (4). *Assume that* (A1)–(A3) *hold and that b and c have the same sign. Then there exists $\mu_0 > 0$ such that, for any μ with $|\mu| < \mu_0$ and $\mu ab < 0$, the map $(2)_\mu$ has an infinite number of solutions $\{q_n(s, \mu)\}_{n\in\mathbb{Z}}$, $s \in \mathbb{R}$, homoclinic to $(0, 0)$ which are the unique homoclinic solutions satisfying*

$$\lim_{\mu\to 0} \sup_{n\in\mathbb{Z}} |q_n(s, \mu) - \bar{q}_n| = 0 \tag{5}$$

in a neighborhood of $\mu = 0$. In particular $(2)_\mu$ does not have homoclinic orbits satisfying (5) *when μ is small and $\mu ab > 0$.*

If (5) is satisfied we say that $\{q_n(s, \mu)\}_{n\in\mathbb{Z}}$ is "near" $\{\bar{q}_n\}_{n\in\mathbb{Z}}$ as it was stated in the Introduction.

We apply now Proposition 1 to the map $(2)_\mu$. To this end we have to compute explicitly a, b, c. As regards a, observe that $q_n^+(0, \mu) = 0$, then $\frac{\partial q_n^+}{\partial \mu}(0, 0) = 0$, for any $n > \bar{n}$, and the component $y_n^+(0, \mu) = 0$ satisfies

$$y_{n+1}^{+}(0,\mu) = f_2(0, y_n^{+}(0,\mu)) + \mu h_2(0, y_n^{+}(0,\mu),\mu)$$

from which it follows

$$\begin{cases} \frac{\partial y_{n+1}^{+}}{\partial \mu}(0,0) = f_{2y}(0,\bar{y}_n)\frac{\partial y_n^{+}}{\partial \mu}(0,0) + h_2(0,\bar{y}_n,0) \\ \frac{\partial y_{\bar{n}+1}^{+}}{\partial \mu} = 0. \end{cases}$$

Then

$$\frac{\partial y_{\bar{n}}^{+}}{\partial \mu}(0,0) = \left[\frac{\partial y_{\bar{n}+1}^{+}}{\partial \mu}(0,0) - h_2(0,\bar{y}_{\bar{n}},0)\right] f_{2y}(0,\bar{y}_{\bar{n}})^{-1} = -\frac{h_2(0,\bar{y}_{\bar{n}},0)}{f_{2y}(0,\bar{y}_{\bar{n}})};$$

then, using the induction

$$\begin{aligned} &\frac{\partial y_1^{+}}{\partial \mu}(0,0) \\ &= -\left\{\frac{h_2(0,\bar{y}_1,0)}{f_{2y}(0,\bar{y}_1)} + \frac{h_2(0,\bar{y}_2,0)}{f_{2y}(0,\bar{y}_1)f_{2y}(0,\bar{y}_2)} + \ldots + \frac{h_2(0,\bar{y}_{\bar{n}},0)}{f_{2y}(0,\bar{y}_1)\cdot\ldots\cdot f_{2y}(0,\bar{y}_{\bar{n}})}\right\} \\ &= -\sum_{k=1}^{\bar{n}} \frac{h_2(0,\bar{y}_k,0)}{f_{2y}(0,\bar{y}_1)\cdot\ldots\cdot f_{2y}(0,\bar{y}_k)}. \end{aligned}$$

Thus,

$$\begin{aligned} a &= -(0,1)\begin{pmatrix} -h_1(0,\bar{y}_0,0) \\ \frac{\partial y_1^{+}}{\partial \mu}(0,0) - h_2(0,\bar{y}_0,0) \end{pmatrix} \\ &= \sum_{k=1}^{\bar{n}} \frac{h_2(0,\bar{y}_k,0)}{f_{2y}(0,\bar{y}_1)\cdot\ldots\cdot f_{2y}(0,\bar{y}_k)} + h_2(0,\bar{y}_0,0). \end{aligned} \tag{6}$$

REMARK. Observe that $a \neq 0$ is a generic condition and that a depends only on the perturbative term $h_2(0,y,0)$ and on a finite number of points, usually very small in concrete examples, $\bar{y}_0,\ldots\bar{y}_{\bar{n}}$, of the unperturbed snap-back orbit. Note also that $f_{2y}(0,\bar{y}_n) \neq 0$ for any $n \neq 0$ since A_n is invertible for any $n \neq 0$.

What concerns b an easy computation gives:

$$b = f_{2yy}(0, \bar{y}_0) \tag{7}$$

and then $b \neq 0$ because of (A3). To compute c we have to evaluate $\frac{\partial q_1^+}{\partial \beta}(0,0) = \begin{pmatrix} 0 \\ \frac{\partial y_1^+}{\partial \beta}(0,0) \end{pmatrix}$. We know that $y_n^+(\beta, 0)$ satisfies $y_{n+1}^+(\beta, 0) = f_2(0, y_n^+(\beta, 0))$; so $\frac{\partial y_1^+}{\partial \beta}(0,0)$ satisfies the linear system:

$$u_{n+1} = f_{2y}(0, \bar{y}_n)u_n$$

with the *initial condition* $u_{\bar{n}+1} = -1$ (see the proof of Proposition 2 in [3]). So

$$\frac{\partial y_{\bar{n}}^+}{\partial \beta}(0,0) = -\frac{1}{f_{2y}(0, \bar{y}_{\bar{n}})},$$

and, using the induction,

$$\frac{\partial y_1^+}{\partial \beta}(0,0) = -\frac{1}{f_{2y}(0, \bar{y}_{\bar{n}}) \cdot \ldots \cdot f_{2y}(0, \bar{y}_1)};$$

from (4) we get

$$c = -(0, 1)\begin{pmatrix} 0 \\ \frac{\partial y_1^+}{\partial \beta}(0,0) \end{pmatrix} = \frac{1}{f_{2y}(0, \bar{y}_{\bar{n}}) \cdot \ldots \cdot f_{2y}(0, \bar{y}_1)}. \tag{8}$$

REMARK. The reason why we do not consider the possibility that $f_{1x}(0, \bar{y}_0) = 0$ (see (A3)) is that this fact implies $\mathcal{N}A_0 = \text{span}\{e_1\}$, $e_1 = (1,0)^*$. Thus $c = 0$ and the condition of Proposition 1 concerning the signs of a, b, c could not be satisfied.

3. An example

It is not difficult to find discrete perturbed systems which satisfy the assumptions (A1), (A2), (A3) of the previous section 2. Consider for example

$$(9)_\mu \quad \begin{cases} x_{n+1} = x_n(x_n y_n + 2) + \mu[x_n + \sin(\mu y_n)] \\ y_{n+1} = 12(x_n^4 + y_n^4) + 11(x_n^2 + y_n^2) - y_n(24y_n^2 - 1) + \mu(\mu x_n^2 + y_n^2) \end{cases}$$

so that, following the notations of section 2, we have:

$$f_1(x,y) = x(xy+2), \quad h_1(x,y,\mu) = x + \sin(\mu y),$$

$$f_2(x,y) = 12(x^4+y^4) + 11(x^2+y^2) - y(24y^2-1), \quad h_2(x,y,\mu) = \mu x^2 + y^2.$$

Simple computations show that (A1) and (A2) of section 2 are satisfied; in particular we get $f_{1x}(0,0) = 2$, $f_{2y}(0,0) = 1$, $f_{2yy}(0,0) = 22 > 0$. Concerning (A3) we have that $(0,0)$ is a fixed point for $(9)_\mu$ for any μ; moreover, the jacobian matrix $A(\mu)$ of $(9)_\mu$ at $(0,0)$ is $A(\mu) = \begin{pmatrix} 2+\mu & \mu^2 \\ 0 & 1 \end{pmatrix}$; then $(0,0)$ is a semi-expanding fixed point of $(9)_\mu$ for small μ near $\mu = 0$. The unperturbed system $(9)_0$, that is

$$(9)_0 \qquad \begin{cases} x_{n+1} = x_n(x_n y_n + 2) \\ y_{n+1} = 12(x_n^4 + y_n^4) + 11(x_n^2 + y_n^2) - y_n(24y_n^2 - 1) \end{cases}$$

has the orbit $\{\bar{q}_n = (0,\bar{y}_n)\}_{n\in\mathbb{Z}}$ homoclinic (snap-back) to $(0,0)$ if we set for $\{\bar{y}_n\}_{n\in\mathbb{Z}}$ the homoclinic (snap-back) orbit of the scalar map $y_{n+1} = f_2(0,y_n) = 12y_n^4 - 24y_n^3 + 11y_n^2 + y_n$ starting from $\bar{y}_0 = 1/2$. Then we have : $\bar{q}_0 = (0,1/2) \neq 0$, $\bar{q}_1 = (0,1)$, $\bar{q}_2 = (0,0)$, $\bar{q}_n = (0,0)$ for any $n > 1$; so $\bar{n} = 1$. We easily get $f_{1x}(0,1/2) = 2 \neq 0$, $f_{2y}(0,1/2) = 0$, $f_{2yy}(0,1/2) = -14 < 0$; then $(A3)$ is satisfied too. Applying Proposition 2 to the present case we see that $b = f_{2yy}(0,1/2) = -14 < 0$, $c = 1/f_{2y}(0,1) = -1 < 0$; hence b and c have the same sign. Moreover,

$$a = \frac{h_2(0,1,0)}{f_{2y}(0,1)} + h_2(0,1/2,0) = -3/4 < 0.$$

Then $ab > 0$ and the original perturbed map $(9)_\mu$ has an infinite number of homoclinic orbits near $\{\bar{q}_n\}_{n\in\mathbb{Z}}$ for any small fixed μ such that $\mu ab < 0$, i.e. for any small fixed $\mu < 0$.

REMARK. Note that the infinite orbits assured by the previous theory generally belong to the plane, while the unperturbed orbit $\{\bar{q}_n\}_{n\in\mathbb{Z}}$ belongs to the y-axis.

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