NEW PHENOMENA RELATED TO THE PRESENCE OF FOCAL POINTS IN TWO-DIMENSIONAL MAPS

GIAN-ITALO BISCHI, LAURA GARDINI AND CHRISTIAN MIRA

To the memory of Professor György Targonski

Abstract. In this paper we consider two-dimensional maps, defined in the whole plane, with the property of mapping a whole curve δ into a single point Q. We relate such property to the fact that at least one inverse map exists with a denominator which can vanish, and assumes the form $0/0$ in Q. This allows us to apply the properties of focal points and prefocal curves in order to explain the dynamic phenomena observed. By an example we show that the iteration of such maps may generate discrete dynamical systems with peculiar attracting sets, characterized by the presence of "knots", where infinitely many phase curves shrink into a single point.

1. Introduction

The concepts of *focal point* and *prefocal curve* have been recently defined in order to characterize geometric and dynamic properties, together with some new kinds of bifurcations, peculiar of maps with a denominator which vanishes in a subset of the phase space (see [4] and references therein). These concepts have been introduced for the first time during the ECIT (European Conference on Iteration Theory) held in Urbino in September 1996, by the authors of this paper in order to describe some global bifurcations, peculiar of maps with denominator, which cause the creation of particular structures of the basin boundaries (see [9, 3, 6]). However, some observations on the role of the vanishing denominator and of the points in which a map (or its

inverses) assumes the form 0/0 were already present in [11] (see also [10], page 25).

Roughly speaking, a *prefocal curve* is a set of points for which at least one inverse exists which maps (or "focalizes") the whole set into a single point, called *focal point*.

However, the theory of focal points and prefocal curves may also be useful to understand some properties observed in maps defined in the whole plane (for example polynomial maps) due to the presence of a vanishing denominator in at least one inverse map. Such maps may have the property that among the points at which the Jacobian vanishes there exists a curve δ which is mapped into one single point Q. Such property, that has been already observed in polynomial maps by some authors (see e.g. [5] or [10] page 197 and page 228) can be related to the fact that the curve δ is a prefocal curve of at least one inverse, the point Q being the corresponding focal point. The existence of focal points of an inverse map can cause the appearance of particular attracting sets because a focal point of an inverse map may behave like a "knot" where infinitely many invariant curves of an attracting set shrink into a single point (an example is given below).

The plan of this paper is the following. In Section 2 we recall the definitions and some properties of focal points and prefocal sets. In Section 3 we propose an example of an invertible quadratic polynomial map of the plane whose inverse has a focal point, and we show how this fact can be used to explain some particular dynamical properties of the map.

2. Focal points and prefocal set

In this section we recall some definitions and some generic properties concerning two-dimensional maps which are not defined in the whole plane, due to the presence of a denominator which can vanish (see [4] for a more complete treatment). Let us consider a map $(x, y) \rightarrow (x', y') = T(x, y)$ of the form

$$(1) \qquad\qquad T : \begin{cases} x' = F(x, y) \\ y' = G(x, y) \end{cases}$$

where x and y are real variables and at least one of the components has the form of a fractional rational function, i.e.

$$(2) \qquad F(x, y) = \frac{N_1(x, y)}{D_1(x, y)} \quad \text{and/or} \quad G(x, y) = \frac{N_2(x, y)}{D_2(x, y)}$$

where the functions $N_i(x, y)$ and $D_i(x, y)$, $i = 1, 2$, are defined in the whole plane $\mathbb{R}^2$. The set of non definition δ_s is the locus of points in which at least one denominator vanishes:

$$(3) \qquad \delta_s = \left\{ (x, y) \in \mathbb{R}^2 \mid D(x, y) = 0 \right\}.$$

As shown in [4], the image $T(\gamma)$ of a bounded arc γ crossing δ_s in a generic point is made up of two disjoint unbounded arcs. In fact, let γ be a simple arc transverse to δ_s represented by the parametric equations

$$(4) \qquad \gamma(\tau): \begin{cases} x(\tau) = x_0 + \xi_1 \tau + \xi_2 \tau^2 + \cdots \\ y(\tau) = y_0 + \eta_1 \tau + \eta_2 \tau^2 + \cdots \end{cases} \quad \tau \neq 0.$$

On taking the image $T(\gamma)$, we assume that the arc γ is deprived of the point (x_0, y_0) in which it intersects δ_s, so that the portion of γ in such a neighborhood of (x_0, y_0) can be seen as the union of two disjoint pieces, say $\gamma = \gamma_- \cup \gamma_+$, where γ- and γ_+ denote the portions of γ located on opposite sides with respect to the singular curve δ_s, obtained from (4) with $\tau < 0$ and $\tau > 0$ respectively. According to the definition (4) of δ_s, at least one denominator vanishes in (x_0, y_0). Without loss of generality we assume that $D_1(x_0, y_0) = 0$. Since, in a generic situation, $N_1(x_0, y_0) \neq 0$, we have

$$(5) \qquad \lim_{\tau \to 0} T(\gamma(\tau)) = \lim_{\tau \to 0} \left(\frac{N_1(\gamma(\tau))}{D_1(\gamma(\tau))}, G(\gamma(\tau)) \right) = (\infty, G(x_0, y_0)).$$

This means that the image $T(\gamma)$ is made up of two disjoint unbounded arcs asymptotic to the line of equation $y = G(x_0, y_0)$. A different situation may occur if the point $(x_0, y_0) \in \delta_s$ is such that not only the denominator but also the corresponding numerator vanishes in it, for example $D_1(x_0, y_0) = N_1(x_0, y_0) = 0$. In this case, in the limit (5) the first component assumes the form $0/0$. This implies that, in contrast with (5), the limit may give a finite value, so that the image $T(\gamma)$ is a bounded arc crossing the line $y = G(x_0, y_0)$ in the point $(x, G(x_0, y_0))$, where

$$(6) \qquad x = \lim_{\tau \to 0} \frac{N_1(\gamma(\tau))}{D_1(\gamma(\tau))}.$$

It is plain that if also $G(x_0, y_0)$ assumes the form $0/0$, then a similar limit should be computed for the y component as well. This leads us to the following definition:

DEFINITION 1. Consider the map (6). A point $Q = (x_0, y_0)$ is a focal point if at least one component of the map T takes the form $0/0$ in Q and there

exist smooth simple arcs $\gamma(\tau)$, with $\gamma(0) = Q$, such that $\lim_{\tau \to 0} T(\gamma(\tau))$ is finite. The set of all such finite values, obtained by taking different arcs $\gamma(\tau)$ through Q, is the prefocal set δ_Q.

Of course, the value of $\lim_{\tau \to 0} T(\gamma(\tau))$ depends on the arc γ. More exactly, in the generic case it is possible to define a one-to-one correspondence

$$(7) \qquad\qquad m \leftrightarrow p = (x(m), y(m))$$

between the slope $m = \eta_1/\xi_1$ of the arc γ in Q and the point $p \in \delta_Q$ in which the image $T(\gamma)$ crosses the prefocal curve δ_Q (see [4] or [2] for more details).

The reasons for the choice of the terms focal and prefocal is related to the particular behavior of the inverse (or the inverses) of T. In fact, at least one inverse must exist, say T_Q^{-1}, which maps points arbitrarily close to the prefocal curve δ_Q into points which are arbitrarily close to the corresponding focal point Q. We say that T_Q^{-1} "focalizes" the whole curve δ_Q into a the focal Q, or, more concisely, $T_Q^{-1}(\delta_Q) = Q$. For each arc ω crossing δ_Q in a point $p \in \delta_Q$ the preimage $T_Q^{-1}(\omega)$ is an arc through the focal point Q whose slope can be obtained by inverting the one-to-one correspondence (7). This implies that if different arcs are considered, all crossing δ_Q in the same point p, then these are mapped by each inverse T_Q^{-1} into different arcs through Q, all with the same tangent. This property of a prefocal curve reminds the properties of a curve in which the Jacobian vanishes (see [8, 7, 10]). Indeed, the Jacobian determinant of T_Q^{-1} must necessarily vanish in the points of δ_Q. In fact, since all the points of the curve δ_Q are mapped by T_Q^{-1} into the single point Q, T_Q^{-1} cannot be locally invertible in the points of δ_Q, being it a many-to-one map, and this implies that its Jacobian cannot be different from zero in the points of δ_Q.

This suggests a different method to find the prefocal curves and the corresponding focal points, provided that the inverse(s) of a map T is (are) explicitly known. For each inverse the locus of points at which its Jacobian determinant vanishes is found. If this locus of points includes a curve δ such that an inverse exists that maps δ into a single point Q, then δ is a prefocal curve for the map T and Q is an associated focal point.

3. A polynomial invertible map whose inverse has a focal point

Let us consider the polynomial map

$$(8) \qquad\qquad T : \begin{cases} x' = axy - ax^2 - b \\ y' = a(y - x) \end{cases}$$

with a, b real parameters. This map is defined in the whole plane. Its Jacobian matrix is

(9)
$$DT(x,y) = \begin{bmatrix} ay - 2ax & ax \\ -a & a \end{bmatrix}$$

hence the Jacobian

(10)
$$\det DT(x,y) = a^2 (y - x)$$

vanishes on the line of equation $y = x$. The image of this line by the map T is a single point, being

(11)
$$T(x,x) = (-b, 0) \qquad \text{for every } x.$$

In order to understand this particular behavior of the map (11) let us consider the inverse map T^{-1}. Indeed, the map (11) can be easily inverted by expressing the variables x and y in terms of x' and y'. This gives a unique map $T^{-1} : (x', y') \to (x, y)$, where T^{-1} is the fractional map

(12)
$$T^{-1} : \begin{cases} x = \frac{x'+b}{y'} \\ y = \frac{x'+b}{y'} + \frac{y'}{a}. \end{cases}$$

This map has a denominator which vanishes along the line $y' = 0$, i.e. the set of non definition is given by

(13)
$$\delta_s = \{(x', y') \,|\, y' = 0\}.$$

A unique point of δ_s exists in which also the numerator vanishes, given by $Q = (-b, 0)$. It is easy to realize that the point Q is a focal point of T^{-1}, with prefocal line δ_Q of equation $y = x$. In fact, let γ be a smooth simple arc through Q defined, in the (x', y') plane, by the parametric equations

$$\gamma(\tau) : \begin{cases} x(\tau) = -b + \xi_1 \tau + \dots \\ y(\tau) = \eta_1 \tau + \dots \end{cases} \qquad \tau \neq 0$$

and let us compute the limit

$$\lim_{\tau \to 0} T^{-1}(\gamma(\tau)) = \lim_{\tau \to 0} \left(\frac{\xi_1 \tau}{\eta_1 \tau}, \frac{\xi_1 \tau}{\eta_1 \tau} + \frac{\eta_1 \tau}{a} \right) = \left(\frac{1}{m}, \frac{1}{m} \right)$$

where $m = \eta_1 / \xi_1$ is the slope of γ in Q. Hence the image of γ by T^{-1} is a bounded curve through δ_Q, and the one-to-one correspondence between the slope m of γ in Q and the point $p \in \delta_Q$ is given by

(14)
$$m \leftrightarrow p = (1/m, 1/m).$$

If we now consider the map T, the properties stated above may be expressed by saying that T maps the whole line $y = x$ into the point $Q = (-b, 0)$, and any arc ω through a point $(\overline{x}, \overline{x}) \in \delta_Q$ is mapped into an arc $T(\omega)$ through which has slope $m = 1/\overline{x}$ in Q ($\overline{x} = 0$ implies $m = \infty$, i.e. $T(\omega)$ has a vertical tangent in Q).

These properties of the map (14) may have important consequences on the dynamic behavior of the recurrence $(x_{n+1}, y_{n+1}) = T(x_n, y_n)$. In particular, these properties are responsible for some very particular structures of the attractors and their basins of attraction.

In order to illustrate this, we consider a particular set of parameters, namely $a = -0.5$ and $b = 2.6$, at which the asymptotic dynamics are characterized by a chaotic attractor A (see fig.1). The influence of the point Q on the shape of the attractor A is rather evident in fig.1. In fact, as some portions of A cross the line $y = x$, their images, which are included in the same attractor, must necessarily belong to phase curves through Q. In fact, every trajectory included inside the attractor is conveyed through Q whenever it crosses the line $y = x$. The number of phase curves crossing through Q become higher and higher as the parameter b is increased. In fact, as b increases, the point indicated by P in fig.1 moves to the right, approaching the line δ_Q, and consequently its image $P' = T(P)$ moves towards Q. At $b = 2.8$ the point P has already crossed the line δ_Q and consequently another phase curve belonging to A passes through Q (see fig.2). We can also notice that the new intersection between the attractor A and the line δ_Q, created by the displacement of the point P, occurs approximately at $x = 2$, and this implies that, according to the correspondence (14), the slope of the new arc of A through Q has slope $m = 1/2$ approximately. In fig.2 the point $P'' = T(P')$ is also evidenced, because it is moving towards δ_Q as the parameter b increases and it is going to intersect δ_Q in a point characterized by a negative value of x. Hence we expect that if b is further increased a new arc of A will cross through Q with a negative slope. In fact a new arc crossing through Q with negative slope can be seen in fig.3, obtained for $b = 2.85$.

As b is further increased, the chaotic attractor A includes more and more new arcs through Q, thus assuming a very peculiar shape, characterized by a sort of "knot" of infinitely many curves that shrink into the point Q, the focal point of the inverse map (see fig.4). However, the "structure" of such an attracting set must be more complex of what we see at a first glance because all the images of the knot-point, that belong to the attracting set, behave in the same way.

In fig.4, obtained with $b = 2.99$, the basin of attraction of A is also shown, represented by the white region, whereas the grey region represents the basin of infinity, i.e. the set of points that generate diverging trajectories. Notice that the presence of the "focalized" line δ_Q also has a remarkable

effect on the shape of the basin of A. In fact, since the point Q is inside the basin of A, also the whole line δ_Q must belong to the same basin, as well as its preimages of any rank. This implies that the basin of A cannot be a bounded set because it must necessarily include a whole line and its preimages, which are asymptotes of the basin boundary.
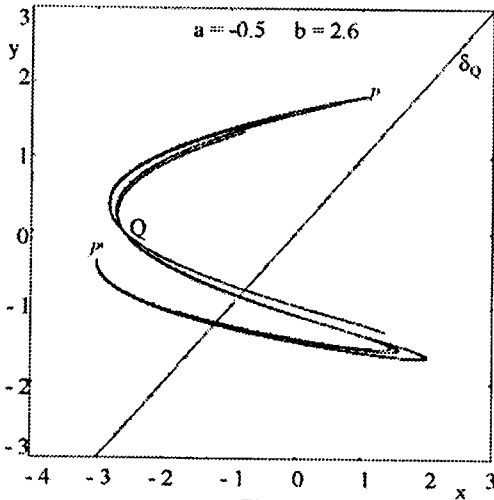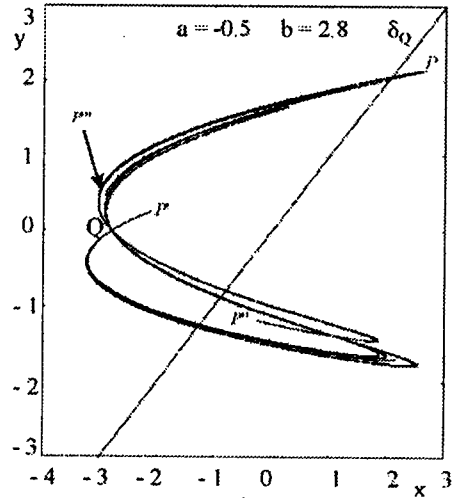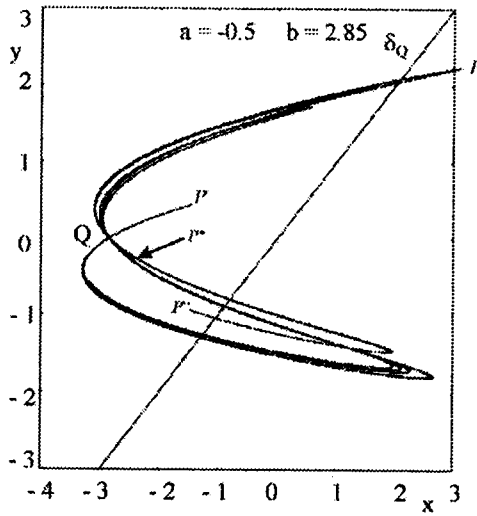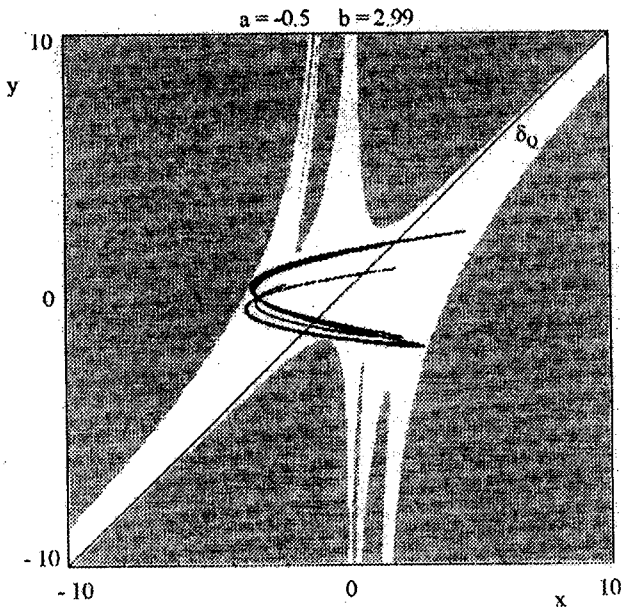

Fig.1

Fig.2

Fig.3

$$a = -0.5 \quad b = 2.99$$

Fig.4

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ISTITUTO DI SCIENZE ECONOMICHE
UNIVERSITÀ DI URBINO
PIAZZA DELLA REPUBBLICA 3
I-61029 URBINO
ITALY
e-mail: bischi@econ.uniurb.it

ISTITUTO DI SCIENZE ECONOMICHE
UNIVERSITÀ DI URBINO
PIAZZA DELLA REPUBBLICA 3
I-61029 URBINO
ITALY
e-mail: gardini@econ.uniurb.it

GESNLA–DGE–INSA
COMPLEXE SCIENTIFIQUE DE RANGUEIL
AVENUE DE RANGUEIL
F-31077 TOULOUSE CEDEX 4
FRANCE
e-mail: mira@dge.insa-tlse.fr