# ON CONJUGACY OF DISJOINT ITERATION GROUPS ON THE UNIT CIRCLE

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*To the memory of Professor György Targonski*

**Abstract.** The aim of this paper is to give a necessary and sufficient condition for conjugacy of some iteration groups $\mathcal{F} = \{F^t : \mathrm{S} \mapsto \mathrm{S}, \quad t \in \mathbb{R}\}$ and $\mathcal{G} = \{G^t : \mathrm{S} \mapsto \mathrm{S}, \quad t \in \mathbb{R}\}$ defined on the unit circle S. Our basic assumption is that they are non-singular, that is at least one element of $\mathcal{F}$ and $\mathcal{G}$ has no periodic point. Moreover, under some further restrictions, we determine all orientation-preserving homeomorphisms $\Gamma : \mathrm{S} \mapsto \mathrm{S}$ such that

$$\Gamma \circ F^t = G^t \circ \Gamma, \qquad t \in \mathbb{R}.$$

Let $\mathrm{S} := \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle with positive orientation. A family $\mathcal{F} = \{F^t : \mathrm{S} \mapsto \mathrm{S}, \quad t \in \mathbb{R}\}$ of homeomorphisms such that

$$F^s \circ F^t = F^{s+t}, \qquad s, t \in \mathbb{R}$$

is said to be a flow or an iteration group.

Definition 1(see [1] and also [6]). An iteration group $\mathcal{F}$ such that for every $t \in \mathbb{R}$, $F^t = \mathrm{id}$ if $F^t$ has a fixed point is said to be disjoint.

Definition 2(see [6]). Let $\mathcal{F} = \{F^t : \mathrm{S} \mapsto \mathrm{S}, \quad t \in \mathbb{R}\}$ and $\mathcal{G} = \{G^t : \mathrm{S} \mapsto \mathrm{S}, \quad t \in \mathbb{R}\}$ be iteration groups. We will say that $\mathcal{F}$ and $\mathcal{G}$ are conjugate if there exists a homeomorphism $\Gamma : \mathrm{S} \mapsto \mathrm{S}$ such that

$$\Gamma \circ F^t = G^t \circ \Gamma, \qquad t \in \mathbb{R}. \tag{1}$$

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The problem of conjugacy of disjoint iteration groups defined on open real intervals has been examined by M. C. Zdun (see [6]). In this paper we give a necessary and sufficient condition for conjugacy of some iteration groups on the unit circle. Moreover, under some further restrictions, we determine all orientation–preserving homeomorphisms $\Gamma : \mathrm{S} \mapsto \mathrm{S}$ fulfilling for these iteration groups condition (1).

Throughout this note the closure of the set $A$ will be denoted by $\mathrm{cl}\ A$ and we write $A^{\mathrm{d}}$ for the set of all cluster points of $A$. $\sim p$ stands for the negation of $p$.

Let $\tilde{\pi} : \mathbb{R} \ni t \mapsto \mathrm{e}^{2\pi i t} \in \mathrm{S}$ and $\pi := \tilde{\pi}_{|[0,\,1)}$. For all $v,\ w,\ z \in \mathrm{S}$, there exist unique $t_1,\ t_2 \in [0,\ 1)$ such that $w\pi(t_1) = z$ and $w\pi(t_2) = v$. Define

$$v \prec w \prec z \qquad \text{if and only if} \quad 0 < t_1 < t_2$$

and

$$v \preceq w \preceq z \qquad \text{if and only if} \quad t_1 \leq t_2 \quad \text{or} \quad t_2 = 0$$

(see [1]).

If $v,\ z \in \mathrm{S}$, $v \neq z$, then there exist $t_v,\ t_z \in \mathbb{R}$ such that $t_v < t_z < t_v + 1$ and $v = \tilde{\pi}(t_v)$, $z = \tilde{\pi}(t_z)$. Put

$$\overrightarrow{(v,\ z)} := \{\tilde{\pi}(t), \quad t \in (t_v,\ t_z)\}.$$

LEMMA 1 (see [3]). *Let* $v,\ w,\ z \in \mathrm{S}$. $v \prec w \prec z$ *if and only if* $w \in \overrightarrow{(v,\ z)}$. *Moreover, if* $v \prec w \prec z$, *then* $v \neq w$, $w \neq z$, $v \neq z$.

LEMMA 2 (see [3]). *For every* $v,\ w,\ z \in \mathrm{S}$ *the following conditions are equivalent:*

(i) $v \prec w \prec z$,

(ii) $w \prec z \prec v$,

(iii) $z \prec v \prec w$.

LEMMA 3 (see [3]). *For every* $v,\ w,\ z \in \mathrm{S}$ *the following conditions are equivalent:*

(i) $\sim (v \prec w \prec z)$,

(ii) $v = w$ *or* $w = z$ *or* $v = z$ *or* $z \prec w \prec v$,

(iii) $z \preceq w \preceq v$,

(iv) $w \preceq v \preceq z$,

(v) $v \preceq z \preceq w$.

Let $A \subset \mathrm{S}$ be such that card$A \geq 3$. We say that the function $\varphi : A \mapsto \mathrm{S}$ is increasing (respectively, strictly increasing, decreasing, strictly decreasing) if for every $v,\ w,\ z \in A$ such that $v \prec w \prec z$ we have $\varphi(v) \preceq \varphi(w) \preceq \varphi(z)$ (respectively, $\varphi(v) \prec \varphi(w) \prec \varphi(z)$, $\varphi(z) \preceq \varphi(w) \preceq \varphi(v)$, $\varphi(z) \prec \varphi(w) \prec \varphi(v)$). According to Lemma 1, the map $\varphi$ is strictly increasing (respectively, strictly decreasing) if $w \in \overrightarrow{(v,\ z)}$ yields $\varphi(w) \in \overrightarrow{(\varphi(v),\ \varphi(z))}$ (respectively, $\varphi(w) \in \overrightarrow{(\varphi(z),\ \varphi(v))}$). The function $\varphi$ is said to be strictly monotonic if $\varphi$ is strictly increasing or strictly decreasing.

A subset $A \subset \mathrm{S}$ is said to be an open arc if $A = \overrightarrow{(v,\ z)}$ for some $v,\ z \in \mathrm{S}$, $v \neq z$. Every open arc is non-empty, different from S, open and connected (see also [1] and [3]).

It is known (see for instance [4]) that for every homeomorphism $F : \mathrm{S} \mapsto \mathrm{S}$ there exists a homeomorphism $f : \mathbb{R} \mapsto \mathbb{R}$ such that

$$F \circ \tilde{\pi} = \tilde{\pi} \circ f$$

and

$$f(x+1) = f(x) + 1, \quad \text{if } f \text{ is strictly increasing}$$

and

$$f(x+1) = f(x) - 1, \quad \text{if } f \text{ is strictly decreasing.}$$

We will say that the function $f$ represents the homeomorphism $F$. If $f$ is strictly increasing we will say that the homeomorphism $F$ preserves orientation.

If $F : \mathrm{S} \mapsto \mathrm{S}$ is an orientation-preserving homeomorphism represented by a function $f$ then the number $\alpha(F) \in [0,\ 1)$ defined by

$$\alpha(F) := \lim_{n\to\infty} \frac{f^n(x)}{n} (\mathrm{mod}\ 1), \qquad x \in \mathbb{R}$$

is said to be the rotation number of $F$. This limit always exists and does not depend on $x$ and $f$. Moreover, $\alpha(F)$ is rational if and only if $F^n(z_0) = z_0$ for a $z_0 \in \mathrm{S}$ and an $n \in \mathbb{Z} \setminus \{0\}$, which means that $z_0$ is a periodic point of $F$.

Definition 3. An iteration group $\mathcal{F}$ is called non-singular if at least one element of $\mathcal{F}$ has no periodic point.

Of course, $\mathcal{F}$ is non-singular if and only if there exists an element of $\mathcal{F}$ with an irrational rotation number. Such iteration groups have been investigated in [1]. Without loss of generality we may assume that the above-mentioned function from $\mathcal{F} = \{F^t : \mathrm{S} \mapsto \mathrm{S},\ t \in \mathbb{R}\}$ is $F^1$, that is $\alpha(F^1) \notin \mathbb{Q}$.

From Remark 2 in [5] it follows that every $F^t \in \mathcal{F}$ and $G^t \in \mathcal{G}$ preserves orientation. Thus, we have the following

REMARK 1 (see [4]). *If the iteration groups* $\mathcal{F} = \{F^t : \mathrm{S} \mapsto \mathrm{S},\ t \in \mathbb{R}\}$ *and* $\mathcal{G} = \{G^t : \mathrm{S} \mapsto \mathrm{S},\quad t \in \mathbb{R}\}$ *are conjugate, then* $\alpha(F^t) = \alpha(G^t),\ t \in \mathbb{R}$.

Let $\mathcal{F}$ and $\mathcal{G}$ satisfy (1). Then, according to Remark 1, $\mathcal{F}$ is non-singular if and only if so is $\mathcal{G}$. Moreover, one can show that $\mathcal{F}$ is disjoint if and only if so is $\mathcal{G}$ (see also [1]).

For a given orientation-preserving homeomorphism $F : \mathrm{S} \mapsto \mathrm{S}$ put

$$C_F(z) := \{F^n(z),\ n \in \mathbb{Z}\}, \qquad z \in \mathrm{S}.$$

If $\alpha(F) \notin \mathbb{Q}$, then the set $L_F := C_F(z)^{\mathrm{d}}$ does not depend on z, is invariant with respect to $F$ (that is $F[L_F] = L_F$) and either $L_F = \mathrm{S}$ or $L_F$ is a perfect nowhere dense subset of S (see for instance [4]).

LEMMA 4. *Let* $\mathcal{F} = \{F^t : \mathrm{S} \mapsto \mathrm{S},\quad t \in \mathbb{R}\}$ *be an iteration group and* $F^{t_0} \in \mathcal{F}$ *be such that* $\alpha(F^{t_0}) \notin \mathbb{Q}$. *Then*

$$F^t[L_{F^{t_0}}] = L_{F^{t_0}}, \qquad t \in \mathbb{R}.$$

PROOF. Fix $t \in \mathbb{R}$, $z \in \mathrm{S}$. By the definition of $C_{F^{t_0}}(z)$ we have $F^t[C_{F^{t_0}}(z)] = C_{F^{t_0}}(F^t(z))$. Hence, using the definition of $L_{F^{t_0}}$ and the fact that $F^t$ is a homeomorphism, we obtain

$$F^t[L_{F^{t_0}}] = F^t[C_{F^{t_0}}(z)^{\mathrm{d}}] = (F^t[C_{F^{t_0}}(z)])^{\mathrm{d}} = (C_{F^{t_0}}(F^t(z)))^{\mathrm{d}} = L_{F^{t_0}}.$$

□

LEMMA 5. *Let* $\mathcal{F} = \{F^t : \mathrm{S} \mapsto \mathrm{S},\ t \in \mathbb{R}\}$ *be an iteration group and* $F^{t_1}$, $F^{t_2} \in \mathcal{F}$ *be such that* $\alpha(F^{t_1}), \alpha(F^{t_2}) \notin \mathbb{Q}$. *Then*

$$L_{F^{t_1}} = L_{F^{t_2}}.$$

PROOF. Fix $t \in \mathbb{R}$, $A \subset \mathrm{S}$ and put

$$C_{F^t}(A) := \bigcup_{w \in A} C_{F^t}(w).$$

Clearly,

$$C_{F^t}(A) = \bigcup_{n \in \mathbb{Z}} F^{nt}[A].$$

Hence and by Lemma 4 we have

$$C_{F^{t_2}}(L_{F^{t_1}}) = \bigcup_{n\in\mathbb{Z}} F^{nt_2}[L_{F^{t_1}}] = L_{F^{t_1}}.$$

Consequently,

$$(C_{F^{t_2}}(L_{F^{t_1}}))^{\mathrm{d}} = (L_{F^{t_1}})^{\mathrm{d}} = L_{F^{t_1}},$$

since the set $L_{F^{t_1}}$ is perfect. Take a $w \in L_{F^{t_1}}$. Using just shown equality we obtain

$$L_{F^{t_2}} = (C_{F^{t_2}}(w))^{\mathrm{d}} \subset (C_{F^{t_2}}(L_{F^{t_1}}))^{\mathrm{d}} = L_{F^{t_1}}.$$

In the same manner we can see that $L_{F^{t_1}} \subset L_{F^{t_2}}$. □

From now on we assume that $\mathcal{F} = \{F^t : \mathrm{S} \mapsto \mathrm{S},\quad t \in \mathbb{R}\}$ (and also $\mathcal{G}$) is a non-singular iteration group. Then, according to Lemma 5, the set $L_{F^t}$ does not depend on the choice of $F^t \in \mathcal{F}$ such that $\alpha(F^t) \notin \mathbb{Q}$. Thus, we can define

$$L_{\mathcal{F}} := L_{F^{t_0}}$$

for an arbitrary $t_0 \in \mathbb{R}$ such that $\alpha(F^{t_0}) \notin \mathbb{Q}$.

Put $L_{\mathcal{F}}(z) := O_{\mathcal{F}}(z)^{\mathrm{d}}$, where $O_{\mathcal{F}}(z)$ denotes the orbit $O_{\mathcal{F}}(z) := \{F^t(z), t \in \mathbb{R}\}$.

REMARK 2 (see [1]). *Let $\mathcal{F} = \{F^t : \mathrm{S} \mapsto \mathrm{S},\quad t \in \mathbb{R}\}$ be a non-singular iteration group and $t_0 \in \mathbb{R}$ be such that $\alpha(F^{t_0}) \notin \mathbb{Q}$. Then*

(i) $L_{\mathcal{F}} = \mathrm{cl}\ C_{F^{t_0}}(z), \qquad z \in L_{\mathcal{F}},$

(ii) $L_{\mathcal{F}} = L_{\mathcal{F}}(z), \quad z \in \mathrm{S}.$

DEFINITION 4 (see also [6]). A non-singular iteration group $\mathcal{F}$ is said to be dense if $L_{\mathcal{F}} = \mathrm{S}$, otherwise $\mathcal{F}$ is called a non-dense group.

LEMMA 6. *Let $\mathcal{F} = \{F^t : \mathrm{S} \mapsto \mathrm{S}, t \in \mathbb{R}\}$ be a non-singular iteration group and $\alpha(F^1) \notin \mathbb{Q}$. Then there exist a unique continuous increasing function $\varphi_{\mathcal{F}} : \mathrm{S} \mapsto \mathrm{S}$ and a uniquely determined function $c_{\mathcal{F}} : \mathbb{R} \mapsto \mathrm{S}$ such that*

(2) $$\varphi_{\mathcal{F}}(F^t(z)) = c_{\mathcal{F}}(t)\varphi_{\mathcal{F}}(z), \qquad z \in \mathrm{S}, \quad t \in \mathbb{R},$$

(3) $$c_{\mathcal{F}}(s+t) = c_{\mathcal{F}}(s)c_{\mathcal{F}}(t), \qquad s,t \in \mathbb{R},$$

(4) $$\varphi_{\mathcal{F}}[L_{\mathcal{F}}] = \mathrm{S},$$

(5) $$\varphi_{\mathcal{F}}(1) = 1$$

*and*

(6) $$c_{\mathcal{F}}(1) = \pi(\alpha(F^1)).$$

*The solution* $\varphi_{\mathcal{F}}$ *of* (2) *is a homeomorphism if and only if the iteration group* $\mathcal{F}$ *is dense.*

Proof. The existence of a continuous increasing function $\varphi_{\mathcal{F}} : \mathrm{S} \mapsto \mathrm{S}$ and a mapping $c_{\mathcal{F}} : \mathbb{R} \mapsto \mathrm{S}$ satisfying conditions (2)-(4) and the fact that $\varphi_{\mathcal{F}}$ is a homeomorphism if and only if $\mathcal{F}$ is dense have been proved by M. Bajger (see Proposition 1 in [1]). Moreover, it is easily seen that the above-mentioned proof gives more, namely $c_{\mathcal{F}}$ satisfies condition (6). Fix $a \in \mathrm{S}$ and observe that $a\varphi_{\mathcal{F}}$ fulfils (4) and (2) with the function $c_{\mathcal{F}}$. Hence, we may assume that $\varphi_{\mathcal{F}}$ satisfy condition (5).

Note now that using (2) and (6) we have

(7) $$\varphi_{\mathcal{F}}(F^1(z)) = \pi(\alpha(F^1))\varphi_{\mathcal{F}}(z), \qquad z \in \mathrm{S}.$$

But in [3] it is proved that for every orientation-preserving homeomorphism with an irrational rotation number there exists a unique up to a multiplicative constant continuous increasing solution of (7). Thus, we have the desired uniqueness of $\varphi_{\mathcal{F}}$. From this it is easy to check that $c_{\mathcal{F}}$ is uniquely determined. □

An immediate consequence of Lemma 6 is the following

Lemma 7 (see also [1]). *If* $\mathcal{F} = \{F^t : \mathrm{S} \mapsto \mathrm{S},\ t \in \mathbb{R}\}$ *is a dense iteration group such that* $\alpha(F^1) \notin \mathbb{Q}$, *then there exist a unique function* $c_{\mathcal{F}} : \mathbb{R} \mapsto \mathrm{S}$ *satisfying* (3) *and* (6) *and a uniquely determined homeomorphism* $\varphi_{\mathcal{F}} : \mathrm{S} \mapsto \mathrm{S}$ *fulfilling* (5) *such that*

$$F^t(z) = \varphi_{\mathcal{F}}^{-1}(c_{\mathcal{F}}(t)\varphi_{\mathcal{F}}(z)), \qquad z \in \mathrm{S}, \quad t \in \mathbb{R}.$$

If $\mathcal{F}$ is a non-dense iteration group, then we have the following unique decomposition

$$\mathrm{S} \setminus L_{\mathcal{F}} = \bigcup_{q \in M} L_q,$$

where $L_q$ for $q \in M$ are open pairwise disjoint arcs and $\mathrm{card} M = \aleph_0$.

Lemma 8 (see [1]). *Let* $\mathcal{F} = \{F^t : \mathrm{S} \mapsto \mathrm{S},\quad t \in \mathbb{R}\}$ *be a non-dense iteration group and* $\alpha(F^1) \notin \mathbb{Q}$. *If* $\varphi_{\mathcal{F}} : \mathrm{S} \mapsto \mathrm{S}$ *is a continuous increasing* *solution of* (2) *satisfying* (4) *and* (5) *with* $c_{\mathcal{F}} : \mathbb{R} \mapsto \mathrm{S}$ *fulfilling* (3) *and* (6), *then:*

(a) *for every* $q \in M$, $\varphi_{\mathcal{F}}$ *is constant on* $L_q$,

(b) *if* $V \subset \mathrm{S}$ *is an open arc and* $\varphi_{\mathcal{F}}$ *is constant on* $V$, *then* $V \subset L_q$ *for some* $q \in M$,

(c) *if* $p \neq q$, *then* $\varphi_{\mathcal{F}}[L_p] \cap \varphi_{\mathcal{F}}[L_q] = \emptyset$,

(d) *for every* $q \in M$ *and every* $t \in \mathbb{R}$, *there exists a* $p \in M$ *such that* $F^t[L_q] = L_p$,

(e) *the sets* $\mathrm{Im}\ c_{\mathcal{F}}$,

$$K_{\mathcal{F}} := \varphi_{\mathcal{F}}[\mathrm{S} \setminus L_{\mathcal{F}}]$$

*are countable,*

(f) $K_{\mathcal{F}} \cdot \mathrm{Im}\ c_{\mathcal{F}} = K_{\mathcal{F}}$.

By Lemma 8 the function

$$\Phi_{\mathcal{F}}(q) := \varphi_{\mathcal{F}}[L_q], \qquad q \in M$$

is a bijection of $M$ onto $K_{\mathcal{F}}$ and the mapping

$$T_{\mathcal{F}}(q,\ t) := \Phi_{\mathcal{F}}^{-1}(\Phi_{\mathcal{F}}(q) c_{\mathcal{F}}(t)), \qquad q \in M,\ t \in \mathbb{R}$$

is well defined. Condition (3) makes it obvious that $T_{\mathcal{F}} : M \times \mathbb{R} \mapsto M$ satisfies the translation equation

$$T_{\mathcal{F}}(T_{\mathcal{F}}(q,\ s),\ t) = T_{\mathcal{F}}(q,\ s+t), \qquad q \in M, \quad s,t \in \mathbb{R}.$$

LEMMA 9 (see [1]). *If* $\mathcal{F} = \{F^t : \mathrm{S} \mapsto \mathrm{S}, \quad t \in \mathbb{R}\}$ *is a non-dense iteration group and* $\alpha(F^1) \notin \mathbb{Q}$, *then*

$$F^t[L_q] = L_{T_{\mathcal{F}}(q,\ t)}, \qquad q \in M,\ t \in \mathbb{R}.$$

The below results show that the strictly monotonic mappings defined on S have many of the properties of strictly monotonic real functions.

Let us first note that an immediate consequence of Lemma 1 is

REMARK 3. *Every strictly monotonic mapping is an injection.*

The following lemma is easy to check

LEMMA 10. *Assume that* $A,\ B,\ C \subset \mathrm{S}$ *are such that* card $A =$ card $B =$ card $C \geq 3$ *and let* $F : A \mapsto B$ *and* $G : B \mapsto C$ *be strictly monotonic. Then:*

(i) *if $F$ has the same type of monotonicity as $G$, then $G \circ F$ is strictly increasing,*

(ii) *if $F$ has different type of monotonicity from $G$, then $G \circ F$ is strictly decreasing.*

The fact that every orientation-preserving homeomorphism is strictly increasing has been shown in [1]. The same proof works for a homeomorphism which revers orientation, so we have

LEMMA 11. *A homeomorphism $F : \mathrm{S} \mapsto \mathrm{S}$ preserves (respectively, revers) orientation if and only if $F$ is strictly increasing (respectively, decreasing).*

LEMMA 12. *Every strictly monotonic function defined on a dense subset of $\mathrm{S}$ can be extended to a strictly monotonic mapping of the entire circle $\mathrm{S}$.*

PROOF. Let $D$ be a dense subset of $\mathrm{S}$ and $F : D \mapsto \mathrm{S}$ be strictly increasing (similar arguments apply to the case of strictly decreasing $F$). Fix $w \in \mathrm{S} \setminus D$ and choose a sequence $\{w_n\}_{n\in\mathbb{N}} \subset D$ such that

$$\overrightarrow{(w_0,\ w_n)} \subset \overrightarrow{(w_0,\ w)},\quad \overrightarrow{(w_0,\ w_n)} \subset \overrightarrow{(w_0,\ w_{n+1})},\qquad n \in \mathbb{N}\setminus\{0\}$$

and

$$\bigcup_{n=1}^{\infty} \overrightarrow{(w_0,\ w_n)} = \overrightarrow{(w_0,\ w)}.$$

As $F$ is strictly increasing on $D$, $\bigcup_{n=1}^{\infty} \overrightarrow{(F(w_0),\ F(w_n))}$ is an open arc, say $\overrightarrow{(F(w_0),\ a)}$. Put $F(w) := a$. It only remains to prove that the definition of $F(w)$ does not depend on the choice of the sequence $\{w_n\}_{n\in\mathbb{N}}$ and that so determined function $F$ is strictly increasing on $\mathrm{S}$. We leave this to the reader. □

The below lemma in the case of strictly increasing mappings can be found in [3]. The same conclusion can be drawn for strictly decreasing functions, so we get

LEMMA 13. *Every strictly monotonic function $F : \mathrm{S} \mapsto \mathrm{S}$ such that the image of $F$ is a dense subset of $\mathrm{S}$ is continuous.*

As an immediate consequence of Lemmas 12 and 13 we have the following

COROLLARY 1. *Let $D_1$, $D_2$ be dense subsets of $\mathrm{S}$ and $F$ be a strictly monotonic mapping from $D_1$ onto $D_2$. Then $F$ can be uniquely extended to a continuous function defined on the entire circle $\mathrm{S}$.*

To prove our main results, we start with

REMARK 4. *Let $\mathcal{F} = \{F^t : \mathrm{S} \mapsto \mathrm{S},\quad t \in \mathbb{R}\}$ and $\mathcal{G} = \{G^t : \mathrm{S} \mapsto \mathrm{S},$ $t \in \mathbb{R}\}$ be non-singular iteration groups such that $\alpha(F^1)$, $\alpha(G^1) \notin \mathbb{Q}$. If $\mathcal{F}$ and $\mathcal{G}$ satisfy* (1) *with a homeomorphism $\Gamma$, then*

$$\Gamma[L_{\mathcal{F}}] = L_{\mathcal{G}}.$$

PROOF. Fix $z \in \mathrm{S}$. By (1) we have

$$\Gamma[C_{F^1}(z)] = C_{G^1}(\Gamma(z)).$$

Hence, using the fact that $\Gamma$ is a homeomorphism,

$$\Gamma[C_{F^1}(z)^{\mathrm{d}}] = C_{G^1}(\Gamma(z))^{\mathrm{d}}$$

and finally $\Gamma[L_{\mathcal{F}}] = L_{\mathcal{G}}$. □

THEOREM 1. *Let the dense iteration groups $\mathcal{F} = \{F^t : \mathrm{S} \mapsto \mathrm{S},\quad t \in \mathbb{R}\}$ and $\mathcal{G} = \{G^t : \mathrm{S} \mapsto \mathrm{S},\quad t \in \mathbb{R}\}$ be such that $\alpha(F^1) = \alpha(G^1) =: \alpha \notin \mathbb{Q}$. Then $\mathcal{F}$ and $\mathcal{G}$ are conjugate if and only if $c_{\mathcal{F}} = c_{\mathcal{G}}$.*

PROOF. Let $\mathcal{F} = \{F^t : \mathrm{S} \mapsto \mathrm{S},\quad t \in \mathbb{R}\}$ and $\mathcal{G} = \{G^t : \mathrm{S} \mapsto \mathrm{S},$ $t \in \mathbb{R}\}$ be dense iteration groups with $\alpha(F^1) = \alpha(G^1) = \alpha \notin \mathbb{Q}$. Then, by Lemma 7, $F^t(z) = \varphi_{\mathcal{F}}^{-1}(c_{\mathcal{F}}(t)\varphi_{\mathcal{F}}(z))$ and $G^t(z) = \varphi_{\mathcal{G}}^{-1}(c_{\mathcal{G}}(t)\varphi_{\mathcal{G}}(z))$ for the homeomorphisms $\varphi_{\mathcal{F}}, \varphi_{\mathcal{G}} : \mathrm{S} \mapsto \mathrm{S}$ fulfilling (5) and the functions $c_{\mathcal{F}}, c_{\mathcal{G}} :$ $\mathbb{R} \mapsto \mathrm{S}$ satisfying conditions (3) and (6). Assume first that $\mathcal{F}$ and $\mathcal{G}$ are conjugate. Putting $\lambda := \varphi_{\mathcal{G}} \circ \Gamma \circ \varphi_{\mathcal{F}}^{-1}$, where $\Gamma$ is a homeomorphism fulfilling (1), it is easy to check that

$$\lambda(zc_{\mathcal{F}}(t)) = \lambda(z)c_{\mathcal{G}}(t), \qquad z \in \mathrm{S},\quad t \in \mathbb{R}. \tag{8}$$

Moreover,

$$c_{\mathcal{F}}(n) = c_{\mathcal{G}}(n) = \pi(\alpha)^n, \qquad n \in \mathbb{Z},$$

since $c_{\mathcal{F}}$ and $c_{\mathcal{G}}$ satisfy (6) and (3). Using now the facts that the set $D :=$ $\{\pi(\alpha)^n,\ n \in \mathbb{Z}\}$ is dense in S (see for instance [2]) and $\lambda$ is continuous, we get by (8)

$$\lambda(zw) = \lambda(z)w, \quad z, w \in \mathrm{S}.$$

Hence and again by (8), $c_{\mathcal{F}} = c_{\mathcal{G}}$.

Conversely, if $c_{\mathcal{F}} = c_{\mathcal{G}}$ then we obtain (1) with $\Gamma := \varphi_{\mathcal{G}}^{-1} \circ \varphi_{\mathcal{F}}$. □

We now give a necessary and sufficient condition for conjugacy of non-dense disjoint iteration groups. It is worth pointing out that in order to get the necessary condition, the assumption that the iteration groups are disjoint can be dropped.

THEOREM 2. *Let the non-dense disjoint iteration groups* $\mathcal{F} = \{F^t : \mathrm{S} \mapsto \mathrm{S},\ \ t \in \mathbb{R}\}$ *and* $\mathcal{G} = \{G^t : \mathrm{S} \mapsto \mathrm{S},\ \ t \in \mathbb{R}\}$ *be such that* $\alpha(F^1) = \alpha(G^1) =: \alpha \notin \mathbb{Q}$. *Then* $\mathcal{F}$ *and* $\mathcal{G}$ *are conjugate if and only if* $c_{\mathcal{F}} = c_{\mathcal{G}}$ *and there exists a* $d \in \mathrm{S}$ *such that*

$$K_{\mathcal{G}} = d \cdot K_{\mathcal{F}}.$$

PROOF. Let $\mathcal{F} = \{F^t : \mathrm{S} \mapsto \mathrm{S},\ \ t \in \mathbb{R}\}$ and $\mathcal{G} = \{G^t : \mathrm{S} \mapsto \mathrm{S},\ \ t \in \mathbb{R}\}$ be non-dense disjoint iteration groups with $\alpha(F^1) = \alpha(G^1) = \alpha \notin \mathbb{Q}$. Then we have the following unique decompositions

$$\mathrm{S} \setminus L_{\mathcal{F}} = \bigcup_{q \in M} L_q \qquad \text{and} \qquad \mathrm{S} \setminus L_{\mathcal{G}} = \bigcup_{q \in M} L'_q, \tag{9}$$

where $L_q$ and $L'_q$ for $q \in M$ are open pairwise disjoint arcs and $\mathrm{card} M = \aleph_0$. Let the continuous increasing functions $\varphi_{\mathcal{F}},\ \varphi_{\mathcal{G}} : \mathrm{S} \mapsto \mathrm{S}$ satisfy conditions (2), (4) and (5) and the functions $c_{\mathcal{F}},\ c_{\mathcal{G}} : \mathbb{R} \mapsto \mathrm{S}$ fulfil (3) and (6). By (6) and (3) the dense set $D = \{\pi(\alpha)^n,\ n \in \mathbb{Z}\}$ is contained in $\mathrm{Im} c_{\mathcal{F}}$ and $\mathrm{Im} c_{\mathcal{G}}$. Hence, using Lemma 8(f), we conclude that the sets $K_{\mathcal{F}}$ and $K_{\mathcal{G}}$ are dense in S.

Lemma 8(a) lets us define

$$\Phi_{\mathcal{F}}(q) := \varphi_{\mathcal{F}}[L_q] \quad \text{and} \quad \Phi_{\mathcal{G}}(q) := \varphi_{\mathcal{G}}[L'_q], \qquad q \in M. \tag{10}$$

Moreover, by Lemma 8(c), (f), we can define

$$T_{\mathcal{F}}(q,\, t) := \Phi_{\mathcal{F}}^{-1}(\Phi_{\mathcal{F}}(q) c_{\mathcal{F}}(t)), \qquad q \in M,\ t \in \mathbb{R}, \tag{11}$$

$$T_{\mathcal{G}}(q,\, t) := \Phi_{\mathcal{G}}^{-1}(\Phi_{\mathcal{G}}(q) c_{\mathcal{G}}(t)), \qquad q \in M,\ t \in \mathbb{R}. \tag{12}$$

It follows from Lemma 9 that

$$F^t[L_q] = L_{T_{\mathcal{F}}(q,\, t)} \quad \text{and} \quad G^t[L'_q] = L'_{T_{\mathcal{G}}(q,\, t)}, \qquad q \in M,\ t \in \mathbb{R}. \tag{13}$$

Assume first that $\mathcal{F}$ and $\mathcal{G}$ are conjugate and let $\Gamma : \mathrm{S} \mapsto \mathrm{S}$ be a homeomorphism satisfying (1). By Remark 4 and (9),

$$\bigcup_{q \in M} \Gamma[L_q] = \bigcup_{q \in M} L'_q.$$

Therefore there exists a bijection $\Phi : M \mapsto M$ such that

$$\Gamma[L_q] = L'_{\Phi(q)}. \tag{14}$$

Using (14), (13) and (1) we get

$$\begin{aligned} L'_{\Phi(T_{\mathcal{F}}(q,\,t))} &= \Gamma[L_{T_{\mathcal{F}}(q,t)}] = \Gamma[F^t[L_q]] = G^t[\Gamma[L_q]] \\ &= G^t[L'_{\Phi(q)}] = L'_{T_{\mathcal{G}}(\Phi(q),\,t)}, \qquad q \in M,\ t \in \mathbb{R} \end{aligned}$$

and consequently

$$\Phi(T_{\mathcal{F}}(q,\,t)) = T_{\mathcal{G}}(\Phi(q), t), \qquad q \in M, \quad t \in \mathbb{R}. \tag{15}$$

Hence by (11) and (12),

$$\Phi(\Phi_{\mathcal{F}}^{-1}(\Phi_{\mathcal{F}}(q)c_{\mathcal{F}}(t))) = \Phi_{\mathcal{G}}^{-1}(\Phi_{\mathcal{G}}(\Phi(q))c_{\mathcal{G}}(t)), \qquad q \in M,\ t \in \mathbb{R}.$$

Putting $q := \Phi_{\mathcal{F}}^{-1}(z)$ for $z \in K_{\mathcal{F}}$ and $\delta := \Phi_{\mathcal{G}} \circ \Phi \circ \Phi_{\mathcal{F}}^{-1}$ we obtain

$$\delta(zc_{\mathcal{F}}(t)) = \delta(z)c_{\mathcal{G}}(t), \qquad z \in K_{\mathcal{F}},\ t \in \mathbb{R}, \tag{16}$$

whence, by (6) and (3), it follows that

$$\delta(z\pi(\alpha)^n) = \delta(z)\pi(\alpha)^n, \qquad z \in K_{\mathcal{F}},\ n \in \mathbb{Z}. \tag{17}$$

It is obvious that $\delta : K_{\mathcal{F}} \mapsto K_{\mathcal{G}}$ is a bijection. We shall prove that it is strictly monotonic. To do this, take $v,\ w,\ z \in K_{\mathcal{F}}$ fulfilling $v \prec w \prec z$ and let $p,\ q,\ r \in M$ be such that $v = \Phi_{\mathcal{F}}(p),\ w = \Phi_{\mathcal{F}}(q),\ z = \Phi_{\mathcal{F}}(r)$. Then

$$\Phi_{\mathcal{F}}(p) \prec \Phi_{\mathcal{F}}(q) \prec \Phi_{\mathcal{F}}(r)$$

and by (10) and the facts that $\varphi_{\mathcal{F}}$ is increasing and $L_q$ for $q \in M$ are open arcs

$$L_p \prec L_q \prec L_r$$

(that is for every $v \in L_p,\ w \in L_q,\ z \in L_r$ we have $v \prec w \prec z$). Now, using Lemma 11 and (14), we get

$$L'_{\Phi(p)} \prec L'_{\Phi(q)} \prec L'_{\Phi(r)}, \qquad \text{if } \Gamma \text{ preserves orientation}$$

and

$$L'_{\Phi(p)} \succ L'_{\Phi(q)} \succ L'_{\Phi(r)}, \qquad \text{if } \Gamma \text{ revers orientation.}$$

Hence, by the fact that $\varphi_{\mathcal{G}}$ is increasing and Lemma 8(a), (c),

$$\varphi_{\mathcal{G}}[L'_{\Phi(p)}] \underset{(\succ)}{\prec} \varphi_{\mathcal{G}}[L'_{\Phi(q)}] \underset{(\succ)}{\prec} \varphi_{\mathcal{G}}[L'_{\Phi(r)}]$$

and from (10)

$$\Phi_{\mathcal{G}}(\Phi(p)) \underset{(\succ)}{\prec} \Phi_{\mathcal{G}}(\Phi(q)) \underset{(\succ)}{\prec} \Phi_{\mathcal{G}}(\Phi(r)).$$

Using now the fact that $p = \Phi_{\mathcal{F}}^{-1}(v)$, $q = \Phi_{\mathcal{F}}^{-1}(w)$, $r = \Phi_{\mathcal{F}}^{-1}(z)$ we have

$$\delta(v) \underset{(\succ)}{\prec} \delta(w) \underset{(\succ)}{\prec} \delta(z).$$

Since the sets $K_{\mathcal{F}}$ and $K_{\mathcal{G}}$ are dense in S, Corollary 1 shows that the function $\delta$ has a continuous extention $\hat{\delta}$ defined on S. By (17), the density of the set $D = \{\pi(\alpha)^n,\ n \in \mathbb{Z}\}$ and the continuity of the function $\hat{\delta}$ we get

$$\hat{\delta}(zw) = \hat{\delta}(z)w, \quad z, w \in \mathrm{S}.$$

Putting $z := 1$ we have $\hat{\delta}(w) = \hat{\delta}(1)w$ for $w \in \mathrm{S}$ and, in consequence,

$$K_{\mathcal{G}} = \delta[K_{\mathcal{F}}] = \hat{\delta}[K_{\mathcal{F}}] = \hat{\delta}(1) \cdot K_{\mathcal{F}}.$$

Moreover, (16) gives $c_{\mathcal{F}} = c_{\mathcal{G}}$. This ends the first part of the proof.

Let now $c_{\mathcal{F}} = c_{\mathcal{G}} =: c$ and $K_{\mathcal{G}} = d \cdot K_{\mathcal{F}}$ for a $d \in \mathrm{S}$. We will prove that $\mathcal{F}$ and $\mathcal{G}$ are conjugate. Actually, we will show even more, namely we shall give the general construction of all orientation–preserving homeomorphisms $\Gamma : \mathrm{S} \mapsto \mathrm{S}$ satisfying (1).

Define the function $\Psi : M \mapsto M$ by

$$\Psi(q) := \Phi_{\mathcal{G}}^{-1}(\Phi_{\mathcal{F}}(q)d), \qquad q \in M.$$

Note that $\Psi$ is a bijection. Moreover,

$$(\Phi_{\mathcal{G}} \circ \Psi \circ \Phi_{\mathcal{F}}^{-1})(z) = zd, \qquad z \in K_{\mathcal{F}}$$

whence

$$(\Phi_{\mathcal{G}} \circ \Psi \circ \Phi_{\mathcal{F}}^{-1})(zc(t)) = (\Phi_{\mathcal{G}} \circ \Psi \circ \Phi_{\mathcal{F}}^{-1})(z)c(t), \qquad z \in K_{\mathcal{F}},\ t \in \mathbb{R},$$

since $zc(t) \in K_{\mathcal{F}}$. From this and (11) we have

$$\begin{aligned}\Phi_{\mathcal{G}}(\Psi(T_{\mathcal{F}}(q,\ t))) &= (\Phi_{\mathcal{G}} \circ \Psi \circ \Phi_{\mathcal{F}}^{-1})(\Phi_{\mathcal{F}}(q)c(t)) \\ &= \Phi_{\mathcal{G}}(\Psi(q))c(t), \qquad q \in M,\ t \in \mathbb{R}.\end{aligned}$$

On the other hand, (12) gives

$$\Phi_{\mathcal{G}}(T_{\mathcal{G}}(\Psi(q),\, t)) = \Phi_{\mathcal{G}}(\Phi_{\mathcal{G}}^{-1}(\Phi_{\mathcal{G}}(\Psi(q))c(t))) = \Phi_{\mathcal{G}}(\Psi(q))c(t), \qquad q \in M,\ t \in \mathbb{R}.$$

Consequently,

(18) $$\Psi(T_{\mathcal{F}}(q,\, t)) = T_{\mathcal{G}}(\Psi(q),\, t), \qquad q \in M,\ t \in \mathbb{R}.$$

Now we introduce the following relation on $M$

$$p\,\mathcal{R}\,q \iff \exists t \in \mathbb{R}\ \ p = T_{\mathcal{F}}(q,\, t).$$

A trivial verification shows that $\mathcal{R}$ is an equivalence relation. Let $E$ be an arbitrary subset of $M$ such that for every $q \in M$, $\operatorname{card}(E \cap [q]) = 1$ (here and subsequently $[q]$ denotes the equivalence class of $q$ with respect to the relation $\mathcal{R}$) and define

$$A(q) := [q] \cap E, \qquad q \in M.$$

Let $W : M \mapsto \mathbb{R}$ be an arbitrary function such that

(19) $$T_{\mathcal{F}}(A(q),\, W(q)) = q, \qquad q \in M.$$

Hence according to (11) we get

$$\Phi_{\mathcal{F}}^{-1}(\Phi_{\mathcal{F}}(A(q))c(W(q))) = q, \qquad q \in M$$

and consequently

$$\Phi_{\mathcal{F}}(A(q)) = \Phi_{\mathcal{F}}(q)\frac{1}{c(W(q))} = \Phi_{\mathcal{F}}(q)c(-W(q)), \qquad q \in M.$$

Hence

$$A(q) = \Phi_{\mathcal{F}}^{-1}(\Phi_{\mathcal{F}}(q)c(-W(q))), \qquad q \in M$$

so, by (11),

(20) $$T_{\mathcal{F}}(q,\, -W(q)) = A(q), \qquad q \in M.$$

Let

(21) $$\Gamma_e : L_e \mapsto L'_{\Psi(e)}, \qquad e \in E$$

be arbitrary strictly increasing homeomorphisms. Define the mapping $\Gamma_0$ by

(22) $$\Gamma_0(z) := (G^{W(q)} \circ \Gamma_{A(q)} \circ F^{-W(q)})(z), \qquad z \in L_q.$$

According to (22), (13), (20), (21), (18) and (19) we have the following equalities

$$\begin{aligned}\Gamma_0[L_q] &= (G^{W(q)} \circ \Gamma_{A(q)} \circ F^{-W(q)})[L_q] = (G^{W(q)} \circ \Gamma_{A(q)})[L_{T_{\mathcal{F}}(q,\, -W(q))}] \\ &= (G^{W(q)} \circ \Gamma_{A(q)})[L_{A(q)}] = G^{W(q)}[L'_{\Psi(A(q))}] = L'_{T_{\mathcal{G}}(\Psi(A(q)),W(q))} \\ &= L'_{\Psi(T_{\mathcal{F}}(A(q),W(q)))} = L'_{\Psi(q)}, \qquad q \in M.\end{aligned}$$

Thus,

(23) $$\Gamma_0[L_q] = L'_{\Psi(q)}, \qquad q \in M.$$

Our next goal is to show that $\Gamma_0 : \bigcup_{q \in M} L_q \mapsto \bigcup_{q \in M} L'_q$ is strictly increasing. In order to do this take $v,\ w,\ z \in \bigcup_{q\in M} L_q$ such that $w \in \overrightarrow{(v,\ z)}$. We shall show that $\Gamma_0(v) \prec \Gamma_0(w) \prec \Gamma_0(z)$. For this purpose, we consider three cases:

(i) $\{v,\ w,\ z\} \subset L_q$ for a $q \in M$. As $G^t$ and $F^t$ for $t \in \mathbb{R}$ preserve orientation (see Remark 2 in [5]), we obtain our claim from (22) and Lemmas 11 and 10.

(ii) $\operatorname{card}(\{v,\ w,\ z\} \cap L_q) = 2$ for a $q \in M$. By Lemmas 1 and 2 we can assume that $v,\ w \in L_q$. Choose $u \in L_q$ such that $w \in \overrightarrow{(v,\ u)}$. Using (i) and (23) we get $\Gamma_0(w) \in \overrightarrow{(\Gamma_0(v),\ \Gamma_0(u))} \subset L'_{\Psi(q)}$. Moreover, $\Gamma_0(z) \notin L'_{\Psi(q)}$, since $z \notin L_q$ and $\Psi$ is a bijection. According to the above remarks, we have $\Gamma_0(w) \in \overrightarrow{(\Gamma_0(v),\ \Gamma_0(z))}$.

(iii) $\operatorname{card}(\{v,\ w,\ z\} \cap L_q) \leq 1$ for every $q \in M$. Suppose that $v \in L_q$, $w \in L_p$, $z \in L_r$ for $p,\ q,\ r \in M$, $p \neq q$, $q \neq r$, $p \neq r$. Let us note that $L_q \prec L_p \prec L_r$. Using the fact that $\varphi_{\mathcal{F}}$ is increasing, Lemma 8 and (10) we have

$$\Phi_{\mathcal{F}}(q)d \prec \Phi_{\mathcal{F}}(p)d \prec \Phi_{\mathcal{F}}(r)d.$$

Hence and by the definition of $\Psi$,

$$\Phi_{\mathcal{G}}(\Psi(q)) \prec \Phi_{\mathcal{G}}(\Psi(p)) \prec \Phi_{\mathcal{G}}(\Psi(r)).$$

Now (10), the facts that $\varphi_{\mathcal{G}}$ is increasing and $L'_q$ for $q \in M$ are open arcs lead to

$$L'_{\Psi(q)} \prec L'_{\Psi(p)} \prec L'_{\Psi(r)},$$

whence by (23) we obtain our claim.

Thus, $\Gamma_0$ is strictly increasing. Moreover, by (9), (23) and the fact that $\Psi$ is a bijection we get

$$\Gamma_0[\mathrm{S} \setminus L_{\mathcal{F}}] = \mathrm{S} \setminus L_{\mathcal{G}}.$$

Since the sets $\mathrm{S} \setminus L_{\mathcal{F}}$ and $\mathrm{S} \setminus L_{\mathcal{G}}$ are dense in S, Corollary 1 shows that $\Gamma_0$ has the unique continuous extention $\Gamma : \mathrm{S} \mapsto \mathrm{S}$.

We will prove that $\Gamma$ satisfies (1). First we show that

(P) if $T_{\mathcal{F}}(p,\, u) = T_{\mathcal{F}}(p,\, v)$ for a $p \in M$, then $F^u = F^v$ and $G^u = G^v$.

In fact, if $T_{\mathcal{F}}(p,\, u) = T_{\mathcal{F}}(p,\, v)$, then by (11), $c(u) = c(v)$ and (3) gives $c(u-v) = 1$. Hence,

$$T_{\mathcal{F}}(p,\, u-v) = p \quad \text{and} \quad T_{\mathcal{G}}(p,\, u-v) = p,$$

which follows from (11) and (12). Set

$$\overrightarrow{(a_p,\, b_p)} := L_p \quad \text{and} \quad \overrightarrow{(a'_p,\, b'_p)} := L'_p.$$

By (13),

$$F^{u-v}[L_p] = L_p \quad \text{and} \quad G^{u-v}[L'_p] = L'_p,$$

whence it follows that $F^{u-v}(a_p) = a_p$ and $G^{u-v}(a'_p) = a'_p$. From this $F^u = F^v$ and $G^u = G^v$, since the iteration groups $\mathcal{F}$ and $\mathcal{G}$ are disjoint.

Fix $q \in M$, $t \in \mathbb{R}$. By (19), the facts that $T_{\mathcal{F}}$ satisfies the translation equation and $A(q) = A(T_{\mathcal{F}}(q,\, t))$ we get

$$\begin{aligned} T_{\mathcal{F}}(q,\, t) &= T_{\mathcal{F}}(T_{\mathcal{F}}(A(q),\, W(q)),\, t) = T_{\mathcal{F}}(A(q),\, W(q)+t) \\ &= T_{\mathcal{F}}(A(T_{\mathcal{F}}(q,\, t)),\, W(q)+t). \end{aligned} \tag{24}$$

Putting $u := T_{\mathcal{F}}(q,\, t)$ in (24) we have $u = T_{\mathcal{F}}(A(u),\, W(q)+t)$ and consequently by (19), $T_{\mathcal{F}}(A(u),\, W(u)) = T_{\mathcal{F}}(A(u),\, W(q)+t)$. Hence by (P),

$$F^{W(q)+t} = F^{W(u)} = F^{W(T_{\mathcal{F}}(q,\, t))}, \quad G^{W(q)+t} = G^{W(u)} = G^{W(T_{\mathcal{F}}(q,\, t))}. \tag{25}$$

Let $z_0 \in \mathrm{S} \setminus L_{\mathcal{F}}$ and $p \in M$ be such that $z_0 \in L_p$. By Lemma 9, $F^t(z_0) \in L_{T_{\mathcal{F}}(p,\, t)}$. Hence from (22) and (25) we conclude that

$$\begin{aligned} (G^t \circ \Gamma)(z_0) &= (G^t \circ G^{W(p)} \circ \Gamma_{A(p)} \circ F^{-W(p)})(z_0) \\ &= (G^{t+W(p)} \circ \Gamma_{A(p)} \circ F^{-t-W(p)} \circ F^t)(z_0) \\ &= (G^{W(T_{\mathcal{F}}(p,\, t))} \circ \Gamma_{A(p)} \circ F^{-W(T_{\mathcal{F}}(p,\, t))})(F^t(z_0)) \\ &= (G^{W(T_{\mathcal{F}}(p,\, t))} \circ \Gamma_{A(T_{\mathcal{F}}(p,\, t))} \circ F^{-W(T_{\mathcal{F}}(p,\, t))})(F^t(z_0)) \\ &= (\Gamma \circ F^t)(z_0). \end{aligned}$$

Therefore, by the density of the set $S\backslash L_{\mathcal{F}}$ and the continuity of the mappings $G^t$, $F^t$ and $\Gamma$ we obtain (1).

Finally, let $\Gamma$ be an orientation–preserving homeomorphism fulfilling (1). Putting $\Gamma_e := \Gamma_{|L_e}$ for $e \in E$ we get

$$\Gamma(z) = (G^{W(p)} \circ \Gamma_{A(p)} \circ F^{-W(p)})(z), \qquad z \in L_p,\ p \in M.$$

Hence it follows that the above-described construction determines all orientation–preserving homeomorphisms $\Gamma$ satisfying (1). □

## References


[1] M. Bajger, *On the structure of some flows on the unit circle*, Aequationes Math. 55 (1998), 106–121.

[2] A. Beck, *Continuous flows in the plane*, Grundlehren 201, Springer–Verlag, Berlin–New York–Heidelberg 1974.

[3] K. Ciepliński, *On the embeddability of a homeomorphism of the unit circle in disjoint iteration groups*, Publ. Math. Debrecen (to appear).

[4] I. P. Cornfeld, S. V. Fomin, Y. G. Sinai, *Ergodic theory*, Grundlehren 245, Springer–Verlag, Berlin–Heidelberg–New York 1982.

[5] M. C. Zdun, *On embedding of homeomorphisms of the circle in a continuous flow*, Iteration theory and its functional equations (Proceedings, Schloss Hofen 1984), Lecture Notes in Mathematics 1163, Springer-Verlag, Berlin–Heidelberg–New York 1985, 218–231.

[6] M. C. Zdun, *On some invariants of conjugacy of disjoint iteration groups*, Results in Math. 26 (1994), 403–410.



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