CHARACTERIZATION OF BASIN BOUNDARIES IN BAIRSTOWS ITERATIVE METHODS

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To the memory of Professor György Targonski

Abstract. In this paper we consider a two-dimensional map with a denominator which can vanish, obtained by applying Bairstow's method, an iterative algorithms to find the real roots of a polynomial based on Newton's method. The complex structure of the basins of attraction of the fixed points is related to the existence of singularities specific to maps with a vanishing denominator, such as sets of non definition, focal points and prefocal curves.

1. Introduction

Many numerical iterative methods to find the roots of equations, based on the Newton's method, require the iteration of rational maps with denominator. However, a well known weakness of Newton's method is that its convergence is only local, so a study of the basins of attraction becomes important for the practical utilization of such numerical algorithms. As stressed in the literature (see e.g. [2, 3]) the question of the delimitation of basins boundaries for the iterated maps arising from Newton's method is an intriguing problem also from the point of view of the global analysis of dynamical systems. In this paper we apply some recent results on the global dynamical properties of maps with denominator to the study of the basins of a map obtained by one of the most known iterative algorithms to find

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roots of polynomials with real coefficients, proposed by Bairstow in 1914 [1]. This numerical method, which involves only real arithmetic, is based on the factorization of the polynomials into products of quadratic functions, whose coefficients are the roots of algebraic equations. The application of the iterative Newton's method to find such roots gives rise to a *two-dimensional rational map*, whose fixed points correspond to the desired coefficients of the quadratic factors (see e.g. [13]). These fixed points are locally attracting, i.e. the iterative method converges provided that the initial condition is sufficiently close to a fixed point, but there are no general results on their basins of attraction (see [9]).

The dynamic behavior of a two-dimensional map coming from Bairstow's method is strongly influenced by the following two general properties: (i) it is a noninvertible map; (ii) it is a fractional rational map with vanishing denominator.

From (ii) it follows that there is a subset of the plane where the map is not defined, also called *singular set* in [2], and such a subset may include points in which the map takes the form 0/0, which are candidate to be *focal points*, following the terminology introduced in [6, 4, 12, 10]. Some properties of these singularities, specific to maps with a vanishing denominator, have been recently investigated (see [6] and references therein) and can be usefully applied in order to study the structure of the basin boundaries.

Particular structures in the basins of the fixed points have been evidenced in [2] for a particular example, obtained by the application of the Bairstow method to a cubic polynomial. We show that these structures are determined by the presence of focal points. Although our analysis is limited to the particular example proposed in [2], we believe that similar properties also hold in other maps deduced in the same way, because they come from the properties (i) and (ii) stated above, which hold for all the maps obtained by the Bairstow method.

2. A particular case: roots of a cubic polynomial

As a prototype of the class of maps obtained by the application of Bairstow's method, we consider the one obtained by applying the method to the one-parameter family of cubic polynomials $P_a(x) = x^3 + (a-1)x - a$. The factorization $P_a(x) = (x^2 + ux + v)(x - u)$ occurs iff u and v are fixed points of the two-dimensional map $T_a : (u, v) \to (u', v')$ given by (see [2]):

$$(1) \qquad T_a : \begin{cases} u' = \frac{u^3+u(v-a+1)+a}{2u^2+v} \\ v' = \frac{v(u^2+a-1)+2au}{2u^2+v} \end{cases} .$$

The domain of definition of the function T_a is $\mathbb{R}^2 \backslash \delta_s$, where δ_s is the *singular set*

$$(2) \qquad \delta_s = \left\{(u,v) \in \mathbb{R}^2 : \quad v = -2u^2\right\}.$$

So the iteration of T_a is well defined provided that the initial condition belongs to the set E given by

$$(3) \qquad E = \mathbb{R}^2 \setminus \Lambda$$

where Λ is the union of the preimages of any rank of the singular set δ_s

$$\Lambda = \bigcup_{k=0}^{\infty} T_a^{-k}(\delta_s).$$

In fact, only the points belonging to the set E generate non interrupted trajectories

$$(4) \qquad \tau(u_0,v_0) = \{(u_n,v_n) = T_a^n((u_0,v_0), n \geq 0\}$$

by the iteration of the map $T_a : E \to E$. We notice that, being the singular set δ_s a curve of $\mathbb{R}^2$, the set Λ of points excluded from the phase space of the recurrence has zero Lebesgue measure in $\mathbb{R}^2$.

Several properties of the map T_a have been already studied in the literature. For example, in [9] it is proved that:

PROPERTY 1. *Let ξ be a real root of $P(x)$. Then the line (L_ξ) in the (u,v) plane of equation $v + \xi u + \xi^2 = 0$ is invariant for T_a. The restriction of T_a to the line (L_ξ) is a one-dimensional map associated with the Newton-function applied to the reduced polynomial $P_\xi(x) = P(x)/(x-\xi)$.*

Furthermore, in [8] it is shown that:

PROPERTY 2. *Let $R = (u^*, v^*)$ be a fixed point of T_a. Then the line $u = u^*$ is mapped by T_a into the fixed point R.*

For $a < 1/4$ the map T_a has three distinct fixed points (associated with three quadratic factors for $P_a(x)$), given by

$$(5) \qquad R_1 = (u_1^*, v_1^*) = (1, a), \qquad R_2 = (u_2^*, u_3^*), \qquad R_3 = (u_3^*, u_2^*)$$

where

$$(6) \qquad u_2^* = \frac{-1+\sqrt{1-4a}}{2}, \qquad u_3^* = \frac{-1-\sqrt{1-4a}}{2}.$$

From Property 1 we deduce that for $a < 1/4$ three invariant lines exist, say L_i, $i = 1, 2, 3$. Each invariant line L_i connects two fixed points, being R_i the excluded one. The line L_1, of equation

$$u + v + 1 = 0 \tag{7}$$

is invariant also in the range $a > 1/4$, when only the fixed point R_1 exists. The equations of the other two invariant lines L_i, $i = 1, 2$, existing for $a < 1/4$, are

$$u = a + \lambda_i(v - 1) \ , \quad i = 2, 3 \tag{8}$$

where $\lambda_2 = \frac{(a-u_2^*)}{(1-u_3^*)}$ and $\lambda_3 = \frac{(a-u_3^*)}{(1-u_2^*)}$.

If $(u, v) \in L_i$, then $(u', v') \in L_i$, and the one-dimensional dynamics embedded into the line L_i is governed by the restriction of T_a to the invariant line, which can be written as a one-dimensional map $u' = F_i(u)$. Each one-dimensional map F_i coincides with the Newton function of the reduced polynomial $P_i(u) = u^2 - s_i u + p_i$, i.e. $F_i(u) = u - P_i(u)/P_i'(u)$, where the coefficients of $P_i(u)$ are given by $(s_1, p_1) = (-1, a)$, $(s_2, p_2) = (-u_2^*, u_3^*)$ and $(s_3, p_3) = (-u_3^*, u_2^*)$. Thus we get:

$$F_1(u) = \frac{u^2 - a}{2u + 1} \ , \quad F_2(u) = \frac{u^2 - u_3^*}{2u + u_2^*} \ , \quad F_3(u) = \frac{u^2 - u_2^*}{2u + u_3^*}.$$

3. Focal points of the map T_a

Some global dynamical properties of two-dimensional maps with denominator have been recently investigated through the definition of new kinds of singularities, such as the sets where a denominator vanishes and the points where the map takes the form 0/0 in at least one component. These concepts have been for the first time introduced during the ECIT conference of 1996 (see [10, 5, 12]), see also [4, 6]. In particular, the concepts of *focal point* and *prefocal curve* have been defined in order to explain some properties and bifurcations of basins boundaries. We recall the definition (see [6]):

DEFINITION 1. Consider a two-dimensional map T. A point Q belonging to the set of non definition δ_s, is a focal point if at least one component of the map T takes the form 0/0 in Q and there exist smooth simple arcs $\gamma(\tau)$, with $\gamma(0)=Q$, such that $\lim_{\tau \to 0} T(\gamma(\tau))$ is finite. The set of all such finite values, obtained by taking different arcs $\gamma(\tau)$ through Q, is called prefocal set δ_Q.

For a short summary of the main geometric properties related to the presence of a focal point see also [7]. For the map (8) we always have the focal point

$$Q_1 = (1, -2) \tag{9}$$

and, for $a \le 1/4$, two further focal points exist, given by

$$Q_2 = (u_2^*, -2u_2^{*2}), \qquad Q_3 = (u_3^*, -2u_3^{*2}) \tag{10}$$

The parametric equation of the prefocal set δ_{Q_1} is given, as a function of the parameter m (slope of the arc) by the equations:

$$u_m = \frac{2 - a + m}{4 + m}, \qquad v_m = \frac{-4 + 2a + ma}{4 + m} \tag{11}$$

and by eliminating the parameter m we get the equation of the prefocal line δ_{Q_1}, given by:

$$\delta_{Q_1} : \ v = 2u - 2 + a. \tag{12}$$

Analogously, for the two further focal points existing for $a < 1/4$, given in (12), we get the parametric equations of the prefocal sets δ_{Q_i} as a function of the parameter m (slope of the arc through Q_i):

$$u_m = \frac{u_i^*(m-1) + 1 - 2a}{4u_i^* + m}, \qquad v_m = \frac{u_i^*(4a - 4 - m) - 2a - m}{4u_i^* + m}$$

and by eliminating the parameter m we get the equations of the prefocal lines δ_{Q_i}, $i = 2, 3$, given by:

$$\delta_{Qi} : \ v = 2u_i^* u + (u_i^* + 2a - 1), \qquad i = 2, 3. \tag{13}$$

Note that the second component of the map T_a, also takes the form $0/0$ in the origin $O = (0, 0)$. However, according to definition 1, this point of δ_s is not a focal point, except for the case $a = 0$, when it merges with the focal point Q_2. In fact, for $a \neq 0$ the first component of the map becomes $a/0$, thus no finite value can be obtained, and O behaves as a generic point, not focal, of the singular set δ_s.

4. Properties of the inverses of the map T_a

For the study of the basins it is important to understand the properties of the inverses of (13). Let (u', v') be a given point of the plane. Then, by solving the system of equations (13) with respect to u and v we can get either two distinct real solutions, called rank-1 preimages of the point (u', v'), or no real solutions. Following the notations of [12], let us call Z_2 and Z_0 the regions of the plane whose points have, respectively, two distinct rank-1 preimages and no preimages. The regions Z_0 and Z_2 are, respectively, inside and outside the parabola of equation

$$v = u^2 + a - 1. \tag{14}$$

For a point $(u', v') \in Z_2$ we denote the inverses of the map T_a by $T_{a,r}^{-1}$ and $T_{a,l}^{-1}$, being the two preimages located one on the right and one on the left of the point (u', v') (symmetric with respect to that point):

$$T_{a,r}^{-1} : \begin{cases} u = u' + \sqrt{u'^2 - v' + a - 1} \\ v = v' + \frac{(u'v'-a)}{\sqrt{u'^2-v'+a-1}} \end{cases},$$
$$T_{a,l}^{-1} : \begin{cases} u = u' - \sqrt{u'^2 - v' + a - 1} \\ v = v' - \frac{(u'v'-a)}{\sqrt{u'^2-v'+a-1}} \end{cases}. \tag{15}$$

We denote by T_a^{-1} both the inverses of T_a, i.e. $T_a^{-1}(u, v) = T_{a,l}^{-1} \cup T_{a,r}^{-1}$. Notice that the boundary $\partial Z_2 = \partial Z_0$ between the two regions, is not a locus of critical points (see [11]) where the two inverses are defined and merge. As explained in [6], this occurrence is related to the fact that such a boundary is the singular set of at least one of the inverses of the map. In fact, from the definition (15) follows that both the inverses are not defined in the set ∂Z_2, since the denominator vanishes. In other words, ∂Z_2 is the singular set, say δ_s', of T_a^{-1}.

As also the inverses have a vanishing denominator we may ask for focal points and prefocal curves of the inverses. But from Property 2 we know that the vertical line through a fixed point is mapped into the fixed point itself, $T_a(u_i^*, v) = R_i$, and, as shown in [7], this implies that the focal points, say Q_i', of the inverse map T_a^{-1} are the fixed points R_i of the map T_a , with related prefocal lines $\delta_{Q_i'}$ of equation $u = u_i^*$. This means that if we consider a small bounded arc η which crosses ∂Z_2 in a point (u', v') which is not a fixed point, and look for the rank-1 preimages of this arc, then the points in Z_0 have no preimages, while the points in Z_2 have two distinct preimages. These preimages, $T_{a,r}^{-1}(\eta)$ and $T_{a,l}^{-1}(\eta)$, are unbounded arcs asymptotic to the straight line $u = u'$, which depends on the point in which η crosses the singular set $\delta'_s (= \partial Z_2)$ of the inverses.

We can consider such a situation similar to the one occurring for one-dimensional maps having a horizontal asymptote which separates portions of the range having different number of preimages. As a horizontal asymptote of a one-dimensional map corresponds to a vertical asymptote for at least one inverse, also in our two-dimensional map T_a we can consider the parabola $\partial Z_2 \equiv \delta'_s$ as a two-dimensional analogue of a horizontal asymptote. On the basis of similar arguments we suggest the following interpretation for the sets δ_s and δ'_s : *the points of the singular set δ_s of the map T_a behave as points of vertical asymptote except for the focal points Q_i, while for the map T_a the points of the singular set $\delta'_s (= \partial Z_2)$ of the inverses behave like points of horizontal asymptote, except for the points R_i.*

Straightforward algebraic computations show that the prefocal lines δ_{Qi} are the tangents to ∂Z_2 in the fixed points R_i. We can so state the following:

PROPOSITION 1. *The fixed points of the map T_a, defined in (15), belong to the boundary of the region Z_2. The focal points $Q_i \in \delta_s \cap (u = u_i^*)$ have prefocal curves δ_{Qi} which are the tangents to ∂Z_2 in the fixed points R_i.*

From this proposition and the properties of the inverses given above, it follows that if we consider a small neighborhood U of the fixed point R_i, then the points belonging to $U \cap Z_0$ have no preimages, while the points in $U \cap Z_2$ have two distinct rank-1 preimages, given by an unbounded area $T_a^{-1}(U)$ (which must include the whole line $u = u_i^*$) whose qualitative shape is shown in fig. 1.

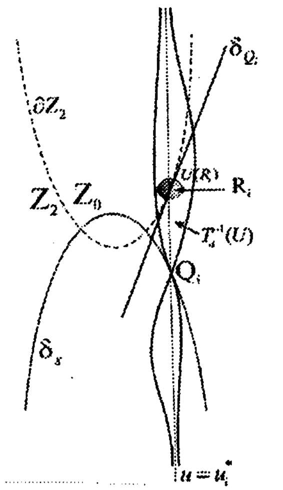

Fig. 1. Qualitative sketch of the rank-1 preimages $T_a^{-1}(U) = T_{a,l}^{-1}(U) \cup T_{a,r}^{-1}(U)$ of the neighborhood U of the fixed point R_i (the grey-shaded circular region)

The particular shape of $T_a^{-1}(U)$ is due to the fact that the focal point Q_i belongs to the line $u = u_i^*$ and that the prefocal curve δ_{Q_i} is tangent to the singular set $\delta_s^{'}(= \partial Z_2)$ in the fixed point R_i. This means that any neighborhood U of R_i intersects δ_{Q_i}, as well as $\delta_s^{'}$, in two distinct points: the preimages of arcs crossing $U \cap \delta_{Q_i}$ are arcs through Q_i, hence $T_a^{-1}(U)$ must shrink into the focal point Q_i, and the preimages of points near $(u^{'}, u^{'2} + a - 1) \in U \cap \delta_s^{'}$ are arbitrarily large, i.e. close to infinity, asymptotic to the line $u = u^{'}$.

This property is at the basis of some complex structures observed in the basins of the map (15).

5. Structure of the basins in a typical situation

The Bairstow's method is clearly a "local method", since the convergence to a given fixed point is only ensured for initial conditions which are sufficiently close to it. A global study of the asymptotic behavior of the trajectories (15), as the initial condition (u_0, v_0) varies in the plane, is still an open problem, studied by many authors in the recent literature (see [9, 8, 2, 3]).

We denote by $\mathcal{B}(R)$ the basin of a given fixed point R, defined as the set of points whose trajectories converge to R

$$\mathcal{B}(R) = \{(u, v) \in E|\ \ T^n(u, v) \to R \text{ as } n \to +\infty\}.$$

As in [2], we only consider the case $a < 1/4$, when three fixed points $R_i, i = 1, 2, 3$, exist, and any point $(u, v) \in E$ belongs to one of the basins $\mathcal{B}(R_i)$. However, as already argued in [8], there are regions of "uncertainty" with respect to the asymptotic behavior, i.e. regions in which, given an initial condition (u_0, v_0), it is difficult to decide to which of the three fixed points the iterations will converge. Our main goal is to describe how the complex structure of the basins can be explained by the properties of the focal points, and related prefocal lines, of the map (15).

The basins' boundaries are sets which are invariant under inverse iteration of the map, that is $T_a^{-1}(\partial\mathcal{B}(R_i)) = \partial\mathcal{B}(R_i)$. Generally a basin's boundary is a set which includes some repelling cycles and their stable sets. However, for $a < 1/4$ we have not found any other cycle except for the stable fixed points R_i. But in two-dimensional maps with vanishing denominator it may occur that the set Λ of the preimages of any rank of the singular set also behaves as a frontier between two or more basins of attraction. This situation is commonly met in the study of one-dimensional maps, where a vertical asymptote may constitute the boundary which separates two basins of attraction. A two-dimensional example where a similar property holds is given in [6]. The map (15) is another example of this property.

Thus, in order to describe the basins' boundaries in this range of the parameter, we have to consider the singular set δ_s and its preimages. The two parabolas δ_s and ∂Z_2 intersect in two points: the portion of δ_s located inside Z_0 has no preimages, whereas the portion located inside Z_2, say $\delta_s \cap Z_2 = \delta_{s,l} \cup \delta_{s,r}$, is made up of two disjoint branches, each one having two rank-1 preimages. Thus $T_a^{-1}(\delta_s)$ is made up of four branches, as shown in fig. 2a. We remark that even if fig. 2 has been obtained for the particular value $a = 0.15$ of the parameter, the qualitative structure of the preimages shown in this figure may be considered as emblematic of the whole range $a < 1/4$.

As $\delta_{s,r}$ is an unbounded curve that intersects the prefocal line δ_{Q_1} in a point with abscissa u_1, then its rank-1 preimage $T_{a,r}^{-1}(\delta_{s,r})$ is an unbounded arc crossing through Q_1 with slope $m(u_1)$ given by (15). Furthermore, since $\delta_{s,r}$ includes Q_1, its preimages must include the corresponding preimages of that focal point. Analogously, as $\delta_{s,r}$ intersects also the prefocal lines δ_{Q_2} and δ_{Q_3} the rank-1 preimage on the left, $T_{a,l}^{-1}(\delta_{s,r})$, must be an unbounded arc crossing through the two focal points Q_2 and Q_3, with known slopes. Following similar arguments, as the arc $\delta_{s,l}$ intersects the prefocal lines δ_{Q_1} and δ_{Q_2} its rank-1 preimage $T_{a,r}^{-1}(\delta_{s,l})$ must be an unbounded arc crossing through Q_1 and Q_2 with given slopes, and the rank-1 preimage on the left, $T_{a,l}^{-1}(\delta_{s,l})$, must be an unbounded arc crossing through the focal point Q_3 and its preimage $Q_{3,-1}$. These four branches are represented in fig. 2a. Many portions of the rank-1 preimages of δ_s belong to the region Z_2, so preimages of higher rank exist and so on. Indeed, the process never ends, and infinitely many preimages are get. Note that whenever an arc is unbounded and intersects the set ∂Z_2 then it also intersects all the three prefocal lines (due to the fact that the prefocal lines are tangent to ∂Z_2 in the fixed points R_i), thus its rank-1 preimage is made up of two branches, $T_a^{-1} = T_{a,l}^{-1} \cup T_{a,r}^{-1}$, which cross through the three focal points Q_i.

It is a numerical evidence that all these arcs, that constitute the set Λ, also separate the three basins of attraction (compare fig. 2a with fig. 2b). But more, each branch of the preimages determined as described above is, from one side, a limit set of other preimages, i.e. a limit set of portions of the three different basins. This numerical result can be explained by the properties of the focal points and prefocal lines. For example, consider in fig. 2a the arc $T_{a,l}^{-1}(\delta_{s,l})$. It crosses the prefocal line δ_{Q_1} in two points (only the upper one is visible in the figure). Thus its rank-1 preimage on the right must be a "loop" issuing from the focal point Q_1. The same property holds for the infinite sequence of preimages having a "parabolic shape" that are located on the left (i.e. in the half-plane $u < 0$). This gives infinitely many "lobes" of different basins issuing from the focal point Q_1, as shown in the enlargement of fig. 3a. Note also that all the lobes issuing from Q_1 intersect both ∂Z_2 and δ_{Q_1}, so that the infinitely many preimages "on the right" must include a "fan" of unbounded arcs issuing from Q_1 and crossing through $Q_{1,-1}$, issuing from $Q_{1,-1}$ and crossing through $Q_{1,-2}$, and similar fans of strips of different colors (i.e. whose points belong to different basins) issue from all the infinitely many preimages of Q_1.

Analogously, the preimage $T_{a,r}^{-1}(\delta_{s,r})$ (or, better, the fun issuing from Q_1, $Q_{1,-1}$, etc.) intersects the prefocal line δ_{Q_3} in two points, and its rank-1 preimage has a "loop" issuing from the focal point Q_3, and so on for the other curves, constituting a fan, intersecting the prefocal line δ_{Q_3} in "arcs", giving rise to infinitely many lobes of the three basins issuing from the focal point Q_3 (see the enlargement in fig. 3b)

From fig. 2 it can be seen the different role played by the focal point Q_2 with respect to the other two focal points. This is due to the fact that $Q_2 \in Z_0$, so that it has no preimages. It follows that whenever some arc crosses through the prefocal line δ_{Q_2} its preimage gives rise to an arc through the focal point Q_2 and here the sequence of preimages stops.

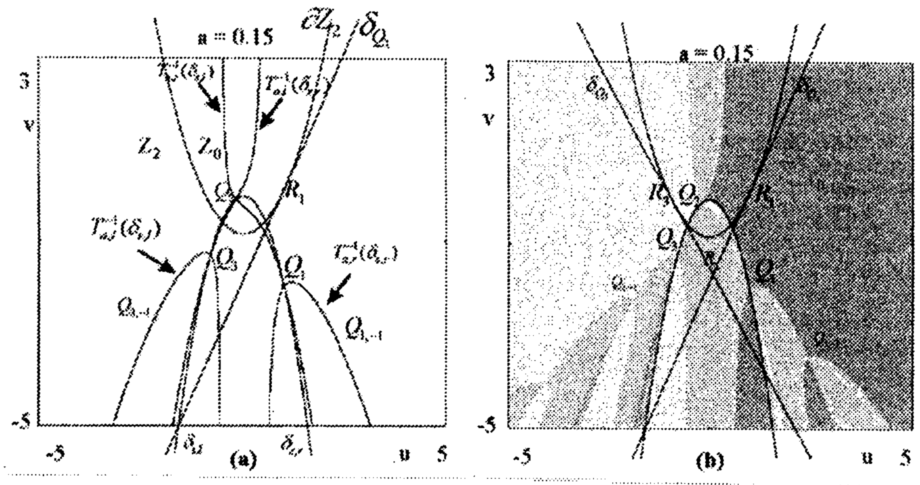

Fig. 2. $a = 0.15 < 1/4$. In (a) the rank-1 preimages of $\{\delta_{s,l}, \delta_{s,r}\} = \delta_s \cap Z_2$ are represented. In (b) three different grey tones are used to represent the three basins of attraction of the fixed points. The dark grey represents the points belonging to $\mathcal{B}(R_1)$, the intermediate grey represents the points belonging to $\mathcal{B}(R_2)$ and the light grey represents the points belonging to $\mathcal{B}(R_3)$. The points Q_1, Q_2 and Q_3 are the focal points

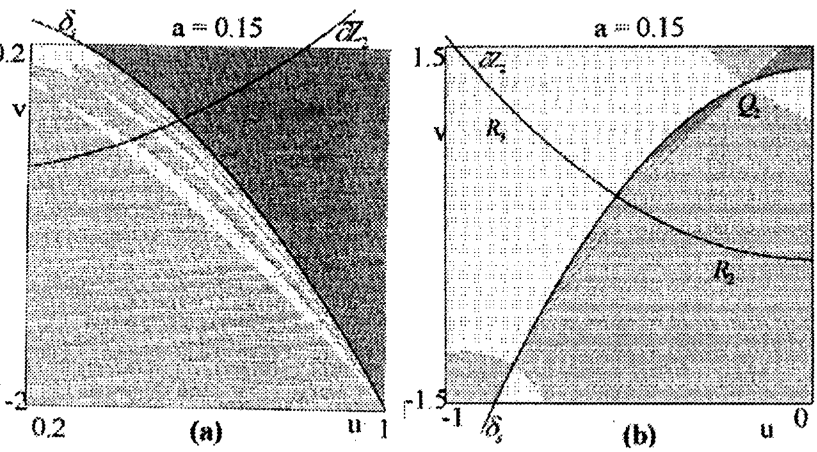

Fig. 3. $a = 0.15 < 1/4$. Enlargements of portions of fig. 2b.

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