ON ITERATIVE EQUIVALENCE OF SOME CLASSES OF MAPPINGS

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To the memory of Professor György Targonski

Abstract. We introduce the notion of iterative equivalence of two classes of mappings on metric spaces and we demonstrate its utility in metric fixed-point theory. In particular, we show that the fixed-point theorem for Matkowski's contractions can be derived from the corresponding theorem for Browder's contractions, though the first class of mappings is essentially wider than the second one.

1. Introduction

A selfmap f of a metric space (X, d) is said to have the *contractive fixed point property* (abbr., CFPP) if f has a unique fixed point $x_0 \in X$ and $\lim_{n\to\infty} f^n(x) = x_0$ for all $x \in X$, where f^n denotes the nth iterate of f. Let us recall the following simple extension of the Banach contraction principle, which may be found, e.g., in Dugundji–Granas [3, p. 17].

PROPOSITION 1. *Let (X, d) be a complete metric space and f be a selfmap of X such that for some positive integer k, f^k is a Banach contraction. Then f has the CFPP.*

In fact, it suffices here to assume that f^k has the CFPP. In this case we may also drop the assumption on completeness of (X, d).

Let F and G denote two classes of selfmaps of metric spaces. We treat elements of these classes as pairs of the form (f, d), where d is a metric for

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the domain of a map f. We say that classes F and G are *iteratively equivalent* if given $(f,d) \in F$ there exists $k \in \mathbb{N}$ such that $(f^k,d) \in G$ and, conversely, given $(g,\rho) \in G$ there is $m \in \mathbb{N}$ such that $(g^m,\rho) \in F$. We say that F has the CFPP if each map in F has the CFPP. The following result is an immediate consequence of a remark, which follows Proposition 1.

PROPOSITION 2. *Let F and G be classes of selfmaps of metric spaces such that F and G are iteratively equivalent. Then F has the CFPP if and only if G has the CFPP.*

Proposition 2 gives us a tool for proving fixed-point theorems involving the CFPP. If we know that some two classes of selfmaps are iteratively equivalent, then it suffices to prove a fixed-point theorem only for one of them and then we may conclude from Proposition 2 that both classes have the CFPP. Moreover, fixed-point theorems for such two classes are, in some sense, equivalent even if one of these classes is a proper subclass of the other. We will discuss this fact with details in a sequel.

Given an $\alpha \in (0,1)$, a selfmap f is said to be an *α-contraction* if $d(fx,fy) \leq \alpha\ d(x,y)$ for all $x,y \in X$. f is a *Banach contraction* if it is an α-contraction for some $\alpha \in (0,1)$.

We give the simplest reasonable example of two iteratively equivalent classes (the simplest one deals with two identical classes; incidentally, the iterative equivalence is an equivalence relation).

EXAMPLE 1. For any fixed $\alpha \in (0,1)$ the class **Ba** of all Banach contractions and the class of all α-contractions are iteratively equivalent.

Some less trivial examples for the iterative equivalence phenomenon will be given in the next sections. We will examine classes of maps, which satisfy a *nonlinear contractive condition*, that is, for each such a map f there exists a function φ from $\mathbb{R}_+$, the set of all nonnegative reals, into $\mathbb{R}_+$, such that $\varphi(t) < t$ for $t > 0$ and

$$d(fx,fy) \leq \varphi(d(x,y)) \quad \text{for all} \quad x,y \in X.$$

We also say then that a map f is *φ-contractive*. In the sequel we will consider the following classes **Br**, **BW** and **M** of maps satisfying nonlinear contractive condition.

A selfmap f of a metric space (X,d) is a *Browder contraction* ($(f,d) \in$ **Br**) if there exists a non-decreasing and right continuous function φ such that f is φ-contractive (cf. [2] or [3, p. 18]).

A selfmap f of a metric space (X,d) is a *Boyd-Wong contraction* ($(f,d) \in$ **BW**) if there exists a right upper semicontinuous function φ such that f is φ-contractive (cf. [1] or [11, p. 38]).

A selfmap f of a metric space (X,d) is a *Matkowski contraction* ($(f,d) \in$ **M**) if there exists a non-decreasing function φ such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t \in \mathbb{R}_+$ and f is φ-contractive (cf. [13], [3,p. 12] or [11, p. 39]).

It is known that each of the above classes has the CFPP. Moreover, **Ba** $\subset$ **Br** $\subset$ **BW** and **Br** $\subset$ **M**, and all these inclusions are proper. Classes **BW** and **M** are incomparable. A comprehensive study of nonlinear contractive conditions is given in [8] (also cf. [6], [9] and references therein). Here we will show that classes **Br** and **M** are iteratively equivalent (cf. Section 3), whereas classes **BW** and **M** are not iteratively equivalent (cf. Section 4). Also we will give a complete characterization of these non-decreasing functions φ, for which $\mathbf{C}_\varphi$, the class of all φ-contractive maps, and the class **Ba** are iteratively equivalent (cf. Section 2).

Throughout this paper the letter φ denotes a function from $\mathbb{R}_+$ into $\mathbb{R}_+$ such that $\varphi(t) < t$ for $t > 0$ and $\varphi(0) = 0$.

2. Comparison of classes Ba, Br and $\mathbf{C}_\varphi$

We omit a standard proof of the following

LEMMA 1. *Let X be a nonempty subset of $\mathbb{R}_+$ and for $x, y \in X$,*

$$d(x,y) := \max\ \{x,y\} \quad \textit{if} \quad x \neq y, \quad \textit{and} \quad d(x,x) := 0.$$

Then (X,d) is a metric space. Moreover, (X,d) is complete if and only if either $0 \notin \bar{X}$, the closure of X in $\mathbb{R}_+$ endowed with the Euclidean topology, or $0 \in X$. Further, let a function φ be non-decreasing and such that $\varphi(X) \subseteq X$. Then a map f defined by $f := \varphi|_X$, the restriction of φ to X, is φ-contractive.

Observe that the metric d defined in Lemma 1 satisfies the inequality

$$d(x,y) \le \max\ \{d(x,z), d(z,y)\} \quad \text{for all} \quad x,y,z \in X.$$

Such a metric is called an *ultrametric* or a *non-Archimedean metric* (cf. [4, p. 504]).

The following result shows that, contrary to the linear case (cf. Example 1), there does not exist a function φ, for which classes $\mathbf{C}_\varphi$ and **Br** would be iteratively equivalent.

THEOREM 1. *Given a function φ, there exist a complete metric space (X,d) and a map $f : X \mapsto X$ such that $(f,d) \in$ **Br** and for all $k \in \mathbb{N}$, f^k is not φ-contractive.*

Proof. We will define a strictly increasing and continuous function $\psi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that $\psi(t) < t$ for $t > 0$ and $\psi^n(n) > \varphi(n)$ for all $n \in \mathbb{N}$. Then if we apply Lemma 1 to the ultrametric space (X, d) with $X := \mathbb{R}_+$ and the map $f := \psi$, we will be able to conclude that

$$d(f^n n, f^n 0) = \psi^n(n) > \varphi(n) = \varphi(d(n,0)) \quad \text{for all} \quad n \in \mathbb{N}$$

so f^n is not φ-contractive, and the proof will be completed.

By induction we will define $\psi|_{(n-1,n]}$ and a family of strictly increasing finite sequences $(t_j^{(n)})_{j=1}^n$ with $t_j^{(n)} \in (n-1, n]$ for $n \in \mathbb{N}$. Let $t_1^{(1)} := 1$. Define $\psi|_{[0,1]}$ as the segment with endpoints $(0,0)$ and $(1, (1+\varphi(1))/2)$. Assume that $n \in \mathbb{N}$ and $\psi|_{(n-1,n]}$ is defined. Then we define a sequence $(t_j^{(n+1)})_{j=1}^{n+1}$ and $\psi|_{(n,n+1]}$ as follows:

$$\max\left\{n, \frac{n+1+\varphi(n+1)}{2}\right\} < t_1^{(n+1)} < n+1, \quad t_{n+1}^{(n+1)} := n+1,$$

$$\text{and for} \quad n \geq 2 \quad \text{and} \quad j = 2, \cdots, n, \quad t_j^{(n+1)} \in \left(t_{j-1}^{(n+1)}, n+1\right);$$

$$\psi\left(t_1^{(n+1)}\right) := \max\left\{n, \frac{n+1+\varphi(n+1)}{2}\right\} \quad \text{and} \quad \text{for} \quad j = 2, \cdots, n+1,$$

$$\psi\left(t_j^{(n+1)}\right) := t_{j-1}^{(n+1)},$$

and $\psi|_{(n,n+1]}$ is the polygonal line with nodes $(n, \psi(n))$ and $(t_j^{(n+1)}, \psi(t_j^{(n+1)}))$ for $j = 1, \cdots, n+1$.

Since all nodes of this polygonal line lie in the convex set $\{(0,0)\} \cup \{(x,y) : x > 0, \quad 0 < y < x\}$, we may infer that $\psi(t) < t$ for $t > 0$. Since for all $n \in \mathbb{N}$ $\psi(n-1) < \psi(t_1^{(n)})$ and both $(t_j^{(n)})_{j=1}^n$ and $(\psi(t_j^{(n)}))_{j=1}^n$ are strictly increasing, so is ψ. Obviously, ψ is continuous. Moreover,

$$\psi^n(n) \geq \frac{n+\varphi(n)}{2} > \varphi(n) \quad \text{for all} \quad n \in \mathbb{N}$$

so ψ has all the properties we need. □

In the sequel we compare classes $\mathbf{C}_\varphi$ and **Ba**.

Theorem 2. *Let a function φ be non-decreasing. The following statements are equivalent:*

(i) *classes* **Ba** *and* $\mathbf{C}_\varphi$ *are iteratively equivalent;*

(ii) $\inf\ \{\varphi(t)/t : t > 0\} > 0$ *and there exists* $k \in \mathbb{N}$ *such that* $\sup\ \{\varphi^k(t)/t : t > 0\} < 1$.

PROOF. (i) $\Rightarrow$ (ii). Consider the Euclidean space $(\mathbb{R}_+, d_e)$ and the map f defined by $fx := x/2$ for $x \in \mathbb{R}_+$. Obviously, $f \in \mathbf{Ba}$ so by (i), there is $k \in \mathbb{N}$ such that $f^k \in \mathbf{C}_\varphi$. In particular,

$$\frac{x}{2^k} = d_e(f^k x, f^k 0) \le \varphi(d_e(x,0)) = \varphi(x) \quad \text{for all} \quad x \in \mathbb{R}_+.$$

Hence $\varphi(x)/x \ge 1/2^k$ for $x > 0$ and since k does not depend on x, we may infer that $\inf\{\varphi(t)/t : t > 0\} > 0$.

Now consider the ultrametric space (X, d) with $X := \mathbb{R}_+$ and the map f as in Lemma 1. Then $f \in \mathbf{C}_\varphi$ so by (i), there exist $k \in \mathbb{N}$ and $\alpha \in (0,1)$ such that f^k is an α-contraction. In particular,

$$\varphi^k(x) = d(f^k x, f^k 0) \le \alpha\, d(x,0) = \alpha\, x \quad \text{for all} \quad x \in \mathbb{R}_+,$$

which implies that $\sup\{\varphi^k(t)/t : t > 0\} \le \alpha < 1$. Therefore, (ii) holds.

(ii) $\Rightarrow$ (i). Let $f \in \mathbf{C}_\varphi$. By monotonicity of φ, f^k is φ^k-contractive and by (ii), $\varphi^k(t) \le \alpha\, t$, where $\alpha := \sup\{\varphi^k(t)/t : t > 0\} < 1$, which implies that $f^k \in \mathbf{Ba}$. On the other hand, if $f \in \mathbf{Ba}$ and α is a contractive constant of f then, by (ii), there is $k \in \mathbb{N}$ such that $\alpha^k \le \inf\{\varphi(t)/t : t > 0\}$. Then $f^k \in \mathbf{C}_\varphi$. □

A natural question arises whether in condition (ii) of Theorem 2 we could substitute the inequality "$\sup\{\varphi(t)/t : t > 0\} < 1$" for the condition "$\sup\{\varphi^k(t)/t : t > 0\} < 1$ for some $k \in \mathbb{N}$". Our Example 2 given below shows that, in general, this is not possible. Nevertheless, such a substitution can be made under some additional assumptions on a function φ as is done in the following lemma. The *right upper Dini derivative* of a function φ is denoted by $D^+\varphi$, that is,

$$(D^+\varphi)(s) := \limsup_{t \to s^+} \frac{\varphi(t) - \varphi(s)}{t - s}.$$

LEMMA 2. *Let a function φ be non-decreasing and such that $\sup\{\varphi(t)/t : t > 0\} = 1$. Then there exists a strictly monotonic sequence $(t_n)_{n=1}^\infty$ of positive reals such that $\lim_{n\to\infty} \varphi(t_n)/t_n = 1$. If $\lim_{n\to\infty} t_{n+1}/t_n = 1$, then for all $k \in \mathbb{N}$ $\sup\{\varphi^k(t)/t : t > 0\} = 1$.*

In particular, such a sequence $(t_n)_{n=1}^\infty$ exists if φ is right continuous and it satisfies one of the following conditions:

(a) φ *is differentiable at 0 and such that either there exists* $\lim_{t\to\infty} \varphi(t)/t$, *or* $\limsup_{t\to\infty} \varphi(t)/t < 1$;

(b) $(D^+\varphi)(0) < 1$ *and there exists* $\lim_{t\to\infty} \varphi(t)/t$.

PROOF. By hypothesis, there is a sequence (t_n) of positive reals such that $\lim \varphi(t_n)/t_n = 1$. Clearly, (t_n) cannot be constant since $\varphi(t)/t < 1$ for $t > 0$. By passing to a subsequence if necessary, we may assume that (t_n) is strictly monotonic, hence convergent to some $a \in \mathbb{R}_+ \cup \{\infty\}$. We will consider the case $t_n \searrow a$; then a similar argument can be used in the case, in which (t_n) is increasing. Let $\lim t_{n+1}/t_n = 1$. We show that there exists $\lim_{t\to a^+} \varphi(t)/t$. Let $s_n \to a^+$. There is a sequence (k_n) of positive integers such that $k_n \to \infty$ and $t_{k_n+1} \le s_n < t_{k_n}$ for sufficiently large n. Then for all such n we have:

$$\frac{\varphi(s_n)}{s_n} \le \frac{\varphi(t_{k_n})}{t_{k_n+1}} = \frac{\varphi(t_{k_n})}{t_{k_n}} \, \frac{t_{k_n}}{t_{k_n+1}} \longrightarrow 1;$$

$$\frac{\varphi(s_n)}{s_n} \ge \frac{\varphi(t_{k_n+1})}{t_{k_n}} = \frac{\varphi(t_{k_n+1})}{t_{k_n+1}} \, \frac{t_{k_n+1}}{t_{k_n}} \longrightarrow 1.$$

Hence we conclude that $\lim_{t\to a^+} \varphi(t)/t = 1$. In particular, $\varphi(t) \to a^+$ as $t \to a^+$ because of monotonicity of φ, so given an integer $k \ge 2$ also $\varphi^j(t) \to a^+$ as $t \to a^+$ for $j = 1, \cdots, k-1$. This easily implies that $\lim_{t\to a^+} \varphi^k(t)/t = 1$ since

$$\frac{\varphi^k(t)}{t} = \prod_{j=0}^{k-1} \frac{\varphi^{j+1}(t)}{\varphi^j(t)} \quad \text{for all} \quad t > 0,$$

whereas $\lim_{t\to a^+} \varphi(\varphi^j(t))/\varphi^j(t) = 1$. In particular, we conclude that $\sup \{\varphi^k(t)/t : t > 0\} = 1$ since by hypothesis $\varphi^k(t)/t < 1$ for $t > 0$.

In the sequel we assume that φ is right continuous. If (t_n) is a strictly monotonic sequence with $\lim \varphi(t_n)/t_n = 1$, then either (t_n) decreases to 0, or (t_n) increases to ∞; for otherwise, $t_n \to a \in (0, \infty)$ and then by monotonicity and right continuity of φ we have

$$\lim_{n\to\infty} \frac{\varphi(t_n)}{t_n} \le \limsup_{t\to a} \frac{\varphi(t)}{t} \le \frac{\varphi(a)}{a} < 1,$$

which yields a contradiction. To finish the proof consider the following two cases.

Assume (a). We will prove that $\lim_{t\to a} \varphi(t)/t = 1$ for either $a = 0$, or $a = \infty$. If $t_n \searrow 0$, then

$$1 = \lim_{n\to\infty} \frac{\varphi(t_n)}{t_n} = \varphi'(0) = \lim_{t\to 0} \frac{\varphi(t)}{t}.$$

If $t_n \nearrow \infty$, then $\limsup_{t\to\infty} \varphi(t)/t = 1$, whence, due to (a), there exists $\lim_{t\to\infty} \varphi(t)/t$. Consequently, in this case $\lim_{t\to\infty} \varphi(t)/t = 1$. Therefore the sequence $(1/n)_{n=1}^{\infty}$ if $a = 0$, or $(n)_{n=1}^{\infty}$ if $a = \infty$, has the required property.

If condition (b) holds, then each sequence (t_n) satisfying $\lim \varphi(t_n)/t_n = 1$ converges to the infinity. Since $\lim_{t\to\infty} \varphi(t)/t$ exists, it equals 1 and then, for example, the sequence $(n)_{n=1}^{\infty}$ has the property we need. □

Recall that a function φ is *subadditive* if

$$\varphi(s+t) \le \varphi(s) + \varphi(t) \quad \text{for all} \quad s, t \in \mathbb{R}_+;$$

φ is *superadditive* if the reverse inequality holds.

LEMMA 3. *Let a function φ be non-decreasing and subadditive. Then φ is continuous and φ satisfies condition (a) of Lemma 2. Moreover, $\lim_{t\to\infty} \varphi(t)/t < 1$.*

PROOF. Since $\varphi(0) = 0$ and φ is continuous at 0, we may conclude by [15, Remark 1] that φ is continuous. Further, by [5, Theorem 7.11.1] there exists $\lim_{t\to 0+} \varphi(t)/t$. Hence φ is differentiable at 0. Finally, by [5, Theorem 7.6.1], there exists $\lim_{t\to\infty} \varphi(t)/t$. Moreover,

$$\lim_{t\to\infty} \varphi(t)/t = \inf\{\varphi(t)/t : t > 0\} \le \varphi(1) < 1.$$

□

LEMMA 4. *Let a function φ be superadditive. Then φ satisfies condition (b) of Lemma 2.*

PROOF. Observe that φ is non-decreasing. By [5] there exist both limits $\lim_{t\to 0+} \varphi(t)/t$ and $\lim_{t\to\infty} \varphi(t)/t$. Moreover,

$$\varphi'(0) = \lim_{t\to 0+} \varphi(t)/t = \inf\{\varphi(t)/t : t > 0\} \le \varphi(1) < 1$$

and thus (b) is satified. □

As an immediate consequence of Theorem 2 and Lemmas 2–4, we obtain the following

COROLLARY 1. *Assume that a function φ is non-decreasing, right continuous and φ satisfies either condition (a), or condition (b) of Lemma 2. The following statements are equivalent:*

(i) *classes* **Ba** *and* $\mathbf{C}_\varphi$ *are iteratively equivalent;*

(ii) *there exist* $\alpha, \beta \in (0,1)$ *such that*

$$\alpha\, t \leq \varphi(t) \leq \beta\, t \quad \text{for all} \quad t \in \mathbb{R}_+.$$

In particular, conditions (i) and (ii) are equivalent if either φ is non-decreasing and subadditive, or φ is superadditive and right continuous.

(Incidentally, it can be proved that in the last statement of Corollary 1 it suffices to assume that φ is superadditive (not necessarily right continuous).)

Also with a help of Lemma 2 we can easily deduce that classes **Br** and **Ba** are not iteratively equivalent. Moreover, we give below a complete characterization of these subadditive or superadditive functions φ, for which there exists a φ-contractive map with the property that none of its iterates is a Banach contraction.

COROLLARY 2. *Let a function φ be non-decreasing and subadditive. The following statements are equivalent:*

(i) $\varphi'(0) = 1$;

(ii) *there exists a map* $f \in \mathbf{C}_\varphi$ *such that for all* $k \in \mathbb{N}$ $f^k \notin$ **Ba**.

Hence classes **Ba** *and* **Br** *are not iteratively equivalent.*

PROOF. (i) $\Rightarrow$ (ii). (i) implies that $\sup\{\varphi(t)/t : t > 0\} = 1$. By Lemmas 2 and 3, we conclude that for all $k \in \mathbb{N}$, $\sup\{\varphi^k(t)/t : t > 0\} = 1$. Consider the ultrametric space (X, d) with $X := \mathbb{R}_+$ and the map f as in Lemma 1. Then $f \in \mathbf{C}_\varphi$. Suppose that $f^k \in$ **Ba** for some $k \in \mathbb{N}$. Then

$$\varphi^k(x) = d(f^k x, f^k 0) \leq \alpha\, d(x, 0) = \alpha\, x$$

for some $\alpha \in (0,1)$, which implies that $\sup\{\varphi^k(t)/t : t > 0\} < 1$, a contradiction. Therefore (ii) holds.

(ii) $\Rightarrow$ (i). By [5, Theorem 7.11.1] $\varphi'(0)$ exists and

$$\varphi'(0) = \sup\{\varphi(t)/t : t > 0\}. \tag{1}$$

Hence $\varphi'(0) = 1$; for otherwise, each map $f \in \mathbf{C}_\varphi$ would be an α-contraction with $\alpha := \varphi'(0)$ because of the inequality $\varphi(t) \leq \alpha\, t$ for all $t \in \mathbb{R}_+$, and this violates (ii).

Finally, observe that $\mathbf{C}_\varphi \subseteq \mathbf{Br}$, since by Lemma 3 φ is continuous. Hence and by (i) $\Rightarrow$ (ii) we may deduce the last statement of Corollary 2.

□

COROLLARY 3. *Let a function φ be superadditive and right continuous. The following statements are equivalent:*

(i) $\lim_{t\to\infty} \varphi(t)/t = 1$;

(ii) *there exists a map $f \in \mathbf{C}_\varphi$ such that for all $k \in \mathbb{N}$ $f^k \notin \mathbf{Ba}$.*

PROOF. (i) $\Rightarrow$ (ii). (i) implies that $\sup\{\varphi(t)/t : t > 0\} = 1$. Then Lemmas 2 and 4 give that for all $k \in \mathbb{N}$ $\sup\{\varphi^k(t)/t : t > 0\} = 1$. To show that (ii) holds it suffices to consider the same function f defined as in the proof of Corollary 2 ((i) $\Rightarrow$ (ii)).

That (ii) implies (i) follows from [10, Theorem 4.7] (it suffices to assume that f is not a Banach contraction). □

The following example shows that we cannot drop the assumption "$\lim t_{n+1}/t_n = 1$" in Lemma 2. Moreover, also we cannot omit the assumptions on a behaviour of $\varphi(t)/t$ as t tends to the infinity, in conditions (a) and (b) of the same lemma. The function φ defined below is even a homeomorphism from $\mathbb{R}_+$ onto $\mathbb{R}_+$, differentiable at 0 with $\varphi'(0) < 1$, and φ has the property that $\sup\{\varphi(t)/t : t > 0\} = 1$, but $\sup\{\varphi^2(t)/t : t > 0\} < 1$. Moreover, $\inf\{\varphi(t)/t : t > 0\} > 0$ so by Theorem 2, classes $\mathbf{C}_\varphi$ and $\mathbf{Ba}$ are iteratively equivalent. Thus, in general, condition (ii) of Corollary 1 is not necessary for the iterative equivalence of classes $\mathbf{C}_\varphi$ and $\mathbf{Ba}$.

EXAMPLE 2. Let φ be the polygonal line with nodes $(0,0), (2^{n+1}-1, 2^n), (2^{n+1}, 2^{n+1}-1)$ for $n \in \mathbb{N}$. Since these nodes lie in the convex set $\{(0,0)\} \cup \{(x,y) : 0 < y < x\}$, we may infer that $\varphi(t) < t$ for all $t > 0$. Since for all $n \in \mathbb{N}$,

$$0 < 2^{n+1} - 1 < 2^{n+1} < 2^{n+2} - 1$$

and

$$\varphi(0) < \varphi(2^{n+1}-1) < \varphi(2^{n+1}) < \varphi(2^{n+2}-1),$$

we see that φ is strictly increasing. Obviously, φ is continuous. Since $\lim_{t\to\infty} \varphi(t) = \infty$, we conclude that φ is a homeomorphism from $\mathbb{R}_+$ onto $\mathbb{R}_+$. Clearly, $\sup\{\varphi(t)/t : t > 0\} \le 1$. Simultaneously, $\lim \varphi(2^{n+1})/2^{n+1} = 1$, which implies that $\sup\{\varphi(t)/t : t > 0\} = 1$ and $\limsup_{t\to\infty} \varphi(t)/t = 1$. Further, φ is differentiable at 0 and $\varphi'(0) = 2/3$.

We will estimate $\sup\{\varphi^2(t)/t : t > 0\}$. For $t \in [0,3]$, $\varphi^2(t)/t = 4/9$. Assume that $t \in [2^{n+1}-1, 2^{n+1}]$. By monotonicity of φ^2,

$$\frac{\varphi^2(t)}{t} \leq \frac{\varphi^2(2^{n+1})}{2^{n+1}-1} = \frac{2^n}{2^{n+1}-1} \leq \frac{2}{3}$$

since the sequence $(2^n/(2^{n+1}-1))_{n=1}^{\infty}$ is decreasing. Hence we get that

$$\sup\left\{\frac{\varphi^2(t)}{t} : t \in \bigcup_{n\in N} [2^{n+1}-1, 2^{n+1}]\right\} \leq 2/3.$$

(In fact, elementary computations show that this supremum equals $1/2$.)

Assume that $t \in [2^{n+1}, 2^{n+2}-1]$. Then $\varphi(t) \in [2^{n+1}-1, 2^{n+1}]$. This time $\varphi^2(2^{n+2}-1)/2^{n+1} \to 1$ and we cannot use a similar argument as in the preceding case. Elementary computations show that

$$\varphi(t) = \frac{t}{2^{n+1}-1} + \frac{2^{2n+2}-3\cdot 2^{n+1}+1}{2^{n+1}-1}, \tag{2}$$

whereas for $t \in [2^{n+1}-1, 2^{n+1}]$,

$$\varphi(t) = (2^n-1)\,t + 2^{n+2} - 2^{2n+1} - 1. \tag{3}$$

By (2) and (3) one can obtain that for $t \in [2^{n+1}, 2^{n+2}-1]$,

$$\varphi^2(t) = \frac{2^n-1}{2^{n+1}-1}\,t + \frac{2^n}{2^{n+1}-1}.$$

Hence, $\max\{\varphi^2(t)/t : t \in [2^{n+1}, 2^{n+2}-1]\} = \varphi^2(2^{n+1})/2^{n+1} = 1/2$.

Combining all the above cases we get that $\sup\{\varphi^2(t)/t : t > 0\} < 1$ (in fact, this supremum equals $1/2$).

Finally, we compute $\inf\{\varphi(t)/t : t > 0\}$. For $t \in [0,3]$, $\varphi(t)/t = 2/3$. By (3),

$$\min\left\{\frac{\varphi(t)}{t} : t \in [2^{n+1}-1, 2^{n+1}]\right\} = \frac{\varphi(2^{n+1}-1)}{2^{n+1}-1} = \frac{2^n}{2^{n+1}-1} \searrow \frac{1}{2}.$$

By (2),

$$\min\left\{\frac{\varphi(t)}{t} : t \in [2^{n+1}, 2^{n+2}-1]\right\} = \frac{\varphi(2^{n+2}-1)}{2^{n+2}-1} = \frac{2^{n+1}}{2^{n+2}-1} \searrow \frac{1}{2}.$$

Therefore we get that $\inf\{\varphi(t)/t : t > 0\} = 1/2$.

3. Iterative equivalence of classes Br and M

Throughout this section we assume that a function φ is non-decreasing.

J. Matkowski and J. Miś [14] gave an example of a function φ for which there exists a fixed-point free φ-contractive map. Clearly, for such a function φ there is a $t_0 > 0$ such that

$$\lim_{n\to\infty} \varphi^n(t_0) > 0. \tag{4}$$

With a help of Lemma 1 we can improve this result by showing that for e v e r y function φ satisfying (4) there exists a φ-contractive map, which has no fixed points.

PROPOSITION 3. *Given a function φ, the following statements are equivalent:*

(i) $\lim_{n\to\infty} \varphi^n(t) = 0$ *for all* $t \in \mathbb{R}_+$*;*

(ii) *given a complete metric space (X, d) and a φ-contractive selfmap f of X, f has a fixed point.*

PROOF. The implication (i) $\Rightarrow$ (ii) follows from Matkowski's theorem (cf. [13, Theorem 1.2] or [3, Theorem 3.2, p. 12]). To prove (ii) $\Rightarrow$ (i) suppose, on the contrary, that for some $t_0 > 0$ (4) holds (this limit exists because of the assumption "$\varphi(t) < t$ for $t > 0$ and $\varphi(0) = 0$"). Denote this limit by r. Set $t_n := \varphi^n(t_0)$ for $n \in \mathbb{N}$. Then the sequence (t_n) is strictly decreasing and $t_n > r$. Define $X := \{t_n : n \in \mathbb{N}\}$ and consider the ultrametric space (X, d) as in Lemma 1. Clearly, $\varphi(X) \subseteq X$. Thus Lemma 1 implies that (X, d) is complete since $0 \notin \bar{X}$, and $f := \varphi|_X$ is a φ-contractive map. Obviously, f is fixed-point free, which violates (ii). □

Given a function φ, define the set $M_+(\varphi)$ by

$$M_+(\varphi) := \left\{ t > 0 : \lim_{s\to t^+} \varphi(s) = t \right\}.$$

LEMMA 5. *Assume that* $\lim_{n\to\infty} \varphi^n(t) = 0$ *for all* $t \in \mathbb{R}_+$*. Then the set* $M_+(\varphi^2)$ *is empty.*

PROOF. Suppose, on the contrary, that $t_0 \in M_+(\varphi^2)$. Since $\varphi^2(s) \le \varphi(s) < s$ for all $s > 0$, we may conclude that $\lim_{s\to t_0^+} \varphi(s) = t_0$, that is, $t_0 \in M_+(\varphi)$. By [8, Theorem 7] there is $\delta > 0$ such that for $s \in (t_0, t_0 + \delta)$

$\varphi(s) = t_0$. Hence

$$t_0 = \lim_{s \to t_0^+} \varphi^2(s) = \varphi(t_0) < t_0,$$

which yields a contradiction. □

THEOREM 3. *Assume that f is a Matkowski contraction on a metric space (X, d). Then f^2 is a Browder contraction. Hence, classes* **Br** *and* **M** *are iteratively equivalent.*

PROOF. By hypothesis, there is a function φ such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t \in \mathbb{R}_+$ and f is φ-contractive. Clearly, f^2 is φ^2-contractive because of monotonicity of φ. By Lemma 5, $M_+(\varphi^2) = \emptyset$ and [8, Theorem 5] implies that f^2 is Browder's contraction. On the other hand, **Br** $\subset$ **M** so we may conclude that these classes are iteratively equivalent. □

REMARK 1. In view of Proposition 2 and Theorem 3, the fixed point theorem of Matkowski [13] can be derived from Browder's theorem [2], though the class **Br** is a proper subclass of **M**.

4. Lack of iterative equivalence of classes M and BW

Given a function φ, define the set $M_-(\varphi)$ by

$$M_-(\varphi) := \left\{ t > 0 : \limsup_{s \to t^-} \varphi(s) = t \right\}.$$

The following result extends Theorem 3 in [8].

THEOREM 4. *Let a function φ be right upper semicontinuous. The following statements are equivalent:*

(i) *$M_-(\varphi) \neq \emptyset$;*

(ii) *there exist a complete metric space (X, d) and a φ-contractive selfmap of X such that for all $k \in \mathbb{N}$ f^k is not a Matkowski contraction;*

(iii) *there exist a metric space (X, d) and a φ-contractive selfmap f of X such that f is not a Browder contraction.*

Hence, classes **M** *and* **BW** *are not iteratively equivalent.*

PROOF. The implication (ii) $\Rightarrow$ (iii) is obvious since **Br** $\subset$ **M**. (iii) $\Rightarrow$ (i) follows from [8, Theorem 3]. We prove that (i) implies (ii). We will use the same argument as in the proof of Theorem 3 in [8]. By (i), there is a strictly increasing sequence $(t_n)_{n=1}^{\infty}$ and $t_0 > 0$ such that $t_n \nearrow t_0$ and $\varphi(t_n) \nearrow t_0$.

Without loss of generality we may assume, by passing to a subsequence if necessary, that $\varphi(t_{n+1}) > t_n$ for $n \in \mathbb{N}$. Set $X := \{t_n : n \in \mathbb{N}\}$ and consider the ultrametric space (X, d) as in Lemma 1. Since $t_n \geq t_1 > 0$, $0 \notin \bar{X}$ so Lemma 1 implies that (X, d) is complete. Further, define a map f by

$$ft_1 := t_1 \quad \text{and} \quad ft_{n+1} := t_n \quad \text{for} \quad n \in \mathbb{N}.$$

Obviously, f is a selfmap of X. If $m, n \in \mathbb{N}$ and $m > n$, then

$$d(ft_n, ft_m) \leq ft_m = t_{m-1} < \varphi(t_m) = \varphi(d(t_n, t_m))$$

so f is φ-contractive. Suppose, on the contrary, that for some $k \in \mathbb{N}$ $f^k \in \mathbf{M}$. Then there is a non-decreasing function $\psi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t \in \mathbb{R}_+$ and $d(f^k x, f^k y) \leq \psi(d(x, y))$. Hence and by monotonicity of ψ, we get in particular, that

$$0 < t_2 = d(f^{kn} t_{kn+2}, f^{kn} t_1) \leq \psi^n(d(t_{kn+2}, t_1)) \leq \psi^n(t_0),$$

which yields a contradiction. Thus none of iterates of f is Matkowski's contraction so (ii) holds.

Obviously, there exist a right continuous function φ satisfying (i) so by (i) $\Rightarrow$ (ii) we may conclude that classes $\mathbf{M}$ and $\mathbf{BW}$ are not iteratively equivalent. □

It turns out that there exists a Boyd-Wong contraction f with a stronger property than that given in condition (ii) of Theorem 4: not only $(f^k, d) \notin \mathbf{M}$ for all $k \in \mathbb{N}$, but also $(f^k, \rho) \notin \mathbf{M}$ for all $k \in \mathbb{N}$ and any metric ρ equivalent to d. The following theorem gives a complete characterization of functions φ, for which such a φ-contractive map exists. Clearly, such a function φ must satisfy condition (i) of Theorem 4.

THEOREM 5. *Let a function φ be right upper semicontinuous and $M_-(\varphi) \neq \emptyset$. The following statements are equivalent:*

(i) $\inf M_-(\varphi) = 0$;

(ii) *there exist a complete metric space (X, d) and a φ-contractive selfmap f of X such that for all $k \in \mathbb{N}$ and any metric ρ, which induces a weaker topology on X than d does, f^k is not a Matkowski contraction on (X, ρ);*

(iii) *there exist a complete metric space (X, d) and a φ-contractive selfmap f of X such that for any metric ρ equivalent to d f is not a Browder contraction.*

PROOF. The implication (ii) $\Rightarrow$ (iii) is obvious. (iii) $\Rightarrow$ (i) follows from [8, Theorem 4]. We show that (i) implies (ii). Again, we will use the same construction as in [8]. By (i) we may conclude that there exist a strictly decreasing sequence $(t_k)_{k=1}^{\infty}$ and strictly increasing sequences $(t_n^{(k)})_{n=1}^{\infty}$ ($k \in \mathbb{N}$) such that

$$t_k \in M_-(\varphi), \quad t_n^{(k)} > t_{k+1}, \quad t_k \searrow 0, \quad t_n^{(k)} \nearrow t_k \quad \text{as} \quad n \to \infty \quad (k \in \mathbb{N}),$$

$$\varphi\left(t_1^{(k)}\right) > t_{k+1}, \quad \varphi\left(t_{n+1}^{(k)}\right) > t_n^{(k)} \quad (k, n \in \mathbb{N}), \quad \varphi\left(t_n^{(k)}\right) \nearrow t_k \quad \text{as} \quad n \to \infty.$$

Set $X := \left\{t_n^{(k)} : k, n \in \mathbb{N}\right\} \cup \{0\}$. Consider the ultrametric space (X, d) as in Lemma 1. Since $0 \in X$, Lemma 1 implies that (X, d) is complete. Further, define a map f on X by

$$f0 := 0, \ ft_{n+1}^{(k)} := t_n^{(k)} \quad \text{and} \quad ft_1^{(k)} := t_1^{(k+1)} \quad \text{for} \quad k, n \in \mathbb{N}.$$

We show that f is φ-contractive. Let $x, y \in X$. We may assume, without loss of generality, that $x < y$. Then $d(x, y) = y$. The following cases are possible.

1. $x = t_n^{(k)}$, $y = t_m^{(k)}$ and $m > n$. Then

$$d(fx, fy) = t_{m-1}^{(k)} < \varphi\left(t_m^{(k)}\right) = \varphi(d(x, y)).$$

2. $x = t_n^{(p)}$, $y = t_m^{(k)}$, $p > k$ and $m \geq 2$. Then $d(fx, fy)$ can be estimated as in case 1.

3. $x = t_n^{(p)}$, $y = t_1^{(k)}$ and $p > k + 1$. Then

$$d(fx, fy) = t_1^{(k+1)} < t_{k+1} < \varphi\left(t_1^{(k)}\right) = \varphi(d(x, y)).$$

4. $x = t_n^{(k+1)}$, $y = t_1^{(k)}$. If $n \geq 2$, then

$$d(fx, fy) = t_{n-1}^{(k+1)} < t_{k+1} < \varphi\left(t_1^{(k)}\right) = \varphi(d(x, y)).$$

If $n = 1$, then $d(fx, fy)$ can be estimated as in case 3.

5. $x = 0$, $y = t_m^{(k)}$. If $m \geq 2$ (resp., $m = 1$), then $d(fx, fy)$ can be estimated as in case 1 (resp., case 3).

Now suppose, on the contrary, that there exist $k \in \mathbb{N}$, a metric ρ, which induces a weaker topology on X than d does, and a non-decreasing function $\psi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ with $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t \in \mathbb{R}_+$, such that f^k is ψ-contractive on (X, ρ). Then there is $r > 0$ such that for $x \in X$, $d(x, 0) < r$ implies $\rho(x, 0) < 1$. Since $t_k \to 0$, there exists $j \in \mathbb{N}$ such that $t_j < r$. Then $t_n^{(j)} < r$ for all $n \in \mathbb{N}$, which implies that $\rho(t_n^{(j)}, 0) < 1$. Since f^k is ψ-contractive, we may conclude that, in particular,

$$0 < \rho\left(t_1^{(j)}, 0\right) = \rho\left(f^{kn} t_{kn+1}^{(j)}, f^{kn} 0\right) \leq \psi^n\left(\rho\left(t_{kn+1}^{(j)}, 0\right)\right) \leq \psi^n(1),$$

which yields a contradiction, since $\psi^n(1) \to 0$. $\square$

5. Remarks on the Meir-Keeler type contractions

Meir and Keeler [16] introduced the following class MK of maps, which also has the CFPP.

A selfmap f of a metric space (X, d) is a *Meir-Keeler contraction* ($(f, d) \in$ MK) if given $\varepsilon > 0$ there is $\delta > 0$ such that for all $x, y \in X$,

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \quad \text{implies that} \quad d(fx, fy) < \varepsilon. \tag{5}$$

REMARK 2. It can be easily verified that each Meir-Keeler contraction satisfies the inequality $d(fx, fy) < d(x, y)$ for all $x, y \in X$ with $x \neq y$. Hence (5) implies that $d(fx, fy) < \varepsilon$ for all $x, y \in X$ with $d(x, y) < \varepsilon + \delta$.

It is easy to show that BW $\subset$ MK. Moreover, Meir and Keeler gave an example of a map $f \in$ MK such that $f \notin$ BW. However, it can be easily verified that the map f from this example has the property that $f^2 \in$ BW (in fact, f^2 is a Banach contraction). Thus the following problem is opened.

QUESTION 1. Are classes MK and BW iteratively equivalent?

Subsequently, the result of Meir and Keeler was extended by Matkowski, who defined the following class Mt (cf. [11, Theorem 1.5.1]).

$(f, d) \in$ Mt if $d(fx, fy) < d(x, y)$ for all $x, y \in X$ with $x \neq y$, and given $\varepsilon > 0$ there is $\delta > 0$ such that for all $x, y \in X$,

$$\varepsilon < d(x, y) < \varepsilon + \delta \quad \text{implies that} \quad d(fx, fy) \leq \varepsilon. \tag{6}$$

It can be easily verified that MK $\subset$ Mt. Moreover, this inclusion is proper (cf. [7], in which also other Meir-Keeler type theorems are compared). However, it turns out that classes MK and Mt are iteratively equivalent according to the following

PROPOSITION 4. *If* $(f, d) \in$ Mt, *then* $(f^2, d) \in$ MK.

PROOF. Let $\varepsilon > 0$. Then there is $\delta > 0$ such that (6) holds. Let $\varepsilon \leq d(x, y) < \varepsilon + \delta$. If $d(x, y) = \varepsilon$ then, by hypothesis, $d(f^2x, f^2y) \leq d(fx, fy) < d(x, y)$ and hence $d(f^2x, f^2y) < \varepsilon$. So let $\varepsilon < d(x, y) < \varepsilon + \delta$. If $f^2x = f^2y$, then we are done. If $f^2x \neq f^2y$, then $fx \neq fy$ and by (6) we get that $d(f^2x, f^2y) < d(fx, fy) \leq \varepsilon$. Thus we may infer that $f^2 \in$ MK. □

A remarkable generalization of the Meir-Keeler theorem was given by Leader [12], who considered the following class Le.

$(f, d) \in \mathbf{Le}$ if f is continuous and given $\varepsilon > 0$ there exist $\delta > 0$ and $k \in \mathbb{N}$ such that for all $x, y \in X$

$$d(x,y) < \varepsilon + \delta \quad \text{implies that} \quad d(f^k x, f^k y) < \varepsilon. \tag{7}$$

Then **Le** has the CFPP. The essential novelty of this definition is that integer k may vary with ε. If a map f is a Meir-Keeler contraction then by Remark 2 f satisfies (7) with $k = 1$. On the other hand, the very special case of (7) with $k = 2$ covers the class **Mt**. We close the paper with the following

QUESTION 2. Are classes **Le** and **Mt** iteratively equivalent?

Since classes **Mt** and **MK** are iteratively equivalent and the relation of iterative equivalence is transitive, we may consider, in lieu of Question 2, the following equivalent

QUESTION 2'. Are classes **Le** and **MK** iteratively equivalent?

We suspect the answer is negative.

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REFERENCES

[1] D. W. Boyd and J. S. W. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc. **20** (1969), 458–464.

[2] F. E. Browder, *On the convergence of successive approximations for nonlinear functional equations*, Indag. Math. **30** (1968), 27–35.

[3] J. Dugundji and A. Granas, *Fixed Point Theory*, Polish Scientific Publishers, Warszawa 1982.

[4] R. Engelking, *General Topology*, Polish Scientific Publishers, Warszawa 1977.

[5] E. Hille and R. S. Philips, *Functional Analysis and Semigroups*, Vol. 31, Amer. Math. Soc. Colloq. Publ., Providence, R. I., 1957.

[6] J. Jachymski, *An iff fixed point criterion for continuous self-mappings on a complete metric space*, Aequationes Math. **48** (1994), 163–170.

[7] J. Jachymski, *Equivalent conditions and the Meir-Keeler type theorems*, J. Math. Anal. Appl. **194** (1995), 293–303.

[8] J. Jachymski, *Equivalence of some contractivity properties over metrical structures*, Proc. Amer. Math. Soc. **125** (1997), 2327–2335.

[9] J. Jachymski, *An extension of A. Ostrovski's theorem on the round-off stability of iterations*, Aequationes Math. **53** (1997), 242–253.

[10] J. Jachymski, *Some consequences of the Tarski-Kantorovitch ordering theorem in metric fixed point theory*, Quaestiones Math. **21** (1998), 89–99.

[11] M. Kuczma, B. Choczewski and R. Ger, *Iterative Functional Equations*, Encyclopedia of Mathematics and its Applications, Vol. 32, Cambridge Univ. Press, Cambridge, 1990.

[12] S. Leader, *Equivalent Cauchy sequences and contractive fixed points in metric spaces*, Studia Math. **66** (1983), 63–67.

[13] J. Matkowski, *Integrable solutions of functional equations*, Diss. Math. **127** (1975).

[14] J. Matkowski and J. Miś, *Examples and remarks to a fixed point theorem*, Facta Univ. (Niš) Ser. Math. Inform. **1** (1986), 53–56.

[15] J. Matkowski and T. Świątkowski, *On subadditive functions*, Proc. Amer. Math. Soc. **119** (1993), 187–197.

[16] A. Meir and E. Keeler, *A theorem on contraction mappings*, J. Math. Anal. Appl. **28** (1969), 326–329.

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