

THE PROBLEM OF CLASSIFICATION OF TRIANGULAR MAPS WITH ZERO TOPOLOGICAL ENTROPY

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To the memory of Professor György Targonski

Abstract. Relations between the following properties of triangular maps $F : I^2 \rightarrow I^2$ are studied in this paper: (P1) the period of any cycle is a power of two; (P2) every cycle is simple; (P3) the topological entropy of F restricted to the set of periodic points is 0; (P4) the topological entropy of F is 0; (P5) every ω -limit set contains a unique minimal set; (P6) F has no homoclinic trajectory; (P7) every ω -limit set either is a cycle or contains no cycle; (P8) no infinite ω -limit set contains a cycle. It is known that for continuous maps of the interval these properties are mutually equivalent. In the case of triangular maps of the square we prove that the properties (P1), (P2) and (P3) are mutually equivalent, and that (P7) is equivalent to (P8). Moreover, we show which of the remaining implications are true and which not. The problem is completely solved, with the following exceptions: we conjecture that (P7) \Rightarrow (P6) but we do not provide the argument; validity of (P7) \Rightarrow (P1) remains open. Our paper gives a partial solution of the problem stated in 1989 by A. N. Sharkovsky.

1. Definitions and preliminary results

As is well-known, there is a long list of properties characterizing continuous maps of the interval, with zero topological entropy. In 1989 A. N. Sharkovsky [13] asked the question which of these properties are equivalent

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in the case of triangular maps of the square. The problem is still open, regardless there are few partial results showing that some of these properties are not equivalent. In this paper we give a summarization of these results, and add new ones. We begin with a survey of known results and related notions.

In the sequel, $I = [0, 1]$ is the unit compact interval, I^2 the unit square, and X a compact metric space with a metric ρ . Let $C(X, X)$ be the set of continuous mappings of X into itself, and \mathbb{N} the set of positive integers. Let $\varphi^n(x)$ denote the n -th iteration of φ at x , for every $n \in \mathbb{N}$ and $x \in X$. Let $\text{Pr} : I^2 \rightarrow I$ be the projection onto x -axis, i.e., $\text{Pr}(x, y) = x$.

DEFINITION 1.1. Let $\varphi \in C(X, X)$, and let $x_n = \varphi^n(x_0)$, for $n \in \mathbb{N}$. The sequence $\{x_n\}_{n=0}^{\infty}$ is the *trajectory* of x_0 . If there is $k \geq 1$ such that $x_k = x_0$ and $x_n \neq x_0$ for every $n \in \{1, \dots, k-1\}$, then the set $\alpha = \{x_n\}_{n=0}^{k-1}$ is a *cycle* of φ , and k is the *period* of α . The set of accumulation points of the trajectory $\{x_n\}_{n=0}^{\infty}$ is the ω -*limit set* of the point x_0 , and it is denoted by $\omega_\varphi(x_0)$. Denote by $\text{Per}(\varphi)$ the union of the periodic orbits of φ , and by $\omega(\varphi)$ the union of the ω -limit sets of φ . A subset M of X is a *minimal set* if $M = \omega_\varphi(z)$, for any $z \in M$.

DEFINITION 1.2. Let $f : I \rightarrow I$ be a continuous function, and $g_x : \{x\} \times I \rightarrow I$, for $x \in I$, a system of mappings depending continuously on x . A continuous mapping $F : I^2 \rightarrow I^2$ such that $F(x, y) = (f(x), g_x(y))$, is a *triangular map*, f is the *base* of F , and the set $I_x := \{x\} \times I$ is the *layer* over x .

Throughout the paper, $F : I^2 \rightarrow I^2$ denotes a triangular map, and $f : I \rightarrow I$ its base.

DEFINITION 1.3. Let $Y(X, X)$ be a space of continuous mappings $X \rightarrow X$. We say that $Y(X, X)$ has the *Sharkovsky ordering property* if any $\varphi \in Y(X, X)$ possessing a cycle of period m has a cycle of period n , for every $n \prec m$ in the following ordering:

$$1 \prec 2 \prec 4 \prec 8 \prec \dots \prec 7 \cdot 2^k \prec 5 \cdot 2^k \prec 3 \cdot 2^k \prec \dots \prec 7 \cdot 2 \prec 5 \cdot 2 \prec \\ 3 \cdot 2 \prec \dots \prec 7 \prec 5 \prec 3.$$

Since the set of triangular maps has the Sharkovsky ordering property [12], the following classification applies also to triangular maps.

DEFINITION 1.4. Let $\varphi \in Y(X, X)$. Then

- (i) φ is of type less than 2^∞ if there is $k < \infty$ such that φ has cycles of periods $1, 2, 4, \dots, 2^k$ only;
- (ii) φ is of type 2^∞ if φ has a cycle of period 2^n for any n , and no other cycle;

(iii) φ is of type greater than 2^∞ if φ has a cycle of period $\neq 2^k$, $k = 0, 1, 2, \dots$

DEFINITION 1.5. (cf. [5], [7]). Let $\varphi \in C(X, X)$. Then $E \subset X$ is an (n, ε) -separated set if, for every two different points z_1, z_2 from E , there is a j , $0 \leq j < n$, with $\rho(\varphi^j(z_1), \varphi^j(z_2)) > \varepsilon > 0$. If M is a compact subset of X denote by $s_n(\varepsilon, M, \varphi)$ the maximum possible number of points in (n, ε) -separated subsets of M . Put $\bar{s}(\varepsilon, M, \varphi) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log s_n(\varepsilon, M, \varphi)$. The topological entropy of the map φ with respect to the compact subset M and the topological entropy of the map φ are defined by $h_\rho(\varphi, M) = \lim_{\varepsilon \rightarrow 0} \bar{s}(\varepsilon, M, \varphi)$ and $h_\rho(\varphi) = h_\rho(\varphi, X)$, respectively. If no confusion can arise we write h instead of h_ρ .

PROPOSITION 1.6. Let $\varphi \in C(X, X)$.

(i) If $A \subset B$ are invariant sets then $h(\varphi, A) \leq h(\varphi, B)$.

(ii) [1] If $A \subset X$ is an invariant, not necessarily compact set then $h(\varphi^n | A) = n \cdot h(\varphi | A)$, for every $n \geq 1$.

(iii) [10], [6] Let $\varphi = F$ and $X = I^2$. If $h(f) = 0$, then $h(F | \text{Per}(F)) \leq \sup_{x \in \text{Per}(f)} h(F | I_x)$.

(iv) [10], [5] Let $\varphi = F$ and $X = I^2$. Then $h(f) + \sup_{x \in I} h(F | I_x) \geq h(F) \geq \max \{h(f), \sup_{x \in I} h(F | I_x)\}$.

DEFINITION 1.7. Let $\varphi \in C(X, X)$ and $x \in X$ be a fixed point of φ . A sequence $\{x_n\}_{n=1}^\infty$ of distinct points in X such that $\varphi(x_{n+1}) = x_n$, for every $n \in \mathbb{N}$, $\varphi(x_1) = x$, and $\lim_{n \rightarrow \infty} x_n = x$, is a homoclinic trajectory related to the point x .

A sequence $\{y_n\}_{n=1}^\infty$ of distinct points in X such that $\varphi(y_{n+1}) = y_n$, for every $n \in \mathbb{N}$, $\varphi(y_1) = y_k$, for some $k \geq 1$ (i.e., $\{y_1, \dots, y_k\}$ is a cycle of period k), and $\lim_{n \rightarrow \infty} y_{kn+i} = y_i$ for $i = 1, 2, \dots, k$, is a homoclinic trajectory related to the cycle $\{y_1, \dots, y_k\}$.

DEFINITION 1.8. Let $\varphi \in C(I, I)$. Let $\alpha = \{x_1, x_2, \dots, x_{2^n}\} \subset I$, where $n \in \mathbb{N}$, be a cycle of φ with period 2^n such that $x_1 < x_2 < \dots < x_{2^n}$. Then α is a simple cycle of φ , if either $n = 0$ (i.e., x is a fixed point), or $n > 0$ and the sets $\{x_1, x_2, \dots, x_{2^{n-1}}\}$, $\{x_{2^{n-1}+1}, \dots, x_{2^n}\}$ are invariant sets with respect to φ^2 , and each of them is a simple cycle of φ^2 .

A cycle α of a triangular map F with period 2^k , $k \in \mathbb{N}$, is a simple cycle of F if $\text{Pr}(\alpha)$ is a simple cycle of the base f with period $2^n = m$, for some $n \leq k$, and for every $z \in \alpha \cap I_x$, where $x \in \text{Pr}(\alpha)$, $\{F^{im}(z); i = 1, 2, \dots, 2^{k-n}\} \subset I_x$ is a simple cycle of $F^m | I_x : I_x \rightarrow I_x$.

Recall that the notion of a simple cycle, for one-dimensional mappings, was introduced in [4].

THEOREM 1.9 (cf. [10]). *For $\varphi \in C(X, X)$, let $A(\varphi, X)$ denote either of the sets $\text{Per}(\varphi)$, $\omega(\varphi)$. Then $\text{Pr}(A(F, I^2)) = A(f, I)$.*

For $\varphi \in C(X, X)$, where $X = I$, or $X = I^2$, and φ is a triangular map, consider the following properties:

- (P1) period of any cycle is a power of two;
- (P2) every cycle is simple;
- (P3) $h(\varphi | \text{Per}(\varphi)) = 0$;
- (P4) $h(\varphi) = 0$;
- (P5) every ω -limit set contains a unique minimal set;
- (P6) φ has no homoclinic trajectory;
- (P7) every ω -limit set either is a cycle or contains no cycle;
- (P8) no infinite ω -limit set contains a cycle.

THEOREM 1.10. *For $\varphi \in C(I, I)$ the properties (P1)–(P8) are mutually equivalent.*

REMARK 1.11. Theorem 1.10 summarizes results proved by several authors. The list of properties equivalent to (P1)–(P8), in the case of one-dimensional maps, is more extensive. It contains more than 20 properties. For details see [9], [13] and [15].

The main aim of this paper is to classify the triangular maps of I^2 with respect to the properties (P1)–(P8). The following theorem shows that they are not equivalent.

THEOREM 1.12 (cf. [10]). *There is a continuous triangular map F of type 2^∞ with positive topological entropy. The base f is of type 2^∞ , and it has a unique infinite ω -limit set \mathcal{K} which is minimal. For every $x \in \mathcal{K}$, $y \in I$ and $n \in \mathbb{N}$, $F^n(x, y) = (f^n(x), g_x^n(y))$, where $g_x^n(y) = 2^{-i(n, x)}g(y)$, g is the tent map $g(y) = 1 - |2y - 1|$, and $i(n, x)$ are non-negative integers such that $\limsup_{n \rightarrow \infty} i(n, x) = \infty$. It follows that $\liminf_{n \rightarrow \infty} g_u^n(v) = 0$, whenever $u \in \mathcal{K}$ and $v \in I$. Consequently, $M = \mathcal{K} \times \{0\}$ is the unique minimal set contained in $\omega_F(u, v)$.*

Note that the original Kolyada's construction [10] gives a map F which is not differentiable at infinitely many points. In [3] there is constructed a C^k -differentiable map with the similar properties, for any $k \in \mathbb{N}$. In [11] there is constructed a C^∞ -map with the similar properties.

THEOREM 1.13 (cf. [8]). *There is a triangular map F with the following properties: F has zero topological entropy, and possesses a recurrent point which is not uniformly recurrent. The map f is of type 2^∞ and has a*

unique infinite ω -limit set \mathcal{K} which is minimal. Finally, $\omega_F(x, 0) = \mathcal{K} \times \{0\}$, $\omega_F(x, 1) = \mathcal{K} \times \{1\}$, and $\omega_F(x, \frac{1}{2}) \supset \mathcal{K} \times \{0, 1\}$, for any $x \in \mathcal{K}$.

LEMMA 1.14. Let $\alpha \subset I^2$ be a cycle of F and $Pr(\alpha) = \{a_1, \dots, a_n\}$. Assume that $F|_{I_{a_1} \cup \dots \cup I_{a_n}}$ has an ω -limit set containing more than one minimal set. Then the same is true for F .

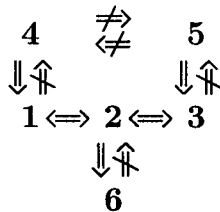
PROOF. Denote $G = F|_{I_{a_1} \cup \dots \cup I_{a_n}}$. Let $z_0 = (a_i, y_0)$ be a point such that $\omega_G(z_0)$ contains more than one minimal set, for some $i \in \{1, \dots, n\}$ and $y_0 \in I$. Let $z_1, z_2 \in \omega_G(z_0)$ be points such that $\omega_G(z_1)$ and $\omega_G(z_2)$ are disjoint subsets of $\omega_G(z_0)$. Then $\omega_F(z_1) \cup \omega_F(z_2) \subset \omega_F(z_0)$ and the sets $\omega_F(z_1), \omega_F(z_2)$ are disjoint. Since each of them contains a minimal set, $\omega_F(z_0)$ contains two minimal sets. □

2. Main results

In this section we are concerned with relations between the statements (P1)–(P8) in the case of triangular maps. Theorem 2.1 gives the complete list of implications between properties (P1)–(P6). Theorem 2.2 contains implications between (P7)–(P8) and relations between them and the first six ones; however, there are still open problems. Proofs of these two theorems are given in Sections 3 and 4, respectively.

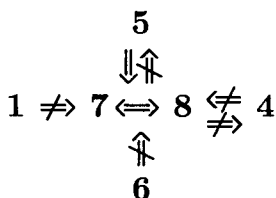
THEOREM 2.1. For any triangular map F , the properties (P1), (P2) and (P3) are mutually equivalent, and each implies but is not implied by (P6). The properties (P4) and (P5) are mutually independent, and each implies but is not implied by (P1).

The statement can be displayed by the following diagram:



THEOREM 2.2. For any triangular map F , the properties (P7) and (P8) are equivalent. They are independent of (P4), weaker than (P5), and are not implied by (P1) either (P6).

The theorem is illustrated by the following diagram (the implications given by Theorem 2.1 are not displayed):



PROBLEM 2.3. Theorems 2.1 and 2.2 give complete classification, with the following exceptions: We cannot decide if the implication (P7) \Rightarrow (P1) is true, and we conjecture that (P7) \Rightarrow (P6) is true but we do not provide a proof.

3. Proof of Theorem 2.1

LEMMA 3.1. (1 \Leftrightarrow 2) *The period of any cycle of F is a power of 2 if and only if every cycle is simple.*

PROOF. Assume that there is a cycle α of F with period $k = 2^s$, $s \in \mathbb{N}$, which is not simple (if k is not a power of 2 the assertion is trivial). Then $\beta = \text{Pr}(\alpha)$ has period $m = 2^n$, for some n . If β is not a simple cycle of f then, according to Theorem 1.10, f has a cycle χ of period p which is not a power of 2. Consider a mapping $F^p|_{I_u} : I_u \rightarrow I_u$, $u \in \chi$; it has a fixed point z . Then z is a periodic point of F with period p which is not a power of 2. If β is a simple cycle of f then according to Definition 1.8 there is a point $z \in \alpha \cap I_x$, $x \in \beta$, such that $\{F^{im}(z); i = 1, 2, \dots, 2^{s-n}\} \subset I_x$ is not a simple cycle of a one-dimensional mapping $F^m|_{I_x} : I_x \rightarrow I_x$. Theorem 1.10 implies that $F^m|_{I_x}$ has a periodic orbit of period $p \neq 2^n$, for any n . Thus, F has a cycle of period mp which is not a power of 2.

The converse implication is obvious by Definition 1.8. \square

LEMMA 3.2. (1 \Leftrightarrow 3) *The period of any cycle of F is a power of 2 if and only if $h(F|\text{Per}(F)) = 0$.*

PROOF. Let the period of any cycle of F be a power of 2. In view of Theorem 1.9, the same is true for f . By Theorem 1.10, $h(f) = 0$, and by Proposition 1.6 (iii), $h(F|\text{Per}(F)) \leq \sup_{x \in \text{Per}(f)} h(F|I_x)$. Let x be a periodic point of f with period q . Denote $G := F^q|_{I_x} : I_x \rightarrow I_x$. In view of the fact that G is a one-dimensional map and F is of type at most 2^∞ , from Theorem 1.10 and Proposition 1.6 (ii) it follows that $0 = h(G) = h(F^q|I_x) = q \cdot h(F|I_x)$. Since x is an arbitrary periodic point, $\sup_{x \in \text{Per}(f)} h(F|I_x) = 0$, and hence, $h(F|\text{Per}(F)) = 0$.

Now assume that there is a cycle α of F with period $k \neq 2^n$, $n = 0, 1, 2, \dots$. If the period of $\text{Pr}(\alpha)$ is not a power of 2, then Theorem 1.10 and [10] imply that $h(F|\text{Per}(F)) \geq h(f|\text{Per}(f)) > 0$. If, on the other hand, the period of $\text{Pr}(\alpha)$ is 2^m , $m \in \mathbb{N}$, let $G = F^{2^m}|_{I_x}$, $x \in \text{Pr}(\alpha)$. This one-dimensional mapping has a cycle with period $k/2^m$, which is not a power of 2. By Theorem 1.10, $h(G|\text{Per}(G)) > 0$. Thus, by Proposition 1.6 (ii), $h(F|\text{Per}(F)) > 0$. \square

LEMMA 3.3. (5 \Rightarrow 1) *If every ω -limit set contains a unique minimal set then the period of every cycle is a power of 2.*

PROOF. Assume that α is a cycle of F of period $k \neq 2^n$, $n = 0, 1, 2, \dots$. Let $\beta = \text{Pr}(\alpha)$. If the period of β is not a power of 2 then, according to Theorem 1.10, there is a point $x_0 \in I$ such that $\omega_f(x_0)$ has two different minimal sets. Thus, there are $x_1, x_2 \in \omega_f(x_0)$ such that $\omega_f(x_1)$ and $\omega_f(x_2)$ are disjoint subsets of $\omega_f(x_0)$. Clearly, $\text{Pr}(\omega_F(z_0)) = \omega_f(x_0)$, for any $z_0 \in I_{x_0}$. Take arbitrary $z_1 \in I_{x_1} \cap \omega_F(z_0)$ and $z_2 \in I_{x_2} \cap \omega_F(z_0)$. Then $\omega_F(z_1)$, $\omega_F(z_2)$ are disjoint subsets of $\omega_F(z_0)$. To finish the argument note that each compact invariant set contains a minimal set.

If, on the other hand, the period of β is 2^m , for some $m \in \mathbb{N}$, fix an $x \in \beta$ and for every $y \in I$ put $G(y) = F^{2^m}(x, y)$. Then G is a one-dimensional mapping $I \rightarrow I$, and any $y \in I$ such that $(x, y) \in \alpha$ is its periodic point of period $k/2^m$, which is not a power of 2. According to Theorem 1.10 there is a point $y_0 \in I$ such that $\omega_G(y_0)$ contains more than one minimal set. By Lemma 1.14, there is an ω -limit set of F containing more than one minimal set. \square

LEMMA 3.4. (1 \Rightarrow 6) *If the period of every cycle of F is a power of 2 then F has no homoclinic trajectory.*

PROOF. Let F have a homoclinic trajectory α . Without loss of generality we may assume that α is a homoclinic trajectory related to a fixed point, since otherwise it suffices to replace F by F^k where k is the period of the cycle related to α . Then $\text{Pr}(\alpha)$ is either a homoclinic trajectory or a fixed point. If $\text{Pr}(\alpha)$ is a homoclinic trajectory then, by Theorem 1.10, f has a cycle with a period that is not a power of 2 and thus, the same is true for F . If $\text{Pr}(\alpha)$ is a fixed point x , then $\alpha \subset I_x$ and $F|_{I_x}$ is a one-dimensional mapping that has a homoclinic trajectory. It follows that $F|_{I_x}$ has a cycle whose period is not a power of 2, and the same is true for F . \square

LEMMA 3.5. (6 $\not\Rightarrow$ 1) *There is a triangular map F with no homoclinic trajectory such that f has positive topological entropy.*

PROOF. Let $F(x, y) = (f(x), \frac{1}{3}y + \tau(x))$, where f is the tent map, and $\tau(x) \in [0, \frac{1}{2}]$, for any x . Let $\{\alpha_i\}_{i=1}^{\infty}$ be the sequence of all periodic orbits of f , let $\alpha_1 = \{0\}$. Put $M_1 = \{0, 1\}$, $\tau_1(0) = 0$ and $\tau_1(1) = \frac{1}{2}$. Then, for any τ such that $\tau|_{M_1} = \tau_1$, F has no homoclinic trajectory related to the fixed point $z = (0, 0)$ of F (which is the only periodic point of F with $\text{Pr}(z) = \alpha_1$). Indeed, let $\{z_n\}_{n=1}^{\infty}$ be such a trajectory. Then $z_1 = (1, y_1)$ and hence, $F(z_1) = (0, \frac{1}{3}y_1 + \frac{1}{2}) \neq z$, for any choice of y_1 . Since z has no improper preimage in I_0 , $F^n(z_1) \neq z$, for any n .

Now assume by induction that there is a finite set $M_k \subset I$ disjoint from α_{k+1} , and a mapping $\tau_k : M_k \rightarrow [0, \frac{1}{2}]$ such that, for any τ with $\tau|_{M_k} = \tau_k$, F has no homoclinic trajectory related to a periodic orbit, whose projection is α_i , $i = 1, \dots, k$. Let p be the period of α_{k+1} and $\{a_1, \dots, a_p\}$ the set of aperiodic preimages of the points in α_{k+1} . Fix sufficiently large integer $m > 0$ depending only on k (below we specify how large). Let $x \in f^{-m}(a_i)$. Then

$$(1) \quad \mu(F^m(I_x)) = \left(\frac{1}{3}\right)^m.$$

Put $N_{k+1} = \bigcup_{j=0}^{m-1} f^{-j}(\alpha_{k+1})$ and $M_{k+1} = M_k \cup N_{k+1}$. Let τ'_{k+1} be the linear extension of τ_k onto M_{k+1} . Define $\tau_{k+1} : M_{k+1} \rightarrow [0, \frac{1}{2}]$ such that

$$(2) \quad \|\tau'_{k+1} - \tau_{k+1}\| < \frac{1}{k+1},$$

and $\tau_{k+1}(x) = \tau'_{k+1}(x)$ whenever $x \notin \{a_1, \dots, a_p\}$. For each i , $1 \leq i \leq p$, $f(a_i) = q_i \in \alpha_{k+1}$. Let s_i denote the point in I such that (q_i, s_i) is a periodic point of F provided that $\tau|_{M_{k+1}} = \tau'_{k+1}$. Because of (1) we can choose $\tau_{k+1}(a_i) \in [0, \frac{1}{2}]$ preserving condition (2) such that, for any $x \in f^{-m}(a_i)$, $s_i \notin Y(x, m) + \tau_{k+1}(a_i)$, where $Y(x, m)$ is the y -projection of $\frac{1}{3}F^m(I_x)$, whenever m is sufficiently large, e.g., whenever $(\frac{2}{3})^m < \frac{1}{k+1}$. To finish the argument, let τ be a continuous map such that $\tau|_{M_k} = \tau_k$, for any k . Since $M = \bigcup_{k=1}^{\infty} M_k$ is a dense subset of I , by the above construction (cf. also (2)) such a τ exists and is uniquely determined.

Now if F would have a homoclinic trajectory $\{z_n\}_{n=1}^{\infty}$ related to a cycle of F in $\alpha_k \times I$, then there is a minimal m such that $z_{m+1} \in M_k \times I$ is eventually periodic, which is by our construction impossible. \square

LEMMA 3.6. (5 \nRightarrow 4) *There is a triangular map F with positive topological entropy such that every ω -limit set of F contains a unique minimal set.*

PROOF. Consider the mapping F from Theorem 1.12, and let W be an infinite ω -limit set of F . If $\text{Pr}(W)$ is infinite then $\text{Pr}(W) = \mathcal{K}$, since \mathcal{K} is the

unique infinite ω -limit set of f . By Theorem 1.12, W contains M , and hence, M is the unique minimal set contained in W . If $\text{Pr}(W) = \alpha$ is a cycle of period p , then $F : \bigcup_{x \in \alpha} I_x \rightarrow \bigcup_{x \in \alpha} I_x$ is (conjugate to) a one-dimensional map of type at most 2^∞ and W is its ω -limit set. By Theorem 1.10, W contains unique minimal set. \square

LEMMA 3.7. (4 $\not\Rightarrow$ 5) *There is a triangular map F with zero topological entropy and with an ω -limit set containing two minimal sets.*

PROOF. Consider the mapping F from Theorem 1.13. If $x \in \mathcal{K}$, $z = (x, \frac{1}{2})$, then $\omega_F(z) \supset \mathcal{K} \times \{0, 1\}$. Thus, $\omega_F(z)$ contains two minimal sets. \square

PROOF OF THEOREM 2.1. (P1) \Leftrightarrow (P2) \Leftrightarrow (P3) follows by Lemmas 3.1 and 3.2. Proposition 1.6 (i) and Theorem 1.12 imply (P4) \Rightarrow (P3) and (P1) $\not\Rightarrow$ (P4). By Lemma 3.3, (P5) \Rightarrow (P1), by Lemma 3.7 and the fact that (P4) \Rightarrow (P3 \Leftrightarrow P1), we have (P1) $\not\Rightarrow$ (P5). By Lemmas 3.6 and 3.7, (P4) and (P5) are mutually independent. Lemmas 3.4 and 3.5 imply that (P1) \Rightarrow (P6) and (P6) $\not\Rightarrow$ (P1).

4. Proof of Theorem 2.2

LEMMA 4.1. (7 \Leftrightarrow 8) *Every ω -limit set of a triangular map F either is a cycle or contains no cycle if and only if no infinite ω -limit set of F contains a cycle.*

PROOF. It follows by the well-known fact that any finite ω -limit set is a cycle [2]. \square

LEMMA 4.2. (5 \Rightarrow 8) *If every ω -limit set of F contains a unique minimal set then no infinite ω -limit set of F contains a cycle.*

PROOF. Assume that there is an infinite ω -limit set $\omega_F(z_0)$, $z_0 \in I^2$, containing a cycle α . By Theorem 1.9, $\text{Pr}(\omega_F(z_0)) = \omega_f(x)$, for some $x \in I$. Assume first that $\omega_f(x)$ is an infinite set containing a cycle $\text{Pr}(\alpha)$. Then, by Theorem 1.10, there is a point $v_0 \in I$ such that $\omega_f(v_0)$ contains two distinct minimal (and hence, disjoint) sets $\omega_f(v_1), \omega_f(v_2)$. According to Theorem 1.9 there are points $y_0, y_1, y_2 \in I$ such that $\text{Pr}(\omega_F(v_i, y_i)) = \omega_f(v_i)$, for $i = 0, 1, 2$, $\omega_F(v_1, y_1) \cap \omega_F(v_2, y_2) = \emptyset$ and $\omega_F(v_1, y_1) \cup \omega_F(v_2, y_2) \subset \omega_F(v_0, y_0)$.

Now let $\omega_f(x)$ be a finite set, i.e., a cycle $\text{Pr}(\alpha) = \{a_1, \dots, a_k\}$ with period k . Since $F(\omega_F(z_0)) = \omega_F(z_0)$, there is a sequence $\{u_n\}_{n=-\infty}^\infty$ such

that $\{u_n\}_{n=-\infty}^0 \subset \omega_F(z_0) \setminus \alpha$, $\{u_n\}_{n=1}^\infty \subset \omega_F(z_0)$, and $F(u_n) = u_{n+1}$. Let $v_F(u_0)$ denote the set of accumulation points of the sequence $\{u_n\}_{n=-\infty}^0$. Then $\omega_F(u_0)$, $v_F(u_0)$ are nonempty compact invariant subsets of $\omega_F(z_0)$. If $\alpha \not\subset \omega_F(u_0)$ or $\alpha \not\subset v_F(u_0)$ then $\omega_F(z_0)$ contains two disjoint compact invariant subsets of F , and hence, two minimal sets. If $\alpha \subset \omega_F(u_0)$ and $\alpha \neq \omega_F(u_0)$ then $G = F|_{I_{a_1} \cup \dots \cup I_{a_k}}$ is (conjugate to) a one-dimensional mapping possessing an infinite ω -limit set with a cycle and, by Theorem 1.10, two minimal sets. If $\omega_F(u_0) = \alpha = v_F(u_0)$ then according to [15, pg. 19], G has a homoclinic trajectory and, by Theorem 1.10 and Lemma 1.14, F has an ω -limit set containing two minimal sets. It remains to consider the case $v_F(u_0) \supset \alpha = \omega_F(u_0) \neq v_F(u_0)$. If $v_F(u_0)$ is finite then it contains two cycles and hence two minimal sets. Let $v_F(u_0)$ be infinite. If there is $z' \in v_F(u_0)$ such that $\omega_F(z') \neq \alpha$ then either $\omega_F(z') \cap \alpha = \emptyset$ or $\alpha \subset \omega_F(z')$, hence either $\omega_F(z_0)$ contains two minimal sets, or G has a homoclinic trajectory and F has an ω -limit set containing two minimal sets. So, assume that $\omega_F(z) = \alpha$, for every $z \in v_F(u_0)$. Then for any $z \in v_F(u_0)$ and any neighbourhood U of α , there is $y \in U$ and $m \in \mathbb{N}$ such that $F^m(y) = z$. This means that F has a homoclinic trajectory and, by Theorem 2.1, F has an ω -limit set containing two minimal sets. \square

LEMMA 4.3. (8 \nrightarrow 5) *There is a triangular map F which has an ω -limit set containing two minimal sets such that no infinite ω -limit set of F contains a cycle.*

PROOF. Consider the mapping F from Theorem 1.13. If W is an infinite ω -limit set of F , then $\text{Pr}(W) = \mathcal{K}$, which contains no cycle. On the other hand, $\omega_F(x, \frac{1}{2})$, for $x \in \mathcal{K}$, contains two minimal sets. \square

LEMMA 4.4. (4 \nrightarrow 8) *There is a triangular map F with zero topological entropy that has an infinite ω -limit set containing a cycle.*

PROOF. Define a triangular map $F(x, y) = (f(x), g_x(y))$ as follows. Let $f(x) = \lambda x$, where $\lambda \in (0, 1)$ is a constant. For $\delta \in (0, 1)$, let $\tau_\delta, \tau_\delta^* : I \rightarrow I$ be such that

$$\begin{aligned} \tau_\delta(x) &= (1 - \delta)x + \delta, \\ \tau_\delta^*(x) &= \begin{cases} 0, & \text{for } x \in [0, \delta], \\ \frac{1}{1-\delta}x + \frac{\delta}{\delta-1}, & \text{for } x \in (\delta, 1]. \end{cases} \end{aligned}$$

Clearly, $\tau_\delta^* \circ \tau_\delta$ is the identity on I . Now put $g_0(y) = y$, for $n = 0, 1, 2, \dots$,

$$g_{f^n(1)}(y) = \begin{cases} \tau_{1/(k+2)}(y), & \text{for } n_k \leq n < \frac{1}{2}(n_k + n_{k+1}), \\ \tau_{1/(k+2)}^*(y), & \text{for } \frac{1}{2}(n_k + n_{k+1}) \leq n < n_{k+1}, \end{cases}$$

for $x \in (f^{n+1}(1), f^n(1))$, $x = \lambda f^{n+1}(1) + (1 - \lambda)f^n(1)$, $\lambda \in (0, 1)$,

$$g_x(y) = \lambda g_{f^{n+1}(1)}(y) + (1 - \lambda)g_{f^n(1)}(y),$$

where $\{n_k\}_{k=0}^{\infty}$ is a sequence of non-negative even numbers such that $n_0 = 0$ and $\lim_{k \rightarrow \infty} \left(1 - \frac{1}{k+2}\right)^{(n_{k+1} - n_k)/2} = 0$. Since f and $F|_{I_x}$ are monotonic, for every $x \in I$, then by Proposition 1.6 (iv), $h(F) = 0$. Take a point $z = (1, 0) \in I^2$. Since $f^n(1) = \lambda^n$, we have $\omega_F(z) = \{0\} \times I$. Since $F(u) = u$, for every $u = (0, y) \in \{0\} \times I$, $\omega_F(z)$ contains infinitely many cycles. \square

PROOF OF THEOREM 2.2. (P7) \Leftrightarrow (P8) follows by Lemma 4.1. Lemmas 4.2 and 4.3 imply (P5) \Rightarrow (P8) and (P8) $\not\Rightarrow$ (P5). By Lemma 4.4, (P4) \Rightarrow (P8). By Lemmas 3.6 and 4.2, (P7) $\not\Rightarrow$ (P4). By the fact that (P4) \Rightarrow (P1) and Lemma 4.4, (P1) $\not\Rightarrow$ (P7). By the fact that (P4) \Rightarrow (P6) and Lemma 4.4, (P6) $\not\Rightarrow$ (P7). \square

REFERENCES

- [1] R. L. Adler, A. G. Konheim, M. H. McAndrew, *Topological entropy*, Trans. Amer. Math. Soc. 114 (1965), 309–319.
- [2] B. Barna, *Über die Iteration reeller Funktionen. I.*, Publ. Math. 7 (1960), 15–40.
- [3] F. Balibrea, F. Esquembre, A. Linero, *Smooth triangular maps of type 2^∞ with positive topological entropy*, Inter. Journal of Bifurcation and Chaos 5 (1995), 1319–1324.
- [4] L. S. Block, *Simple periodic orbits of mappings of the interval*, Trans. Amer. Math. Soc. 254 (1979), 391–398.
- [5] R. Bowen, *Entropy for group endomorphism and homogeneous spaces*, Trans. Amer. Math. Soc. 153 (1971), 401–414.
- [6] R. Bowen, *Topological entropy and axiom A*, Global Anal., Proc. Symp. in pure Math., Amer. Math. Soc. 14 (1970), 23–41.
- [7] E. I. Dinaburg, *Connection with various entropy characterizations of dynamical systems*, Izvestija AN SSSR 35 (1971), 324–366 (Russian).
- [8] G. L. Forti, L. Paganoni, J. Smítal, *Strange triangular maps of the square*, Bull. Austral. Math. Soc. 51 (1995), 395–415.
- [9] V. V. Fedorenko, A. N. Sharkovsky, J. Smítal, *Characterizations of weakly chaotic maps of the interval*, Proc. Amer. Math. Soc. 110 (1990), 141–148.
- [10] S. F. Kolyada, *On dynamics of triangular maps of the square*, Ergodic Theory & Dynam. Syst. 12 (1992), 749–768.
- [11] S. F. Kolyada, M. Misiurewicz, Ľ. Snoha, *Topological entropy of nonautonomous piecewise monotone dynamical systems on the interval*, Fund. Math. (to appear).
- [12] P. E. Kloeden, *On Sharkovsky's cycle coexistence ordering*, Bull. Austr. Math. Soc. 20 (1979), 171–177.

- [13] S. F. Kolyada, A. N. Sharkovsky, *On topological dynamics of triangular maps of the plane*, in Proceedings of ECIT-89, Batschuns, Austria, World Scientific, Singapore 1991, 177–183.
- [14] M. Misiurewicz, *Horseshoes for mappings of the interval*, Bull. Acad. Pol. Sci. **27** (1979), 167–169.
- [15] A. N. Sharkovsky, S. F. Kolyada, A. G. Sivak, V. V. Fedorenko, *Dynamics of one-dimensional mappings*, Mathematics and its Applications (Dordrecht). Kluwer Academic Publishers 1997.

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