RENORMALIZATION ON ITERATED CUBIC MAPS

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To the memory of Professor György Targonski

Abstract. This communication will discuss the dynamics of iterated cubic maps from the real line to itself, and will describe the renormalization of the parameter space for such maps using methods of symbolic dynamics.

1. Introduction

In the past two decades one-dimensional iterative maps have been subject of intense study. In spite of their structural simplicity, the dynamics of these simple nonlinear discrete dynamical systems are extremely rich and complex, providing thus a tool for the modeling and simulation of the dynamics of dissipative higher dimensional systems. We know now that many features observed in dissipative systems that typically occur in physics, chemistry and biology, can also be found in one-dimensional processes, the most well-known of which being, probably, the so called bifurcation route to chaos and its universal scaling laws, studied in detail in the literature, see [Mi 87].

Of particular relevance among one-dimensional iterative maps are the piecewise monotone maps of the interval, particularly those possessing a single smooth extremum. However, with an already reasonable understanding of the dynamics of that family of maps, recent research has focused some attention on the analysis of maps with two extrema, sometimes by trying to generalize results valid for quadratic maps. Following previous work, see

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[La-Se-SR 99], this is a report of recent advances on our attempt to get a generalization of the symbolic $\star$-product introduced by Derrida, Gervois and Pomeau, see [De-Ge-Po 78], for the family of quadratic maps of the interval.

We begin, in Section 2, with a brief summary of the basic tools of Milnor-Thurston's kneading theory, adapted for bimodal maps, i.e., the notions of address and itinerary of a point of the interval, kneading data of a bimodal map, even and odd bimodal sequences, lexicographical symbolic order, shift operator and admissible pair of sequences. It is assumed here that the reader is already familiar with these ideas (see [Mi-Th 88], [Co-Ec 80], [Ri-Tr 95] or [Mi-Tr 98], for details). Then, in Section 3, we present some results concerning the subset $\mathcal{F}BLMA$ of all kneading data obtained from the product of $BLMA$ with any other bimodal kneading data, namely, its admissibility and topological entropy. This case is important since we do not have to impose any condition, necessary, in the general situation, for the admissibility of the sequence resulting from the product, and thus we have that the subset $\mathcal{F}_{BLMA}$ is self-similar with the all set of bimodal kneading data $\mathcal{F}_{KS}$.

2. Bimodal maps and kneading theory

Consider a continuous map f from the interval $I = [c_0, c_3]$ into itself such that there exist two points c_1 and c_2 in its interior such that f is increasing in the subintervals $[c_0, c_1)$ and $(c_2, c_3]$ and decreasing in the subinterval (c_1, c_2). This is called a $\{+,-,+\}$ bimodal map, or a $\{+,-,+\}$-bimodal map. From [Sk et al 83], a suitable family of these maps is given by the two-parameter family of cubic maps

$$f_{ab}(x) = a\,x^3 + b\,x^2 + (1-a)\,x - b\,, \qquad a \neq 0\,,$$

where a and b are parameters such that, for the map f_{ab} to be a map of the interval $I = [-1, 1]$, (a, b) must lie inside the region $\Omega \subset \mathbb{R}^2$, whose boundary $\partial\Omega$ is given by the following curves of the plane assuming that the critical points c_1, c_2 and $f(c_1)$, $f(c_2)$ belong to the interval $[-1, 1]$:

$$b = \pm(2\sqrt{a} - a), \quad 1/4 \leq a \leq 4,$$
$$4(a - 1/2)^2 + 4/3\,b^2 = 1.$$

The critical points of f_{ab} can be easily computed, and we have that

$$c_{1,2} = (-b \mp \sqrt{b^2 + 3\,a\,(a-1)})/(3\,a).$$

The parameter-plane curves that correspond to the existence of pairs of superstable orbits of the critical points c_1 and c_2, i.e. $f^n(c_1) = c_1$ and

$f^m(c_2) = c_2$, or the existence of doubly superstable , i.e., $f^n(c_1) = c_2$ and $f^m(c_2) = c_1$, have been termed, respectively, the bones and the ligaments of the region in Ω where the orbits are stable. The analysis of such curves of the parameter region Ω was done by [Ri-Sc 91] in order to study the bifurcation structure of a family of bimodal maps of the interval.

Following [Mi-Th 88], one can code the dynamics of bimodal maps in the following way: consider the alphabet $\mathcal{A} = \{L, A, M, B, R\}$; we say that the address $\mathsf{A}(x)$ of a point x of the interval I is

$$\mathsf{A}(x) = \begin{cases} L & \text{if} \quad x \in [c_0, c_1) \\ A & \text{if} \quad x = c_1 \\ M & \text{if} \quad x \in (c_1, c_2) \\ B & \text{if} \quad x = c_2 \\ R & \text{if} \quad x \in (c_2, c_3] \end{cases} .$$

Then, we associate to an orbit of a point $x \in I$ the infinite symbolic sequence

$$\mathsf{I}(f, x) = \mathsf{A}(x)\mathsf{A}(f(x))\mathsf{A}(f^2(x)) \; \ldots \mathsf{A}(f^n(x)) \; \ldots$$

called the itinerary of x. When the map f is obvious from the context, we shall use the abbreviated notation $\mathsf{I}(x)$ for the itinerary of a point $x \in I$. The itineraries of the images of both critical points of the map play a special role within the kneading theory:

DEFINITION 1. Let f denote a bimodal map of the interval, with critical points c_1 and c_2. We designate by kneading data of f the pair of symbolic sequences $\mathcal{K}(f) = (\mathcal{K}^+(f), \mathcal{K}^-(f)) = (\; \mathsf{I}(f(c_1)), \mathsf{I}(f(c_2)))$, with $\mathcal{K}^+(f)$ and $\mathcal{K}^-(f)$ often called the kneading sequences of f.

The importance of the kneading data of a map $\mathcal{K}(f)$ lies in the sharp restrictions it imposes on which itineraries can actually occur for that map f, see [Mi-Tr 98]. But clearly, the key point of the kneading theory is the introduction of an order relation between symbolic sequences closely related with the order of the interval: take the following order on the alphabet $\mathcal{A}$, naturally induced from the order of the interval:

$$L \prec A \prec M \prec B \prec R.$$

The next step is to introduce a parity function $\rho(S)$, for any finite sequence S, as $+1$ if S contains an even number of symbols M, and -1 otherwise, and let $\underline{\mathcal{A}}$ denote the set of all sequences written with alphabet $\mathcal{A}$. Then, we can define an ordering $\prec$ on $\underline{\mathcal{A}}$ in the following manner:

DEFINITION 2. Given two sequences $P = P_1 P_2 \ldots$ and $Q = Q_1 Q_2 \ldots$ from $\underline{\mathcal{A}}$, let n be the first integer such that $P_n \neq Q_n$. Denote by $S =$

$S_1 \dots S_{n-1}$ the common first subsequence of both P and Q. Then, we say that $P \prec Q$ if $P_n \prec Q_n$ and $\rho(S) = +1$ or if $Q_n \prec P_n$ and $\rho(S) = -1$. If no such n exists, we say that $P = Q$.

One can easily see that this symbolic order relation is compatible with the order of the interval, in the sense that it satisfies $x < y \Longrightarrow \mathsf{I}(x) \preceq \mathsf{I}(y)$ and $\mathsf{I}(x) \prec \mathsf{I}\ (y) \Longrightarrow x < y$, for any two points x, y of the interval. With this lexicographical symbolic order one can characterize the elements of $\underline{A}$ that are admissible as kneading sequences of a bimodal map. But first, consider the shift map σ on the space of symbolic sequences, defined by $\sigma(S_1 S_2 S_3 \dots) = S_2 S_3 \dots$, and state that a sequence $S \in \underline{A}$ is minimal if $S \preceq \sigma^n(S)$ and maximal if $\sigma^n(S) \preceq S$, for all $n > 0$. Then, we will say that:

Proposition 1. *A pair (P, Q) of sequences is admissible, or realizable, as kneading data of a bimodal map if and only if the following three conditions are satisfied: (i) the sequence P is maximal, (ii) the sequence Q is minimal and (iii) they are such that $Q \preceq \sigma^n(P)$ and $\sigma^n(Q) \preceq P$, for all $n > 0$.*

Denote by $\mathcal{F}_{KS}$ the set of all kneading data of bimodal maps and adopt the following convention: given a kneading sequence S, truncate S after the first symbol A or B, if any. Thus, for example, the class of pairs of finite sequences of the form $(P_1 \dots P_{n-1} A,\ Q_1 \dots Q_{m-1} B)$ corresponds to the kneading data of a map whose critical points belong to two different periodic orbits, whereas the class of pairs of the form $(P_1 \dots P_{n-1} B,\ Q_1 \dots Q_{m-1} A)$ corresponds to the kneading data of a map with both critical points belonging to the same periodic orbit. It will be simpler to denote these kneading data as a unique sequence, i.e., as $P_1 \dots P_{n-1} B Q_1 \dots Q_{m-1} A$, being inferred from the context that it is actually a pair of kneading invariants.

To illustrate some of the concepts and results of bimodal kneading theory, we find convenient to construct a two-entry table, in which the minimal sequences will be placed in decreasing order and the maximal sequences in increasing order, considering only those corresponding to periodic kneading sequences with period smaller than a given one. Next, we give a table for maximal and minimal periodic sequences with period not greater than 4. But first, in order to be able to present a simplified table, consider the following correspondence between numbers and extremal, i.e., maximal and minimal, sequences:

n	1	2	3	4	5	6	7	8	9	10	11
row	B	RLLA	RLA	RLMA	RLB	RA	RMRA	RMB	RMMA	RMMB	RMA
col	A	LRRB	LRB	LRMB	LRA	LB	LMLB	LMA	LMMB	LMMA	LMB

n	12	13	14	15	16	17	18	19	20	21	22
row	RMLB	RMLA	RB	RRLA	RRLB	RRA	RRMB	RRMA	RRB	RRRA	RRRB
col	LMRB	LMRB	LA	LLRB	LLRA	LLB	LLMA	LLMB	LLA	LLLB	LLLA

Then, we have

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
1	○	•	•	•	•	•	•	○	•	○	•	•	•	○	•	•	•	○	•	○	•	○
2	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	○	•
3	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	○	•	○	•	○	•
4	•	•	•	•	•	•	•	•	○	•	○	•	•	•	•	•	○	•	○	•	○	•
5	•	•	•	•	•	•	•	○	•	○	•	•	•	○	•	•	•	○	•	○	•	○
6	•	•	•	•	•	○	○	•	○	•	○	•	•	•	•	•	○	•	○	•	○	•
7	•	•	•	•	•	○	○	•	○	•	○	•	•	•	•	•	○	•	○	•	○	•
8	○	•	•	•	○	•	•	○	•	○	•	○	•	○	•	○	•	○	•	○	•	○
9	•	•	•	○	•	○	○	•	○	•	○	•	•	•	•	•	○	•	○	•	○	•
10	○	•	•	•	○	•	•	○	•	○	•	○	•	○	•	○	•	○	•	○	•	○
11	•	•	•	○	•	○	○	•	○	•	○	•	•	•	•	•	○	•	○	•	○	•
12	•	•	•	•	•	•	•	○	•	○	•	○	•	○	•	○	•	○	•	○	•	○
13	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	○	•	○	•	○	•
14	○	•	•	•	○	•	•	○	•	○	•	○	•	○	•	○	•	○	•	○	•	○
15	•	•	•	•	•	•	•	•	•	•	•	•	•	•	○	•	○	•	○	•	○	•
16	•	•	•	•	•	•	•	○	•	○	•	○	•	○	•	○	•	○	•	○	•	○
17	•	•	○	○	•	○	○	•	○	•	○	•	○	•	○	•	○	•	○	•	○	•
18	○	•	•	•	○	•	•	○	•	○	•	○	•	○	•	○	•	○	•	○	•	○
19	•	•	○	○	•	○	○	•	○	•	○	•	○	•	○	•	○	•	○	•	○	•
20	○	•	•	•	○	•	•	○	•	○	•	○	•	○	•	○	•	○	•	○	•	○
21	•	○	○	○	•	○	○	•	○	•	○	•	○	•	○	•	○	•	○	•	○	•
22	○	•	•	•	○	•	•	○	•	○	•	○	•	○	•	○	•	○	•	○	•	○

where a circle indicates a pair of sequences that it is kneading data of a bimodal map and a point means simply a pair of sequences that does not satisfy the third condition of Proposition 1 (remember that both sequences are, *a priori*, extremal). It is worth noting that, despite the finite character of such table, it does not mean that our results are only valid for such strong restriction on the set of bimodal kneading data.

Although it was proven in [La-Se-SR 99] that the $\star$-product of two bimodal kneading data is always a bimodal kneading data, it is not yet clear if all *reducible* pairs of $\mathcal{F}_{KS}$ are the ones coming from that definition of $\star$-product. The main result of this report deals exactly with this problem, since we are going to study in detail the subset of kneading data $\mathcal{F}_{BLMA}$ one obtains from the product of $BLMA$ with any element of $\mathcal{F}_{KS}$. In fact, for this particular kneading data one finds that there is no need to consider any restriction for the $\star$-product to be admissible, therefore, the set $\mathcal{F}_{BLMA}$ will be self-similar with the set of bimodal kneading sequences $\mathcal{F}_{KS}$ [1)].

3. Self-similarity of the set $\mathcal{F}_{BLMA}$

It is known for some time that for each unimodal map with a periodic critical orbit of period k there exists a countably infinite number of unimodal maps with the same topological entropy and with a periodic critical orbit whose period is some multiple of k. However, for a bimodal family of maps f_{ab}, the subsets of the parameter space such that one of the critical points is periodic with a specified order type, are no longer points, as in the unimodal case, but smooth curves in the parameter plane. Moreover, in [Da-Ga-Mi-Tr 93] was conjectured that those subsets could not have any connected component which was a simple closed curve. This would immediately imply the connectedness of the topological entropy level sets or isentropes in the parameter region Ω and a certain monotonicity condition to the entropy of bimodal maps. But this is still an open question and therefore rises the interest to study bimodal maps with identical entropy.

The topological entropy h which was defined in [Ad-Ko-Mc 65] is an invariant of topological conjugacy. For multimodal maps, piecewise monotone maps, we can use the Misiurewicz–Szlenk–Rothschild result

$$h(f) = \lim_{k\to\infty} \frac{\log \ell(f^k)}{k}$$

where $\ell(f^{\ k})$ is the lap number of f^k. From [Mi-Th 88] we know already that, for the family of bimodal of maps, the topological entropy function $(a, b) \to h(f_{a,b})$ is continuous, with values in interval $[0, \log 3]$. Now, for each fixed $h \in [0, \log 3]$, define the h-entrope as the set of parameter values $(a, b) \in \Omega$ for which the topological entropy $h(f_{a,b})$ is constant and equal to h. For example, it is easy to verify that the $\log 3$-entrope is a single point of Ω , but one can expect countable many isentropes to be connected regions with nonempty interior and the rest just simple arcs. A nice monotonicity property would be to find a curve through each point of the parameter space Ω such that the dynamical complexity of the corresponding bimodal maps would increase monotonically. We think that, as for the unimodal family of maps, the introduction of a bimodal $\star$-product between kneading data can

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1) As should be evident from the symmetry of the symbolic set $\mathcal{F}_{KS}$, there is another subset of kneading data with analogous properties as this one: it is the subset $\mathcal{F}_{ARMB}$ of the kneading data obtained from the $\star$-product of $ARMB$ with any kneading data. All results valid for $\mathcal{F}_{BLMA}$ can be easily given for $\mathcal{F}_{ARMB}$.

be a good contribution to understand how the topological entropy change within a family of bimodal maps. But let us now present our results.

For simplicity, let us begin with the definition of Type 3 and Type 4 bimodal symbolic $\star$-product introduced in [La-Se-SR 99]. Denote by $\mathcal{F}_1$ the following subset of kneading data: $(S,T) \in \mathcal{F}_1$ if and only if $S \preceq RM^\infty$ and $LM^\infty \preceq T$.

DEFINITION 3. Let $PBQA$ be a kneading data, with P and Q both finite sequences of $\mathcal{B}$ and consider the following two possibilities:

Type 3. $\forall\,(XA,YB) \in \mathcal{F}_1$

$$PBQA\star(XA,YB) = (P\psi_P(X_1)Q\eta_Q(X_2)P\ldots A, Q\eta_Q(Y_1)P\psi_P(Y_2)Q\ldots B)\,,$$

Type 4. $\forall\, XBYA \in \mathcal{F}_1$

$$PBQA\star XBYA = P\psi_P(X_1)Q\eta_Q(X_2)P\ldots BQ\eta_Q(Y_1)P\psi_P(Y_2)Q\ldots A\,,$$

where X and Y are both finite sequences and ψ_P, η_Q are functions defined, for every sequence S of $\mathcal{B}$, by:

$$\eta_S(L) = \begin{cases} L & \text{if } \rho(S) = +1 \\ M & \text{if } \rho(S) = -1 \end{cases}\;;\quad \eta_S(M) = \begin{cases} M & \text{if } \rho(S) = +1 \\ L & \text{if } \rho(S) = -1 \end{cases}\;;$$

$$\psi_S(R) = \begin{cases} R & \text{if } \rho(S) = +1 \\ M & \text{if } \rho(S) = -1 \end{cases}\;;\qquad \psi_S(M) = \begin{cases} M & \text{if } \rho(S) = +1 \\ R & \text{if } \rho(S) = -1 \end{cases}\,.$$

It should be noted that the restriction imposed to the neading data appearing as second factor implies that $X_{even}, Y_{odd} \in \{L,M\}$ and $X_{odd}Y_{even} \in \{R,M\}$. Thus, it is enough to define the functions η_Q and ψ_P for those symbols. From [La-Se-SR 99] it is known that both $PBQA\star(XA,YB)$ and $PBQA\star XBYA$ are admissible kneading data. The important question now is that if we do $PBQA = BLMA$ there is no need to impose any condition to the second factor for the product to be a kneading data, as we will show, but it is not possible to write this extended product as was done in Definition 3. Therefore, we are forced to introduce a different definition for both types $BLMA\star(XA,YB)$ and $BLMA\star XBYA$, but we will show later that if (XA,YB) or $XBYA$ belong to $\mathcal{F}_1$ we get the same sequences as before (thus, we can consider this new definition of $\star$-product as an extension of Definition 3).

PROPOSITION 2. *Given the kneading data $BLMA$, consider the product: Type* **A**.

$$BLMA\star(XA,YB) = (\hat{X}_1\hat{X}_2\ldots\hat{A},\ \hat{Y}_1\hat{Y}_2\ldots\hat{B}) \qquad \text{for} \quad (XA,YB)\in\mathcal{F}_{KS};$$

Type **B**.

$$BLMA\star XBYA = \hat{X}_1\hat{X}_2\ldots\hat{B}\hat{Y}_1\hat{Y}_2\ldots\hat{A} \qquad \text{for} \quad XBYA\in\mathcal{F}_{KS},$$

with $\hat{L}=MM$, $\hat{A}=MA$, $\hat{M}=ML$, $\hat{B}=BL$ and $\hat{R}=RL$. Then, both $BLMA\star(XA,YB)$ and $BLMA\star XBYA$ are admissible kneading data.

PROOF. First, we will show that, if one restricts the second factor to $\mathcal{F}_1$, the result of this new product is equal to the one presented before. In fact, with P the empty sequence and $Q=LM$, for which $\rho(P)=+1$ and $\rho(Q)=-1$, accordingly, we have, from Definition 3, that, if $(XA,YB)\in\mathcal{F}_1$, the first sequence of the pair $BLMA\star(XA,YB)$ is given by

$$\psi(X_1)LM\eta(X_2)\psi(X_3)LM\eta(X_4)\ldots\psi(X_{2n-1})LMA\,.$$

Now, if we look at this sequence as pairs of symbols we can see that there are only two kinds of pairs: $\psi(X_{odd})L$ or $M\eta(X_{even})$. Then, one can easily verify the identities $\hat{X}_{odd}=\psi(X_{odd})L$, for $X_{odd}=M,R$, and $\hat{X}_{even}=M\eta(X_{even})$, for $X_{even}=L,M$. For the second sequence of $BLMA\star(XA,YB)$ the procedure is similar, but, for convenience, instead of the sequence given in Definition 3, it is easier to work with the shifted sequence

$$M\eta(Y_1)\psi(Y_2)LM\eta(Y_3)\psi(Y_4)\ldots LM\eta(Y_{2m-1})BL\,,$$

for which one can immediately find the same equalities $\hat{X}_{odd}=\psi(X_{odd})L$, for $X_{odd}=M,R$, and $\hat{X}_{even}=M\eta(X_{even})$, for $X_{even}=L,M$. The admissibility of $BLMA\star(XA,YB)$, for $(XA,YB)\in\mathcal{F}_{KS}$ follows easily once one verifies that the transformation $\hat{S}$ is compatible with the order relation,

$$\hat{L}=MM\prec\hat{A}=MA\prec\hat{M}=ML\prec\hat{B}=BL\prec\hat{R}=RL\,,$$

and satisfies $\sigma(\hat{S})\prec\hat{R}$ and $\hat{L}\prec\sigma(\hat{S})$, with S any symbol from $\underline{\mathcal{B}}$. The same arguments lead us straightforward to the admissibility of $BLMA\star XBYA$.

For the sake of simplicity, we use the same notation, $\mathcal{F}_{BLMA}$, for the subset of kneading data of the form $BLMA\star(P,Q)$, with $(P,Q)\in\mathcal{F}_{KS}$. In the following table we give some examples of the new products one can form:

Type **A**.

$$BLMA * (RRA, LMB) = (RLRLMA, LMMMLB)$$

$$BLMA * (RRLMA, LLLB) = (RLRLMMMLMA, LMMMMMMB)$$

$$BLMA * (RMMMA, LMB) = (RLMLMLMLMA, LMMMLB)$$

Type **B**.

$$BLMA * RRBLMA = RLRLBLMMMLMA$$

$$BLMA * RRLMBLLMA = RLRLMMMLBLMMMMMLMA$$

$$BLMA * RMMBLMMA = RLMLMLBLMMMLMLMA$$

We can now characterize the symbolic subset $\mathcal{F}_{BLMA}$: it is the set of kneading data (P, Q) satisfying the following inequalities:

$$B \preceq P \preceq (RL)^{\infty},$$

$$L(M)^{\infty} \preceq Q \preceq LMA.$$

In terms of the symbolic table suggested before, this subset is the rectangle with vertices $BLMA$, $BL(M)^{\infty}$, $((RL)^{\infty}, LMA)$ and $((RL)^{\infty}, L(M)^{\infty})$. An easy consequence of the previous result is the following:

COROLLARY 1. *The set $\mathcal{F}_{BLMA}$ is isomorphic to the set of all admissible kneading data $\mathcal{F}_{KS}$ (a one-to-one and order preserving correspondence).*

Next, we characterize the topological entropy of any element of the subset $\mathcal{F}_{BLMA}$: denote by $h(P, Q)$ the topological entropy of the bimodal map f_{ab} whose kneading data is $\mathcal{K}(f_{ab}) = (P, Q)$.

THEOREM 1. *Given an arbitrary kneading data $(P, Q) \in \mathcal{F}_{KS}$ we have*

$$h(BLMA \star (P, Q)) = \frac{1}{2} h(P, Q).$$

PROOF. From Misiurewicz–Szlenk–Rothschild's theorem, it is known that we can deduce the topological entropy $h(f)$ of a m-modal map f from its growth number $s(f)$. Therefore, one can use the kneading matrix defined in [Mi-Th 88], since the smallest zero t of its determinant $D(t)$, a formal power series with odd integer coefficients, equals the inverse of the growth number of the map, i.e., $t = 1/s(f)$. Moreover, when both kneading sequences P and Q are periodic, the sum of this formal power series is a rational function and one can evaluate the growth number from an expression introduced in [La-SR 89]. Otherwise, we can approximate it by one. Thus, consider an arbitrary periodic kneading data

$$(P, Q) = (P_1 \ldots P_{p-1}X, Q_1 \ldots Q_{q-1}Y),$$

with $X, Y \in \{A, B\}$. Given a sequence S denote by $S^{(i)}$ the subsequence of S obtained from its truncation after the i-th symbol, that is, $S^{(i)} = S_1 \ldots S_i$. For simplicity reasons, it is most suitable to work with the polynomial $d_{(P,Q)}(t)$, instead of the kneading determinant $D(t)$, defined by

$$d_{(P,Q)}(t) = (1-t)(1-\rho(P)\,t^p)(1-\rho(Q)\,t^q)\,D(t).$$

Thus, from the definition of kneading determinant $D(t)$ of a periodic kneading data (P, Q), we have

$$d_U(t) = (1 - \sum_{i=1}^{p} \delta(P_i)\rho(P^{(i)})\,t^i) \times (1 - \sum_{i=1}^{q} \nu(Q_i)\rho(Q^{(i)})\,t^i) \\ - \sum_{i=1}^{p} \nu(P_i)\rho(P^{(i)})\,t^i \times \sum_{i=1}^{q} \delta(Q_i)\rho(Q^{(i)})$$

with $\delta(L) = -\nu(L) = -1$, $\delta(A) = 0$, $\nu(A) = 1$, $\delta(M) = \nu(M) = 1$, $\delta(B) = 1$, $\nu(B) = 0$ and $\delta(R) = -\nu(R) = 1$. Therefore, from the definition of the $\star$-product of $BLMA$ with another periodic kneading data (S, T), we have that

$$d_{BLMA\star(S,T)}(t) = (1-t)\,d_{(S,T)}(t^2).$$

Thus, the smallest zero of the $d_{BLMA\star(S,T)}(t)$ is just the smallest zero of the polynomial $d_{(S,T)}(t^2)$ (remember that all zeros of $d_{KS}(t)$ are not greater than 1) and the growth number of the product $BLMA \star (S,T)$ comes as $s(BLMA \star (S,T)) = \sqrt{s(S,T)}$.

4. Discussion

Despite the particular character of the results presented, we think that they are really important for a future definition of a $\star$-product between bimodal kneading data. In fact, now we know that the formal straightforward generalization of the unimodal symbolic product, as the definition given in [La-Se-SR 99], is no longer appropriate for all situations and probably these results can give the new perspective necessary for the extension of the former definition. Finally, we would like to stress once more the relevance of the identification of a second factor, as was obvious in Theorem 1. From our point of view it is not enough to present a family of kneading data associated with a given one, being much more interesting to write them as a product of two elements of $\mathcal{F}_{KS}$ and characterize them in terms of both factors of that product.

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