TOPOLOGICAL TRANSITIVITY FOR EXPANDING MONOTONIC MOD ONE TRANSFORMATIONS WITH TWO MONOTONIC PIECES

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To the memory of György Targonski and Martin Grinč

Abstract. Consider a continuous and strictly increasing function $f : [0,1] \to [0,2]$, and define $T_f x = f(x) (\text{mod } 1)$. Then T_f is a monotonic mod one transformation with two monotonic pieces, if and only if $f(0) < 1 < f(1)$. It is proved that T_f is topologically transitive, if f is piecewise differentiable and $\inf_{x\in[0,1]} f'(x) \geq \sqrt{2}$.

Introduction

We consider a continuous strictly increasing function $f : [0,1] \to [0,2]$. Define $T_f x := f(x) \pmod 1$, and let $\mathcal{Z}_f$ be the collection of all maximal open subintervals U of $[0,1]$ with $f(U) \cap \mathbb{Z} = \emptyset$. Obviously $\operatorname{card} \mathcal{Z}_f \leq 2$. A finite partition $\mathcal{Z}$ of $[0,1]$ is a collection of finitely many pairwise disjoint open intervals with $\bigcup_{Z\in\mathcal{Z}} \overline{Z} = [0,1]$. Note that $\mathcal{Z}_f$ is a finite partition. We assume that there exists a finite partition $\mathcal{Y}$ of $[0,1]$, such that for every $Y \in \mathcal{Y}$ the function $f|_Y$ is differentiable. If $\inf_{x\in[0,1]} f'(x) > 1$, then $\operatorname{card} \mathcal{Z}_f = 2$.

The map T_f is called topologically transitive, if there exists an $x \in [0,1]$, such that $\{T_f{}^n x : n \in \mathbb{N}\}$ is dense in $[0,1]$. Properties of topologically transitive dynamical systems can be found in [5] or [13]. For general monotonic mod one transformations $T_f : [0,1] \to [0,1]$ it has been proved in [12] that T_f is topologically transitive, if $\inf_{x\in[0,1]} f'(x) > 2$. If $T_f : [0,1] \to [0,1]$ is a monotonic mod one transformation with three intervals of monotonicity

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and $\inf_{x\in[0,1]} f'(x) \geq 2$, then it is proved in [12] that T_f is topologically transitive. In our situation these results are not applicable, as there are no monotonic mod one transformations T_f with two monotonic pieces satisfying $\inf_{x\in[0,1]} f'(x) > 2$, and the only monotonic mod one transformation T_f with two monotonic pieces satisfying $\inf_{x\in[0,1]} f'(x) \geq 2$ is $T_f x = 2x \pmod 1$ (the topological transitivity in this case is well known, see e.g. [13]).

As the main result of this paper we obtain in Theorem 1 that T_f is topologically transitive, if $\inf_{x\in[0,1]} f'(x) \geq \sqrt{2}$. This paper is organized as follows. In Section 1 we give some basic definitions. Then we describe the Markov diagram (see also [1], [3] and [8]). We prove in Lemma 2 that the topological transitivity of T_f is implied by a certain property of the Markov diagram. Section 2 is devoted to counterexamples. For every $\lambda < \sqrt{2}$ an example of a monotonic mod one transformation T_f with two monotonic pieces satisfying $\inf_{x\in[0,1]} f'(x) \geq \lambda$ is given, where T_f is not topologically transitive. The main result of this paper is contained in Section 3. A special case, where Lemma 2 does not work, is investigated in Lemma 3. Otherwise using Lemma 4 we can apply Lemma 2 and get Theorem 1.

1. Monotonic mod one transformations and their Markov diagram

For a continuous strictly increasing function $f : [0,1] \to \mathbb{R}$ set

$$T_f x := f(x) \pmod 1 := f(x) - [f(x)], \tag{1}$$

where $[y]$ denotes the largest integer smaller than or equal to y. Furthermore let $\mathcal{Z}_f$ be the collection of all nonempty intervals among $f^{-1}(n, n+1)$ for an $n \in \mathbb{Z}$.

We call $\mathcal{Z}$ a *finite partition* of $[0,1]$, if $\mathcal{Z}$ consists of finitely many pairwise disjoint open intervals with $\bigcup_{Z\in\mathcal{Z}} \overline{Z} = [0,1]$. If $T : [0,1] \to [0,1]$ is a transformation and $\mathcal{Z}$ is a finite partition of $[0,1]$ satisfying $T|_Z$ is strictly monotone and continuous for all $Z \in \mathcal{Z}$, then T is called a *piecewise monotonic map* with respect to $\mathcal{Z}$. A map $T : [0,1] \to [0,1]$ is called a *monotonic mod one transformation*, if there exists a continuous strictly increasing function $f : [0,1] \to \mathbb{R}$ with $T = T_f$. Then $\mathcal{Z}_f$ is a finite partition of $[0,1]$ and T_f is a piecewise monotonic map with respect to $\mathcal{Z}_f$. We say that a monotonic mod one transformation $T_f : [0,1] \to [0,1]$ has n *monotonic pieces*, if $\operatorname{card} \mathcal{Z}_f = n$.

Consider a function $f : [0,1] \to \mathbb{R}$. The function f is called *piecewise differentiable*, if there exists a finite partition $\mathcal{Y}$ of $[0,1]$, such that for every $Y \in \mathcal{Y}$ the function $f|_Y$ is differentiable. For a piecewise differentiable function define $\inf f' := \inf \{ f'(x) : x \in \bigcup_{Y\in\mathcal{Y}} Y \}$. We call a monotonic mod one transformation $T_f : [0,1] \to [0,1]$ *expanding*, if $f : [0,1] \to \mathbb{R}$ is continuous, piecewise differentiable and $\inf f' > 1$.

If X is a compact metric space and $T : X \to X$ is continuous, then (X, T) is called a *topological dynamical system*. Let $x \in X$. Then the *ω-limit set* of x is defined as the set of all limit points of the sequence $(T^n x)_{n \in \mathbb{N}}$, and it is denoted by $\omega(x)$. The map T is called *topologically transitive*, if there exists an $x \in X$ with $\omega(x) = X$.

Suppose that $T : [0,1] \to [0,1]$ is a piecewise monotonic map. As T need not be continuous, $\big([0,1], T\big)$ need not be a topological dynamical system. In order to get a topological dynamical system we use a standard doubling points construction as in [6] (see [6] or [8] for the details).

The Markov diagram of a piecewise monotonic map $T : [0,1] \to [0,1]$ with respect to the finite partition $\mathcal{Z}$ of $[0,1]$ was introduced by Franz Hofbauer (see e.g. [1] and [3]). It is an at most countable graph describing the orbit structure of T. Suppose that $D \subseteq Z_0$ for a $Z_0 \in \mathcal{Z}$. A nonempty C is called *successor* of D, if there exists a $Z \in \mathcal{Z}$ with $C = TD \cap Z$. In this case we write $D \to C$. Now let $\mathcal{D}$ be the smallest set with $\mathcal{Z} \subseteq \mathcal{D}$ satisfying $D \in \mathcal{D}$ and $D \to C$ imply $C \in \mathcal{D}$. Then the oriented graph $(\mathcal{D}, \to)$ is called the *Markov diagram* of T with respect to $\mathcal{Z}$. We get that $\mathcal{D}$ is at most countable and its elements are open subintervals of elements of $\mathcal{Z}$.

Let $\mathcal{C} \subseteq \mathcal{D}$. If $C_0, C_1, \dots, C_n \in \mathcal{C}$ and $C_{j-1} \to C_j$ for $j \in \{1, 2, \dots, n\}$, then $C_0 \to C_1 \to \dots \to C_n$ is called a *finite path* in $\mathcal{C}$. A subset $\mathcal{C} \subseteq \mathcal{D}$ is called *irreducible*, if for every $C, D \in \mathcal{C}$ there exists a finite path $C_0 \to C_1 \to \dots \to C_n$ in $\mathcal{C}$ with $C_0 = C$ and $C_n = D$. We call an irreducible $\mathcal{C} \subseteq \mathcal{D}$ *maximal irreducible*, if no $\mathcal{C}'$ with $\mathcal{C} \subsetneq \mathcal{C}' \subseteq \mathcal{D}$ is irreducible.

In the special case of a monotonic mod one transformation $T_f : [0,1] \to [0,1]$ the Markov diagram of T_f has a special structure. For more details of the Markov diagram of a monotonic mod one transformation see [2] (cf. also [7]).

Now we consider expanding monotonic mod one transformations with two monotonic pieces. If $T_f : [0,1] \to [0,1]$ is an expanding monotonic mod one transformation with two monotonic pieces, then we may assume that $f : [0,1] \to [0,2]$ is a continuous strictly increasing function. Furthermore there exists a unique $c \in (0,1)$ with $f(c) = 1$. For an interval $C \subseteq [0,1]$ denote by $|C|$ the length of C.

LEMMA 1. *Let $f : [0,1] \to [0,2]$ be a continuous and piecewise differentiable function, such that* $\inf f' > 1$. *Denote by $(\mathcal{D}, \to)$ the Markov diagram of T_f with respect to $\mathcal{Z}_f$, and let $D \in \mathcal{D}$. Then there exists a finite path $C_0 \to C_1 \to \dots \to C_n$ in $\mathcal{D}$, such that $C_0 = D$ and C_n has two different successors in $\mathcal{D}$.*

PROOF. Set $d := \inf f'$. Assume that $C_0 \to C_1 \to \ldots \to C_n$ is a finite path in $\mathcal{D}$, such that C_j has only one successor in $\mathcal{D}$ for all $j \in \{0, 1, \ldots, n-1\}$. By the mean value theorem we obtain $|C_n| \geq d^n |C_0|$. As $d > 1$ this implies the desired result. □

For expanding monotonic mod one transformations with two monotonic pieces we can prove the following result on the topological transitivity of T_f.

LEMMA 2. *Let $f : [0,1] \to [0,2]$ be a continuous and piecewise differentiable function, such that $\inf f' > 1$. Denote by $(\mathcal{D}, \to)$ the Markov diagram of T_f with respect to $\mathcal{Z}_f$. Suppose that for every $D \in \mathcal{D}$ there exists a finite path $C_0 \to C_1 \to \ldots \to C_n$ in $\mathcal{D}$ with $C_0 = D$ and $C_n \in \mathcal{Z}_f$. Then T_f is topologically transitive.*

PROOF. For $n \in \mathbb{N}_0$ set

$$T_f{}^n 0 := \lim_{x \to 0^+} T_f{}^n x \quad \text{and} \quad T_f{}^n 1 := \lim_{x \to 1^-} T_f{}^n x.$$

Observe that $T_f(0, c) = (T_f 0, 1)$ and $T_f(c, 1) = (0, T_f 1)$. As $\inf f' > 1$ we get $T_f 1 - T_f 0 > 0$. Hence $(0, c) \to (c, 1)$ or $(c, 1) \to (0, c)$. Assume that $(0, c) \to (c, 1)$ (the case $(c, 1) \to (0, c)$ is analogous), and set $E := (c, 1)$. By our assumptions for every $D \in \mathcal{D}$ there exists a finite path $C_0 \to C_1 \to \ldots \to C_n$ in $\mathcal{D}$ with $C_0 = D$ and $C_n = E$.

Denote by $\mathcal{C}$ the set of all $C \in \mathcal{D}$, such that there exists a finite path $C_0 \to C_1 \to \ldots \to C_n$ in $\mathcal{D}$ with $C_0 = E$ and $C_n = C$. Then $\mathcal{C}$ is a maximal irreducible subset of $\mathcal{D}$. By Lemma 1 in [7] (cf. also [2]) $\mathcal{C}$ contains all $D \in \mathcal{D}$ with $\sup D = T_f{}^n 1$ for an $n \in \mathbb{N}_0$, and using also Lemma 1 only finitely many $D \in \mathcal{D}$ with $\inf D = T_f{}^n 0$ for an $n \in \mathbb{N}_0$ may be not contained in $\mathcal{C}$. Again using Lemma 1 in [7] we obtain that $\mathcal{D} \setminus \mathcal{C}$ is finite and contains no irreducible subset. Hence $\mathcal{C}$ is the unique maximal irreducible subset of $\mathcal{D}$.

By Lemma 1 there exists an $n \in \mathbb{N}$ and there exists a finite path $C_0 \to C_1 \to \ldots \to C_n$ in $\mathcal{D}$ with $C_0 = E$, such that C_j has only one successor in $\mathcal{D}$ for all $j \in \{0, 1, \ldots, n-2\}$, C_{n-1} has two different successors in $\mathcal{D}$, and $\inf C_n = T_f{}^{n-1} 0$. Since $T_f 0 < T_f 1$ we get by induction that $T_f{}^j 0 \in [0, c]$, $T_f{}^j 1 \in [0, c]$ and $C_j = \left(T_f{}^{j-1} 0, T_f{}^j 1\right)$ for all $j \in \{1, 2, \ldots, n-1\}$, $T_f{}^j 0 < T_f{}^j 1$ for all $j \in \{1, 2, \ldots, n\}$, and $C_n = \left(T_f{}^{n-1} 0, c\right)$. Therefore $\bigcup_{j=0}^n \overline{C_j} = [0, 1]$. It follows from Theorem 11 in [3] (cf. also Theorem 2 in [8]) that $T_f|_L$ is topologically transitive, where $L = \bigcup_{C \in \mathcal{C}} \overline{C}$. Observing that $C_j \in \mathcal{C}$ for all $j \in \{0, 1, \ldots, n\}$ we get $L = [0, 1]$, and hence T_f is topologically transitive. □

2. Counterexamples

If $f : [0,1] \to \mathbb{R}$ is a continuous and piecewise differentiable function with $\inf f' > 2$, then Theorem 1 in [12] (or Corollary 1.1 in [12]) implies that T_f is topologically transitive. In Section 3 of [12] an example of a continuous and piecewise differentiable function $f : [0,1] \to \mathbb{R}$ with $\inf f' \geq 2$ is given, such that T_f is not topologically transitive. Next consider a continuous, strictly increasing and piecewise differentiable function $f : [0,1] \to \mathbb{R}$, such that T_f has three monotonic pieces. It is proved in Theorem 2 of [12] that T_f is topologically transitive, if $\inf f' \geq 2$. The example given in (2) of [12] shows that for every $\lambda < 2$ there exists a continuous, strictly increasing and piecewise differentiable function $f : [0,1] \to \mathbb{R}$ with $\inf f' \geq \lambda$, such that T_f has three monotonic pieces and is not topologically transitive.

Consider a continuous, strictly increasing and piecewise differentiable function $f : [0,1] \to \mathbb{R}$, such that T_f has two monotonic pieces. It will be shown in Theorem 1 that T_f is topologically transitive, if $\inf f' \geq \sqrt{2}$. Now we give for every $\lambda < \sqrt{2}$ an example with $\inf f' \geq \lambda$, such that T_f is not topologically transitive. Let $\lambda \in (1, \sqrt{2})$. Define

$$f(x) := \lambda x + \left(1 - \frac{\lambda}{2}\right) . \tag{2}$$

Then $T_f : [0,1] \to [0,1]$ is a monotonic mod one transformation with two monotonic pieces satisfying $\inf f' = \lambda$. Set

$$A := \left[0, \frac{\lambda^2 - \lambda}{2}\right] \cup \left[1 - \frac{\lambda}{2}, \frac{\lambda}{2}\right] \cup \left[1 + \frac{\lambda - \lambda^2}{2}, 1\right] .$$

By the choice of λ we have $[0,1] \setminus A \neq \emptyset$. Furthermore $T_f A \subseteq A$, and hence T_f is not topologically transitive.

3. Topological transitivity of monotonic mod one transformations

Suppose that $f : [0,1] \to \mathbb{R}$ is a continuous, strictly increasing and piecewise differentiable function with $\inf f' \geq \sqrt{2}$, such that T_f has two monotonic pieces. We will prove that T_f is topologically transitive. To this end we need the following result.

LEMMA 3. *Assume that* $f : [0,1] \to [0,2]$ *is a continuous and piecewise differentiable function with* $\inf f' \geq \sqrt{2}$. *Furthermore suppose that there exist open intervals* $A_1, A_2, A_3, A_4 \subseteq [0,1]$ *with* $A_1, A_2 \subseteq (0, c)$, $A_3, A_4 \subseteq (c, 1)$,

$T_fA_1 = T_fA_4 = A_2 \cup A_3 \cup \{c\}$, $T_fA_2 = A_4$ and $T_fA_3 = A_1$. Then $f(x) = \sqrt{2}x + 1 - \frac{1}{\sqrt{2}}$ for all $x \in [0,1]$, and T_f is topologically transitive.

PROOF. Note that $|T_f{}^2 A_2| \geq 2|A_2|$. Hence our assumptions imply $|A_2| = |A_3|$ and $|A_1| = |A_4| = \sqrt{2}|A_2|$. Using the mean value theorem we get that for each $j \in \{1,2,3,4\}$ there exists an α_j, such that $f(x) = \sqrt{2}x + \alpha_j$ for all $x \in A_j$.

Define $B_1 := (0,c)\setminus(\overline{A_1 \cup A_2})$ and $B_2 := (c,1)\setminus(\overline{A_3 \cup A_4})$. Assume that $B_1 \neq \emptyset$. Then our assumptions give $T_fB_1 = B_2$ and $T_fB_2 = B_1$. This implies $T_f{}^2 B_1 = B_1$. By the mean value theorem we obtain $|T_f{}^2 B_1| \geq 2|B_1|$, which is a contradiction. Therefore $B_1 = \emptyset$ and analogously $B_2 = \emptyset$. Moreover we have $f(x) = \sqrt{2}x + \alpha$ for all $x \in [0,1]$.

Observe that $\inf A_1 = 0$, and hence $\inf A_2 = \alpha$. Then $\inf A_4 = (\sqrt{2}+1)\alpha$ and $\alpha = \inf A_2 = (\sqrt{2}+3)\alpha - 1$. Hence $\alpha = 1 - \frac{1}{\sqrt{2}}$.

The Markov diagram $(\mathcal{D}, \to)$ of T_f with respect to $\mathcal{Z}_f$ satisfies $\mathcal{D} = \mathcal{Z}_f \cup \{A_1, A_2, A_3, A_4\}$. Furthermore $\{A_1, A_2, A_3, A_4\}$ is a maximal irreducible subset of $\mathcal{D}$, $\overline{A_1 \cup A_2 \cup A_3 \cup A_4} = [0,1]$, and there are no arrows $A_j \to Z$ in $\mathcal{D}$ with $j \in \{1,2,3,4\}$ and $Z \in \mathcal{Z}_f$. Now Theorem 11 in [3] (cf. also Theorem 2 in [8]) implies that T_f is topologically transitive. □

We will also need the following result.

LEMMA 4. *Assume that $f : [0,1] \to [0,2]$ is a continuous and piecewise differentiable function with* $\inf f' \geq \sqrt{2}$. *Suppose that $f(x) \neq \sqrt{2}x + 1 - \frac{1}{\sqrt{2}}$ for an $x \in [0,1]$. Denote by $(\mathcal{D}, \to)$ the Markov diagram of T_f with respect to $\mathcal{Z}_f$. Let $C \in \mathcal{D}$ with $c \in \overline{C}$. Then there exists a finite path $C_0 \to C_1 \to \ldots \to C_n$ in $\mathcal{D}$ with $C_0 = C$ and $c \in \overline{C_n}$, such that $C_n \in \mathcal{Z}_f$ or $|C_n| \geq \sqrt{2}|C|$.*

PROOF. Set $C_0 := C$. If C_0 has two different successors in $\mathcal{D}$, then there exists a $C_1 \in \mathcal{Z}_f$ with $C_0 \to C_1$. For the rest of this proof we assume that C_0 has a unique successor C_1 in $\mathcal{D}$. By Lemma 1 there exists a finite path $C_0 \to C_1 \to \ldots \to C_l$ in $\mathcal{D}$, such that C_l has two different successors in $\mathcal{D}$ and C_j has a unique successor in $\mathcal{D}$ for $j \in \{0,1,\ldots,l-1\}$. Suppose that $l > 1$. By the mean value theorem we get $|T_fC_l| \geq 2^{\frac{l+1}{2}}|C| \geq 2\sqrt{2}|C|$. Hence there is a $C_{l+1} \in \mathcal{D}$ with $C_l \to C_{l+1}$, $c \in \overline{C_{l+1}}$ and $|C_{l+1}| \geq \sqrt{2}|C|$.

It remains to consider the case $l = 1$. Let $Z_0, Z_1 \in \mathcal{Z}_f$ with $C \subseteq Z_0$ and $C \cap Z_1 = \emptyset$.

First assume that $|T_fC_1 \cap Z_0| > |C|$. Set $\delta := |T_fC_1 \cap Z_0| - |C| > 0$ and $C_2 := T_fC_1 \cap Z_0$. Next we prove by induction that for every $k \in \mathbb{N}$ there exists a finite path $C_0 \to C_1 \to \ldots \to C_{2k}$ in $\mathcal{D}$ with $C \subseteq C_{2k}$, such that $C_{2k} \in \mathcal{Z}_f$ or $|C_{2k}| \geq |C| + 2^{k-1}\delta$. In the case $C_{2k-2} \in \mathcal{Z}_f$ we get $C_{2k-4} \subseteq C_{2k-2}$, hence we can find $C_{2k-3} \subseteq C_{2k-1}$ and $C_{2k-2} \subseteq C_{2k}$ with $C_{2k-2} \to C_{2k-1} \to C_{2k}$, and therefore $C_{2k} \in \mathcal{Z}_f$. If C_{2k-2} has two different successors in $\mathcal{D}$, then

$C_{2k-2} \to Z_1$ and $Z_1 \to Z_0$, as $C_1 \subseteq Z_1$ and C_1 has two different successors in $\mathcal{D}$. Otherwise C_{2k-2} has a unique successor C_{2k-1} in $\mathcal{D}$. Since the case sup $C = c$ is analogous we may assume inf $C = c$. Therefore sup $C_{2k-2} -$ sup $C \geq 2^{k-2}\delta$. Setting $C_{2k} := T_f C_{2k-1} \cap Z_0$ the mean value theorem gives sup $C_{2k} -$ sup $C_2 = {T_f}^2$ sup $C_{2k-2} - {T_f}^2$ sup $C \geq 2($sup $C_{2k-2} -$ sup $C) \geq 2^{k-1}\delta$. As sup $C_2 \geq$ sup C this gives $|C_{2k}| \geq |C| + 2^{k-1}\delta$.

Choose a k with $|C| + 2^{k-1}\delta \geq \sqrt{2}|C|$, and set $n := 2k$. Then $C_n \in \mathcal{Z}_f$ or $|C_n| \geq \sqrt{2}|C|$. Note that we have $c \in \overline{C_n}$.

Next we consider the case $|T_f C_1 \cap Z_0| \leq |C|$, and set $C_2 := T_f C_1 \cap Z_1$. Then $|C_2| \geq |C|$, since $|T_f C_1| \geq 2|C|$ by the mean value theorem. If C_2 has two different successors in $\mathcal{D}$, then there exists a $C_3 \in \mathcal{Z}_f$ with $C_2 \to C_3$. It remains to consider the case that C_2 has a unique successor C_3 in $\mathcal{D}$. Assume that also C_3 has a unique successor in $\mathcal{D}$. By Lemma 1 there exists a finite path $D_0 \to D_1 \to \ldots \to D_q$ in $\mathcal{D}$ with $D_0 = C_2$, $q \geq 2$, D_j has a unique successor in $\mathcal{D}$ for $j \in \{0, 1, \ldots, q-1\}$, and D_q has two different successors in $\mathcal{D}$. As $|T_f D_q| \geq 2^{\frac{q+1}{2}}|D_0|$ by the mean value theorem, we get that D_q has a successor D_{q+1} with $c \in \overline{D_{q+1}}$ and $|D_{q+1}| \geq \sqrt{2}|D_0| \geq \sqrt{2}|C|$.

From now on we suppose that C_3 has two different successors in $\mathcal{D}$. If $|T_f C_3 \cap Z_1| > |C_2|$, then a proof analogous to the proof above in the case $|T_f C_1 \cap Z_0| > |C|$ shows the existence of a finite path $C_0 \to C_1 \to \ldots \to C_n$ in $\mathcal{D}$ with $c \in \overline{C_n}$, such that $C_n \in \mathcal{Z}_f$ or $|C_n| \geq \sqrt{2}|C_2| \geq \sqrt{2}|C|$. For the rest of this proof we assume $|T_f C_3 \cap Z_1| \leq |C_2|$ and set $C_4 := T_f C_3 \cap Z_0$. As $|T_f C_3| \geq 2|C_2|$ by the mean value theorem, we get $|C_4| \geq |C_2| \geq |C|$. If C_4 has two different successors in $\mathcal{D}$, then there exists a $C_5 \in \mathcal{Z}_f$ with $C_4 \to C_5$. It remains to consider the case that C_4 has a unique successor C_5 in $\mathcal{D}$. Since $C \subseteq C_4$ we get $C_1 \subseteq C_5$ and C_5 has two different successors in $\mathcal{D}$, one of which is C_2. Set $C_6 := T_f C_5 \cap Z_0$.

We claim that $|C_6| > |C_4|$. The mean value theorem implies

$$(3) \qquad |C_6| = |T_f C_5| - |C_2| = |{T_f}^2 C_4| - |C_2| \geq 2|C_4| - |C_2| \geq |C_4| \ ,$$

as $|C_4| \geq |C_2|$. Assume that $|C_6| = |C_4|$. Then $C_6 = C_4$, and by (3) we get $|C_2| = |C_4|$. Since $T_f C_3 \cap Z_1 \subseteq C_2$ and

$$|T_f C_3 \cap Z_1| = |T_f C_3| - |C_4| = |{T_f}^2 C_2| - |C_2| \geq 2|C_2| - |C_2| = |C_2|$$

by the mean value theorem, we get $T_f C_3 \cap Z_1 = C_2$. Hence $C_3 \subseteq Z_0$, $C_4 \subseteq Z_0$, $C_2 \subseteq Z_1$, $C_5 \subseteq Z_1$, $T_f C_3 = T_f C_5 = C_2 \cup C_4 \cup \{c\}$, $T_f C_2 = C_3$ and $T_f C_4 = C_5$. By Lemma 3 $f(x) = \sqrt{2}x + 1 - \frac{1}{\sqrt{2}}$ for all $x \in [0, 1]$, contradicting our assumption $f(x) \neq \sqrt{2}x + 1 - \frac{1}{\sqrt{2}}$ for an $x \in [0, 1]$.

Therefore $|C_6| > |C_4|$. A proof analogous to the proof above in the case $|T_f C_1 \cap Z_0| > |C|$ shows the existence of a finite path $C_0 \to C_1 \to \ldots \to C_n$ in $\mathcal{D}$ with $c \in \overline{C_n}$, such that $C_n \in \mathcal{Z}_f$ or $|C_n| \geq \sqrt{2}|C_4| \geq \sqrt{2}|C|$. □

Now we are able to prove our main theorem.

THEOREM 1. *Let $f : [0,1] \to [0,2]$ be a continuous and piecewise differentiable function, such that* $\inf f' \geq \sqrt{2}$. *Then T_f is topologically transitive.*

PROOF. Assume at first that $f(x) = \sqrt{2}x + 1 - \frac{1}{\sqrt{2}}$ for all $x \in [0,1]$. Then Lemma 3 gives that T_f is topologically transitive.

For the rest of this proof we assume that $f(x) \neq \sqrt{2}x + 1 - \frac{1}{\sqrt{2}}$ for an $x \in [0,1]$. Let $(\mathcal{D}, \to)$ be the Markov diagram of T_f with respect to $\mathcal{Z}_f$, and let $D \in \mathcal{D}$. By Lemma 1 there exists a finite path $C_0 \to C_1 \to \ldots \to C_q$ in $\mathcal{D}$, such that $C_0 = D$ and C_{q-1} has two different successors in $\mathcal{D}$. Hence $c \in \overline{C_q}$. Set $l_0 := q$.

Next we claim that for every $k \in \mathbb{N}$ there exists a finite path $C_0 \to C_1 \to \ldots \to C_{l_k}$ in $\mathcal{D}$ with $C_0 = D$ and $c \in \overline{C_{l_k}}$, such that $C_{l_k} \in \mathcal{Z}_f$ or $|C_{l_k}| \geq (\sqrt{2})^k |C_q|$. We prove this by induction. If $C_{l_{k-1}} \in \mathcal{Z}_f$, then set $l_k := l_{k-1}$. Otherwise we have $|C_{l_{k-1}}| \geq (\sqrt{2})^{k-1}|C_q|$. By Lemma 4 there exists a finite path $C_{l_{k-1}} \to C_{l_{k-1}+1} \to \ldots \to C_{l_k}$ in $\mathcal{D}$ with $c \in \overline{C_{l_k}}$, such that $C_{l_k} \in \mathcal{Z}_f$ or

$$|C_{l_k}| \geq \sqrt{2}|C_{l_{k-1}}| \geq (\sqrt{2})^k |C_q| \ .$$

Now choose a $k \in \mathbb{N}$ with $(\sqrt{2})^k|C_q| > 1$, and set $n := l_k$. Then $C_0 \to C_1 \to \ldots \to C_n$ is a finite path in $\mathcal{D}$ with $C_0 = D$. Since $|C_n| \leq 1$ we obtain $C_n \in \mathcal{Z}_f$. Therefore T_f is topologically transitive by Lemma 2. □

REMARK. If $f : [0,1] \to [0,2]$ is a continuous and piecewise differentiable function with $\inf f' > 1$, such that $\lim_{x \to 0^+} T_f x = 0$ or $\lim_{x \to 1^-} T_f x = 1$, then T_f is topologically transitive by Theorem 3 in [12] (see Corollary 3.1 in [12]).

The topological transitivity of T_f has nice consequences for the behaviour of perturbations of T_f (see [9] and [11], perturbations of monotonic mod one transformations are also investigated in [7] and [10]).

Finally we consider the density of periodic orbit measures. We say *the periodic orbit measures are dense*, if for every nonempty subset U of the set of all T_f–invariant Borel probability measures, which is open in the weak star–topology, there exists an $x \in [0,1]$ and an $n \in \mathbb{N}$ with $T_f^{\,n} x = x$, such that $\mu_p \in U$, where $\mu_p(B) := \frac{1}{n}\sum_{j=0}^{n-1} 1_B(T_f^{\,j} x)$ for every Borel set $B \subseteq [0,1]$. The following result is an easy consequence of Theorem 2 in [4] and Theorem 1.

THEOREM 2. *Let $f : [0,1] \to [0,2]$ be a continuous and piecewise differentiable function, such that* $\inf f' \geq \sqrt{2}$. *Then the periodic orbit measures are dense.*

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