

TWO-PARAMETER FAMILIES OF DISCONTINUOUS ONE-DIMENSIONAL MAPS

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To the memory of Professor György Targonski

Abstract. The aim of this paper is to establish the existence of a “box-within-a-box” bifurcation structure for monotone families of Lorenz maps and to study its combinatorics.

1. Introduction

This is the first of two papers where we intend to describe all the combinatorial structure of the bifurcation skeleton of Lorenz maps. Guckenheimer and Williams in [2] showed that, adding some new hypothesis, the study of the dynamics of a Lorenz flow (a Lorenz flow with this extra structure is usually called a *geometric Lorenz flow*) can be reduced to the study of the dynamics of a piecewise continuous interval map, with a unique discontinuity and strictly increasing in each of the branches of continuity. We will be interested in *universal families of Lorenz maps* (see [6]). Leonov, in the early sixties, studied the existence of a “boxes in file” bifurcation structure for this kind of families of maps (see [3]). Unlike the bifurcation structure that we will study here, this structure is related with the irreducibility of the kneading invariant and gives a notion of degree of complexity. This two structures, acting together, control all the combinatorics of families of Lorenz maps.

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In this paper, after presenting some preliminary definitions and results in section 2, we will, in section 3, present a Singer-like theorem that, as in the context of continuous maps, states that the critical orbits control all the attractive periodic orbits (see [8]). Considering this we describe the different possible kinds of bifurcation points. We finish this section with a pictorial presentation of the bifurcation skeleton. Finally, in section 4, using symbolic dynamics we will establish the existence of a box-within-a-box bifurcation structure in the space of kneading invariants of Lorenz maps and study its combinatorics. In a forthcoming paper we will study how this two structures (box-within-a-box and boxes in files) interact between them.

2. Preliminaries

DEFINITION 1. Let $P < 0 < Q$. A Lorenz map of class C^r from $[P, Q]$ to $[P, Q]$ is a pair (f_-, f_+) where

1. $f_- : [P, 0] \rightarrow [P, Q]$ and $f_+ : (0, Q] \rightarrow [P, Q]$ are strictly increasing maps of class C^r .

2. $f_-(P) = P$ and $f_+(Q) = Q$.

Let us denote the set of Lorenz maps of class C^r by \mathcal{L}^r .

DEFINITION 2. Let $\Lambda \subset \mathbb{R}^2$ be closed. A Lorenz family is a continuous map $F : \Lambda \rightarrow \mathcal{L}^r$, $r \geq 0$

$$F_\lambda = \{P_\lambda, Q_\lambda, (f_-)_\lambda, (f_+)_\lambda\}.$$

DEFINITION 3. A monotone Lorenz family is a C^3 Lorenz family such that

1. $(f_-)_\lambda$ and $(f_+)_\lambda$ have negative Schwarzian derivative for all $\lambda \in \Lambda$.
2. $\Lambda = [0, 1] \times [0, 1]$.
3. $F : (a, b) \rightarrow \{-1, 1, (f_-)_a, (f_+)_b\}$.
4. If $a_1 < b_2$ then $(f_-)_{a_1}(x) < (f_-)_{b_2}(x)$ for all $x \in [-1, 0]$ and if $b_1 < b_2$ then $(f_+)_{b_1}(x) < (f_+)_{b_2}(x)$ for all $x \in [0, 1]$.
5. $(f_-)_0(0) = 0$, $(f_-)_1(0) = 1$, $(f_+)_0(0) = -1$ and $(f_+)_1(0) = 0$.
6. $(f_-)_a(-1) > 1$ and $(f_+)'_b(-1) > 1$ for all $(a, b) \in [0, 1] \times [0, 1]$.
7. $\lim_{x \uparrow 0} (f_-)'_a(x) = \lim_{x \downarrow 0} (f_+)'_b(x) = 0$ for all $(a, b) \in [0, 1] \times [0, 1]$.

EXAMPLE 1. The family

$$f_{(a,b)}(x) = \begin{cases} (-a-1)x^2 + a & \text{if } x < 0 \\ (2-b)x^2 + b-1 & \text{if } x > 0 \end{cases}$$

with $(a, b) \in [0, 1] \times [0, 1]$, is a monotone Lorenz family. From now on all the graphics are related to this family of Lorenz maps.

To simplify the notation we will represent an element of such a family as $f_{(a,b)} = (f_a, f_b) = ((f_-)_a, (f_+)_b)$, with $(a, b) \in [0, 1] \times [0, 1]$.

Given a monotone Lorenz family $f = f_{(a,b)} = (f_a, f_b)$, define recursively

$$f(a, b, x) = \begin{cases} f_a(x) & \text{if } x < 0 \\ f_b(x) & \text{if } x > 0 \end{cases} \quad \text{and } f^n(a, b, x) = f(a, b, f^{n-1}(a, b, x)).$$

We can also define $f^n(a, b, x^+) = \lim_{y \downarrow x} f^n(a, b, y)$ and $f^n(a, b, x^-) = \lim_{y \uparrow x} f^n(a, b, y)$ where the limits are taken over all the y 's such that $f^j(a, b, y) \neq 0$ for all $j \leq n$.

Kneading theory is a standard tool for studying maps of the interval (see [7]) and has been developed for Lorenz maps in [9]. For simplicity we will take Lorenz maps $f : [-1, 1] \rightarrow [-1, 1]$ with discontinuity point 0. Let f be a Lorenz map and let $x \in [-1, 0) \cup (0, 1]$ such that $f^n(x) \neq 0$ for all $n \in \mathbb{N}$. Define the kneading sequence $k(x) \in \{L, R\}^{\mathbb{N}}$ of x to be the sequence $k_0(x), k_1(x), k_2(x), \dots$ where

$$k_0(x) = \begin{cases} L & \text{if } x < 0 \\ R & \text{if } x > 0 \end{cases}$$

and $k_i(x) = k_0(f^i(x))$. Imposing the relation $L < R$, these sequences can be ordered using the standard lexicographical order, that is $k(x) < k(y)$ if and only if $\exists_{0 \leq r}$ such that $k_i(x) = k_i(y)$ for all $i < r$ and $k_r(x) < k_r(y)$. Furthermore, in the topology induced by the standard metric

$$d(k(x), k(y)) = \sum |k_i(x) - k_i(y)| / 2^i$$

where

$$|k_i(x) - k_i(y)| = \begin{cases} 0 & \text{if } k_i(x) = k_i(y) \\ 1 & \text{if } k_i(x) \neq k_i(y) \end{cases}$$

the limits

$$k(x^+) = \lim_{y \downarrow x} k(y) \quad \text{and} \quad k(x^-) = \lim_{y \uparrow x} k(y)$$

over the y 's such that $f^n(y) \neq 0$ for all $n \in \mathbb{N}$, exist for all $x \in [-1, 1]$. The kneading invariant $k(f)$ of f is the pair $(k^+(f), k^-(f)) = (k(0^+), k(0^-))$ (when we are in the context of Lorenz families we usually denote $k(f_{(a,b)})$ as $k(a, b)$). We have that:

- If $x \in [-1, 1]$ and $f^n(x) \neq 0$ for all $n \in \mathbb{N}$ then $k(x^-) = k(x) = k(x^+)$.

- For all $x \in [-1, 1]$, $k(x^-) \leq k(x^+)$.
- $x, y \in [-1, 1]$ and $x < y \Rightarrow k(x^+) \leq k(y^-)$.

Taking now a monotone Lorenz family $f_{(a,b)}$ with $(a, b) \in [0, 1] \times [0, 1]$, we want to know which pairs of symbolic sequences can be the kneading invariant of some $f_{(a,b)}$. We will call this set *the set of admissible kneading invariants* and will denote it by Σ^+ . Let σ be the usual shift operator, defined by

$$\sigma(k_0 k_1 \dots) = k_1 k_2 k_3 \dots$$

In [4] Hubbard and Sparrow show that if a pair of sequences (k^+, k^-) is admissible then $k_0^+ = R$, $k_0^- = L$ and

$$(1) \quad \sigma(k^+) \leq \sigma^n(k^+) \leq \sigma(k^-), \quad \sigma(k^+) \leq \sigma^n(k^-) \leq \sigma(k^-) \quad \text{for } n \in \mathbb{N}.$$

Furthermore, in [5] Martens and Melo show that: if $f_{(a,b)}$ with $(a, b) \in [0, 1] \times [0, 1]$ is a monotone Lorenz family, then for any pair (k^+, k^-) that satisfies 1, there exists $(a, b) \in [0, 1] \times [0, 1]$ such that $k(a, b) = (k^+, k^-)$. In [4] it is also proved that, for all $x \in [-1, 1]$,

$$(2) \quad \sigma(k^+(f)) \leq \sigma^n(k_f(x^\pm)) \leq \sigma(k^-(f)) \quad \text{for } n \in \mathbb{N}$$

and conversely, that any sequence satisfying 2 is the upper or lower kneading sequence of some point $x \in [-1, 1]$.

From these results we can conclude that Σ^+ is exactly the set of pairs of sequences that satisfy 2, and also that the kneading invariant establishes exactly which sequences can occur as kneading sequences of some point of the interval. In section 4, we will study the structure of the set Σ^+ .

3. Bifurcation

It is well known that, for C^3 maps with negative Schwarzian derivative each attractive periodic orbit attracts at least one of the critical orbits (Singer Theorem, see [8]), so it is natural to expect that the same happens for Lorenz maps with the orbits $f^i(0^-)$ and $f^i(0^+)$ playing the role of critical orbits. Indeed we have:

THEOREM 1 (Singer like). *Let f be a Lorenz map with negative Schwarzian derivative. Then:*

1. *Each attractive or semi attractive periodic orbit attracts at least one of the critical orbits.*

2. *If 0^+ (resp. 0^-) is attracted to a periodic point of period p , then k^+ (resp. k^-) is a periodic sequence of period p .*

3. If k^+ (resp. k^-) is a periodic sequence of period p , then $f^{np}(0^+)$ (resp. $f^{np}(0^-)$) converges in n to an attractive periodic point.

The proof of the theorem is based on the next three lemmas that we will state without proof, because their proofs are just standard applications of the minimum principle for maps with negative Schwarzian derivative (see [8] or [6]).

Based on this theorem, to study bifurcations in the context of periodic orbits we just have to study the bifurcations of the critical orbits, whose combinatorics are given by the kneading invariants.

REMARK. The kneading invariants also determine the combinatorics of the non-attractive periodic orbits.

Let us now denote $f^n(a_0, b_0, x)$ by $g(x)$.

LEMMA 1. In each branch of continuity of $g(x)$ there exists at most three fixed points of $g(x)$.

LEMMA 2. If $x_1 < x_2 < x_3$ are fixed points of g in the same branch of continuity, then $g'(x_2) > 1$, $g'(x_1) < 1$ and $g'(x_3) < 1$.

LEMMA 3. In a branch of continuity with just two fixed points $x_1 < x_2$, one of the following is verified:

1. $g'(x_1) < 1$ and $g'(x_2) > 1$.
2. $g'(x_1) = 1$ and $g'(x_2) < 1$.
3. $g'(x_1) > 1$ and $g'(x_2) < 1$.
4. $g'(x_1) < 1$ and $g'(x_2) = 1$.

PROOF OF THEOREM 1. The previous lemmas prove 1. To prove 2, let $x_0 < p$ such that $f^n(a_0, b_0, p) = p$, $f^j(a_0, b_0, x_0) = 0$, $j \leq n$, $f^n(a_0, b_0, x) \geq x$ for all $x \in (x_0, p]$, $f^n(a_0, b_0, x_0)|_{(x_0, p]}$ continuous and $\left| \frac{\partial f^n}{\partial x}(a_0, b_0, x) \right| < 1$ for all $x \in (x_0, p)$. Then $f^{rn}(a_0, b_0, x_0^+)$ converges monotonically to p when $r \rightarrow \infty$ and $f^j(a_0, b_0, x)|_{(x_0, p]}$ is continuous for all $j \leq n$. Therefore $f^j(a_0, b_0, f^{rn}(a_0, b_0, x_0^+)) = f^{rn}(a_0, b_0, 0^+)$ converges monotonically to $f^j(a_0, b_0, p)$. Let us now suppose that $0 \in (f^k(a_0, b_0, 0^+), f^{k+j}(a_0, b_0, p))$ for some minimal $k \geq 1$, then $0 \in (f^{k+j}(a_0, b_0, x_0^+), f^{k+j}(a_0, b_0, p))$ and $\exists_{z \in (x_0, p)}$ such that $f^{k+j}(a_0, b_0, z) = 0$. Take r such that $rn > k + j$, then $f^{rn}(a_0, b_0, x)$ has a discontinuity at z which is absurd.

To prove 3, let k^+ be periodic of period p , this implies that $k_p^+ = R$ and so $f^p(a_0, b_0, 0^+) \geq 0$, besides, 0^+ and $f^p(a_0, b_0, 0^+)$ have the same itinerary, so $f^n(a_0, b_0, x)|_{(0, f^p(a_0, b_0, 0^+))}$ is continuous and monotonically increasing for all n and we obtain iteratively $f^{np}(a_0, b_0, 0^+) \leq f^{(n+1)p}(a_0, b_0, 0^+)$ so

$f^{np}(a_0, b_0, 0^+)$ converges in n to x_0 and $f^p(a_0, b_0, x_0^-) = \lim_{n \rightarrow \infty} f^p(a_0, b_0, f^{(n-1)p}(a_0, b_0, 0^+)) = \lim_{n \rightarrow \infty} f^{(n+1)p}(a_0, b_0, 0^+) = x_0$, that is x_0 is periodic and attracts the interval $(0, x_0)$. □

In the product of the phase space by the parameters space we have essentially three kinds of bifurcation points:

1. (a_0, b_0, x_0) such that $f^n(a_0, b_0, x_0) = x_0$, $\frac{\partial f^n}{\partial x}(a_0, b_0, x_0) = 1$ and $\frac{\partial^2 f^n}{\partial x^2}(a_0, b_0, x_0) = 0$.

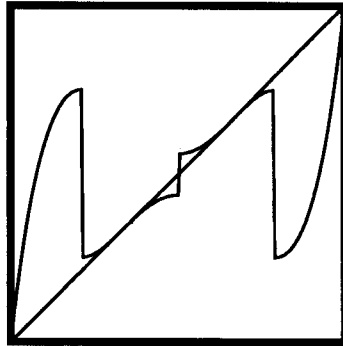


Fig. 1. Graph of $f^2_{(a,b)}$ with $a = 0.5, b = 0.5$

2. (a_0, b_0, x_0) such that $f^n(a_0, b_0, x_0) = x_0$, $\frac{\partial f^n}{\partial x}(a_0, b_0, x_0) = 1$ and $\frac{\partial^2 f^n}{\partial x^2}(a_0, b_0, x_0) \neq 0$.

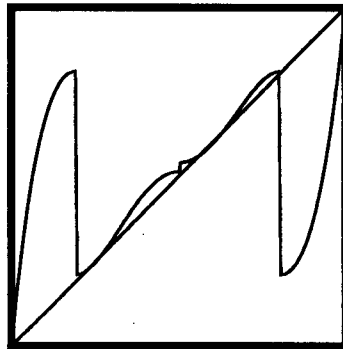


Fig. 2. Graph of $f^2_{(a,b)}$ with $a = 0.618, b = 0.423$

3. $(a_0, b_0, 0)$ such that $f^n(a_0, b_0, 0^\pm) = 0$.

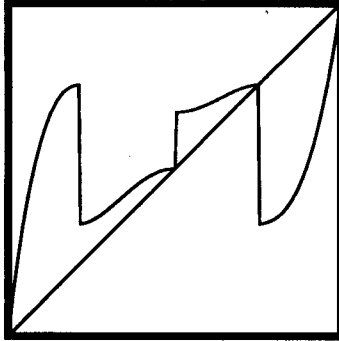


Fig. 3 a. Graph of $f_{(a,b)}^2$ with $a = 0.5, b = 0.6666$

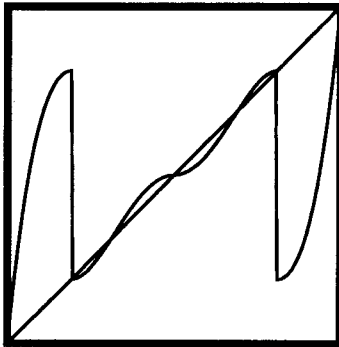


Fig. 3 b. Graph of $f_{(a,b)}^2$ with $a = 0.618, b = 0.382$

REMARK. The points of the first type (transition from just one attractive periodic orbit to the case of Lemma 3) exist only if $k^+ = \sigma^j(k^-)$ for some $j \leq n$ and do not imply any change on the kneading invariant (that is, in some neighborhood of (a_0, b_0) the kneading invariant does not change). The points of the second type imply boxes in file-type changes on the kneading invariant that we will not study in this paper. The points of the third type can imply boxes in file-type changes (if there is no other fixed point in the same branch of continuity (fig. 3 a), or in cases 2 and 4 of Lemma 3), and box-within-a-box-type changes on the kneading invariant (transition from the case of Lemma 3 (fig. 3 b) or the cases 1 and 3 of Lemma 3 to one only repulsive periodic orbit of period n). Those are the main object of this paper.

The projection of the set of points of type 3 in the parameters plane gives us the following picture that is usually called *the bifurcation skele-*

ton: the curves $f^n(a_0, b_0, 0^+) = 0$ are called the *+bones* and the curves $f^n(a_0, b_0, 0^-) = 0$ are called the *-bones*; notice that we may have different bones with the same period (e. g. $(LRRL)^\infty$ and $(LRRR)^\infty$ are two different kneading sequences associated with 0^- (critical) with period 4). The bifurcation points in which we are mainly interested are those that are intersections of a *+bone* with a *-bone*.

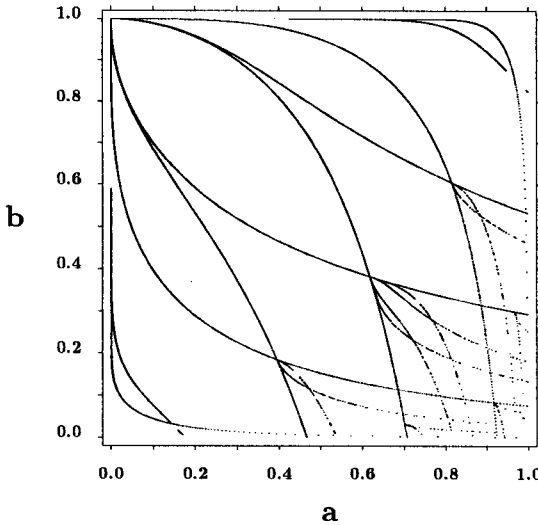


Fig. 4. Part of the Bifurcation skeleton for the family of example (the *+bones* are those that end on the right side and the *-bones* are those that end at the bottom)

4. Symbolic self similarity or box-within-a-box structure

In this section, introducing a generalization of the $*$ operation introduced in [1] we will demonstrate the existence of subsets of Σ^+ isomorphic to the full set Σ^+ , each of this subsets containing other subsets with this property, and that isomorphisms preserve the order in each of the horizontal and vertical “lines” of the symbolic square Σ^+ . From now on we will use capital letters to denote sequences or blocks of letters and small letters to denote elements of sequences or elements of blocks.

REMARK. If $(K^+, K^-) = (k_0^+ k_1^+ \dots, k_0^- k_1^- \dots)$ is an admissible kneading invariant, from 2 we can conclude that $k_0^+ k_1^+ = RL$ and $k_0^- k_1^- = LR$.

DEFINITION 4. Let $K = ((k_0^+ \cdots k_{m-1}^+)^{\infty}, (k_0^- \cdots k_{n-1}^-)^{\infty})$ and $S = (s_0^+ s_1^+ \cdots, s_0^- s_1^- \cdots)$ admissible kneading invariants. Define the product

$$K * S = (K_0^+ K_1^+ \cdots, K_0^- K_1^- \cdots)$$

where

$$K_i^+ = \begin{cases} k_0^- \cdots k_{n-1}^- & \text{if } s_i^+ = L \\ k_0^+ \cdots k_{m-1}^+ & \text{if } s_i^+ = R \end{cases}$$

$$K_i^- = \begin{cases} k_0^- \cdots k_{n-1}^- & \text{if } s_i^- = L \\ k_0^+ \cdots k_{m-1}^+ & \text{if } s_i^- = R. \end{cases}$$

THEOREM 2. If $K = ((k_0^+ \cdots k_{n-1}^+)^{\infty}, (k_0^- \cdots k_{m-1}^-)^{\infty})$ and $S = (s_0^+ s_1^+ \cdots, s_0^- s_1^- \cdots)$ are admissible kneading invariants, then $K * S$ is an admissible kneading invariant.

PROOF. Let $K * S = (a_0 a_1 \cdots, b_0 b_1 \cdots)$. We want to prove that for all p , $a_1 a_2 \cdots \leq b_{p+1} b_{p+2} \cdots \leq b_1 b_2 \cdots$ and $a_1 a_2 \cdots \leq a_{p+1} a_{p+2} \cdots \leq b_1 b_2 \cdots$.

Take $n_k^- = (\#\{i < k : s_i^- = L\})n + (\#\{i < k : s_i^- = R\})m$. We will just prove the first two inequalities, for the case $s_k^- = L$, because the rest of the proof is completely analogous.

If $p = n_k^-$ and $s_k^- = L$, we have $b_{p+1} b_{p+2} \cdots = k_1^- \cdots k_{n-1}^- \cdots = R \cdots > a_1 \cdots = L \cdots$. To prove the other inequality, $b_1 \cdots = k_1^- \cdots k_{n-1}^- k_0^+ \cdots k_{m-1}^+ \cdots$ so, if $s_k^- \cdots = s_1^- \cdots$, $b_{p+1} b_{p+2} \cdots = b_1 b_2 \cdots$, if $s_k^- \cdots < s_1^- \cdots$ then $\exists r$ such that $\forall 0 \leq i < r$, $s_{k+i}^- = s_{1+i}^-$ and $s_{k+r}^- = L < s_{1+r}^- = R$, and $b_{p+1} b_{p+2} \cdots b_{n_{k+r}^-} k_0^- \cdots < b_1 \cdots b_{n_{1+r}^-} k_0^+$. If $s_k^- = R$, $b_{p+1} \cdots = k_1^+ \cdots k_{m-1}^+ \cdots = L \cdots < b_1 \cdots = R \cdots$. To prove the other inequality, $a_1 \cdots = k_1^+ \cdots k_{m-1}^+ k_0^- \cdots k_{n-1}^- \cdots$ so, if $s_{k+1}^- = R$ it is immediate, if $s_{k+1}^- = L = s_1^+$, the proof follows from $s_{k+1}^- \cdots \geq s_1^+ \cdots$.

Let us now take p such that, for some k , $n_{k-1}^- < p < n_k^-$. If $s_{k-1}^- = L$, $b_{p+1} \cdots = k_{l+1}^- \cdots$ for some $l \in \{-1, 1, 2, \dots, n-2\}$. Since $k_{l+1}^- \cdots k_{n-1}^- k_0^- \cdots < k_1^- \cdots k_{n-1}^- k_0^- \cdots$, there exists $s \leq n-l$ such that for all $i < s$, $k_{l+i}^- = k_i^-$ and $k_{l+s}^- < k_s^-$ (otherwise we would have $k_{l+1}^- \cdots k_{n-1}^- k_0^- = k_1^- \cdots k_{n-l-1}^- k_{n-l}^-$ and, as $k_{n-l+1}^- \cdots < k_1^- \cdots$, then $k_{l+1}^- \cdots k_{n-1}^- k_0^- k_1^- \cdots > k_1^- \cdots k_{n-l-1}^- k_{n-l}^- k_{n-l+1}^- \cdots$ which is impossible). If $s < n-l$, the proof follows immediately. If $s = n-l$, then $k_s^- = R$, so if $s_k^- = L$, $a_{p+1} \cdots = k_{l+1}^- \cdots k_{n-1}^- k_0^- \cdots < k_1^- \cdots k_{s-1}^- k_s^- \cdots$. If $s_k^- = R$, we want to compare the following two sequences

$$(3) \quad k_{l+1}^- \cdots k_{n-1}^- k_0^+ k_1^+ \cdots$$

$$k_1^- \cdots k_{n-l-1}^- k_{n-l}^- k_{n-l+1}^- \cdots$$

with $k_{n-l+1}^- \cdots \geq k_1^+ \cdots$. We will divide this part of the proof in four cases:

1. $\exists s < \min \{m, l\}$ such that $k_s^+ < k_{n-l+s}^-$, and in this case the proof follows immediately.

2. $l < m$ and $k_{n-l+1}^- \cdots k_{n-1}^- = k_1^+ \cdots k_{l-1}^+$. In this case $k_l^+ = L$ (otherwise we have $k_1^+ \cdots k_{l-1}^+ k_l^+ \cdots > k_{n-l+1}^- \cdots k_{n-1}^- k_0^- \cdots$), then since $k_0^+ = R$ and $s_1^- = R$, $k_{l+1}^- \cdots k_{n-1}^- k_0^+ k_1^+ \cdots k_{l-1}^+ k_l^+ \cdots < k_1^- \cdots k_{n-l-1}^- k_{n-l}^- k_{n-l+1}^- \cdots k_{n-1}^- k_0^+ \cdots$.

3. $l = m$ and $k_{n-l+1}^- \cdots k_{n-1}^- = k_1^+ \cdots k_{m-1}^+$. In this case the proof follows from the fact that $s_{k+1}^- s_{k+2}^- \cdots \leq s_1^- s_2^- \cdots$.

4. $m < l$ and $k_{n-l+1}^- \cdots k_{n-l+m-1}^- = k_1^+ \cdots k_{m-1}^+$. This implies that $k_{n-l+m}^- = R$. If $s_{k+1}^- = L$, as $s_1^- = R$, the proof follows immediately. If $s_{k+1}^- = R$ we want to compare the sequences

$$k_{l+1}^- \cdots k_{n-1}^- k_0^+ k_1^+ \cdots k_{m-1}^+ k_0^+ \cdots$$

$$k_1^- \cdots k_{n-l-1}^- k_{n-l}^- k_{n-l+1}^- \cdots k_{n-l+m-1}^- k_{n-l+m}^- \cdots$$

that is completely analogous to 3 so we can repeat all the previous process, but since m is finite there exists α (minimum) such that $n - l + \alpha m \geq n$ and so, after at most α steps we will have one of the situations 1, 2 or 3.

The second inequality is proved, let us now see the first one, that is, $b_{p+1} \cdots \geq a_1 \cdots$. Since $k_{l+1}^- \cdots k_{n-1}^- k_0^- \cdots \geq k_1^+ \cdots k_{m-1}^+ k_0^+ \cdots$ there are three possible cases, as the case $n - l = m$ and $k_{l+i}^- = k_i^+$ for all $i < m$ can not happen, since in that case we would have $k_{l+1}^- \cdots k_{n-1}^- k_0^- \cdots < k_1^+ \cdots k_{m-1}^+ k_0^+ \cdots$.

1. $\exists s < \min \{n-l, m\}$ such that $k_s^+ < k_{l+s}^-$, and in this case the proof follows immediately.

2. $n - l < m$ and $k_{l+1}^- \cdots k_{n-1}^- = k_1^+ \cdots k_{n-l-1}^+$. In this case $k_{n-l}^+ = L$, so if $s_k^- = R$ the proof follows immediately. If $s_k^- = L$ we want to compare the sequences

$$k_{l+1}^- \cdots k_{n-1}^- k_0^- k_1^- \cdots$$

$$k_1^+ \cdots k_{n-l-1}^+ k_{n-l}^+ k_{n-l+1}^+ \cdots$$

and the situation is analogous to 3.

3. $m < n - l$ and $k_{l+1}^- \cdots k_{l+m-1}^- = k_1^+ \cdots k_{m-1}^+$. In this case $k_{l+m}^- = R$, and, since $s_1^+ = L$, we have immediately $k_{l+1}^- \cdots k_{l+m-1}^- k_{l+m}^- \cdots > k_1^+ \cdots k_{m-1}^+ k_0^+ \cdots$. □

PROPOSITION 1. Let $K = ((K^+)^{\infty}, (K^-)^{\infty}) = ((k_0^+ \cdots k_{m-1}^+)^{\infty}, (k_0^- \cdots k_{n-1}^-)^{\infty})$, $X = (X^+, X^-)$ and $Y = (Y^+, Y^-)$ admissible kneading

invariants such that $X^+ < Y^+$ and $X^- < Y^-$. Then if $K * X = (A^+, A^-)$ and $K * Y = (B^+, B^-)$, $A^+ < B^+$ and $A^- < B^-$.

PROOF. $X^+ < Y^+ \Leftrightarrow \exists_r$ such that $x_i^+ = y_i^+$ for all $i < r$ and $x_r^+ = L < y_r^+ = R$, so $A^+ = K_0^+ K_1^+ \dots$, $B^+ = K_0^{'+} K_1^{'+} \dots$ where $K_i^+ = K_i^{'+}$ for all $i < r$ and $K_r^+ = k_0^- \dots k_{n-1}^- = Lk_1^- \dots k_{n-1}^- < Rk_1^+ \dots k_{m-1}^+ = k_0^+ \dots k_{m-1}^+ = K_r^{'+}$. The proof of the other inequality is analogous.

REMARK. The proof of the inequality for the first members of the pair do not depend of the second members X^- and Y^- and vice versa.

PROPOSITION 2. Let $K = ((K^+)^{\infty}, (K^-)^{\infty}) = ((k_0^+ \dots k_{m-1}^+)^{\infty}, (k_0^- \dots k_{n-1}^-)^{\infty})$, $S = ((S^+)^{\infty}, (S^-)^{\infty}) = ((s_0^+ \dots s_{q-1}^+)^{\infty}, (s_0^- \dots s_{p-1}^-)^{\infty})$ and $T = (t_0^+ t t_1^+ \dots, t_0^- t_1^- \dots)$ admissible kneading invariants then $(K * S) * T = K * (S * T)$.

PROOF. Let $XL = (X^+, X^-)L = X^-$, $XR(X^+, X^-)R = X^+$ and $Xy_1y_2 \dots (X^+, X^-)y_1y_2 \dots = Xy_1Xy_2 \dots$ then $((K^+, K^-) * (S^+, S^-)) * (T^+, T^-) = (Ks_0^+ \dots Ks_{q-1}^+, Ks_0^- \dots Ks_{p-1}^-) * (T^+, T^-) = ((Ks_0^+ \dots Ks_{q-1}^+, Ks_0^- \dots Ks_{p-1}^-) t_0^+ (Ks_0^+ \dots Ks_{q-1}^+, Ks_0^- \dots Ks_{p-1}^-) t_1^+ \dots, (Ks_0^+ \dots Ks_{q-1}^+, Ks_0^- \dots Ks_{p-1}^-) t_0^- (Ks_0^+ \dots Ks_{q-1}^+, Ks_0^- \dots Ks_{p-1}^-) t_1^- \dots)$.
 On the other hand $(K^+, K^-) * ((S^+, S^-) * (T^+, T^-)) = (K^+, K^-) * (St_0^+ St_1^+ \dots, St_0^- St_1^- \dots) = (K(St_0^+)K(St_1^+) \dots, K(St_0^-)K(St_1^-) \dots)$ but, since

$$(Ks_0^+ \dots Ks_{q-1}^+, Ks_0^- \dots Ks_{p-1}^-) t_i^{\pm} = \begin{cases} Ks_0^+ \dots Ks_{q-1}^+ & \text{if } t_i^{\pm} = R \\ Ks_0^- \dots Ks_{p-1}^- & \text{if } t_i^{\pm} = L \end{cases}$$

and

$$K(St_i^{\pm}) = \begin{cases} KS^+ & \text{if } t_i^{\pm} = R \\ KS^- & \text{if } t_i^{\pm} = L \end{cases} = \begin{cases} Ks_0^+ \dots Ks_{q-1}^+ & \text{if } t_i^{\pm} = R \\ Ks_0^- \dots Ks_{p-1}^- & \text{if } t_i^{\pm} = L \end{cases}$$

the proposition is proved.

Let $K = (K^+, K^-)$ be an admissible kneading invariant and let $C^+(K^+, K^-) = \{(S^+, S^-) \in \Sigma^+ \text{ such that } K^+ \leq S^+ \text{ and } K^- < S^-\}$ analogously let $C^-(K^+, K^-) = \{(S^+, S^-) \in \Sigma^+ \text{ such that } K^+ > S^+ \text{ and } K^- \geq S^-\}$. We can now state our main result.

THEOREM 3. If $K = ((K^+)^{\infty}, (K^-)^{\infty}) = ((k_0^+ \dots k_{m-1}^+)^{\infty}, (k_0^- \dots k_{n-1}^-)^{\infty})$, is an admissible kneading invariant, then

$$C^+(k_0^+ \dots k_{m-1}^+ (k_0^- \dots k_{n-1}^-)^{\infty}, (k_0^- \dots k_{n-1}^-)^{\infty}) \cap C^-(k_0^+ \dots k_{m-1}^+ (k_0^- \dots k_{n-1}^-)^{\infty}, k_0^- \dots k_{n-1}^- (k_0^+ \dots k_{m-1}^+)^{\infty}) = \{K * S : S \in \Sigma^+\}.$$

PROOF. The inclusion \supseteq follows trivially from the previous proposition.

In order to prove the other inclusion, we will take an admissible kneading invariant (A, B) such that

$$k_0^+ \cdots k_{m-1}^+ (k_0^- \cdots k_{n-1}^-)^\infty \leq A < (k_0^+ \cdots k_{m-1}^+)^\infty$$

$$(k_0^- \cdots k_{n-1}^-)^\infty < B \leq k_0^- \cdots k_{n-1}^- (k_0^+ \cdots k_{m-1}^+)^\infty$$

and prove that $A = A_0 A_1 \cdots A_{p-1} A_p \cdots$, $B = B_0 B_1 \cdots B_{p-1} B_p \cdots$ where

$$A_i = \begin{cases} k_0^- \cdots k_{n-1}^- & \text{if } (a_i)_0 = L \\ k_0^+ \cdots k_{m-1}^+ & \text{if } (a_i)_0 = R \end{cases}$$

$$B_i = \begin{cases} k_0^- \cdots k_{n-1}^- & \text{if } (b_i)_0 = L \\ k_0^+ \cdots k_{m-1}^+ & \text{if } (b_i)_0 = R. \end{cases}$$

The proof follows by induction on p : it clearly holds if $p = 0$. Let us see $p = 1$:

Since $B_0 = k_0^- \cdots k_{n-1}^-$, $B \leq k_0^- \cdots k_{n-1}^- (k_0^+ \cdots k_{m-1}^+)^\infty \Rightarrow B_1 \leq k_0^+ \cdots k_{m-1}^+$. On the other hand, since $A_0 = k_0^+ \cdots k_{m-1}^+$ and A is minimal, $B_1 \geq k_0^+ \cdots k_{m-1}^+$ so $B_1 = k_0^+ \cdots k_{m-1}^+$. Analogously we see that $A_1 = k_0^- \cdots k_{n-1}^-$.

We will now suppose that the claim is valid for all q such that $1 \leq q \leq p$ and show that

$$B_{p+1} = \begin{cases} k_0^- \cdots k_{n-1}^- & \text{if } (b_{p+1})_0 = L \\ k_0^+ \cdots k_{m-1}^+ & \text{if } (b_{p+1})_0 = R. \end{cases}$$

The proof for A_{p+1} is analogous.

If $(b_{p+1})_0 = L$, since B is maximal $B_{p+1} \cdots \leq k_0^- \cdots k_{n-1}^- k_0^+ \cdots \Rightarrow B_{p+1} \leq k_0^- \cdots k_{n-1}^-$. If $B_p = k_0^+ \cdots k_{m-1}^+$ then, because A is minimal $B_p B_{p+1} \cdots = k_0^+ \cdots k_{m-1}^+ B_{p+1} \cdots \geq k_0^+ \cdots k_{m-1}^+ k_0^- \cdots k_{n-1}^- \cdots \Rightarrow B_{p+1} \geq k_0^- \cdots k_{n-1}^-$ so $B_{p+1} = k_0^- \cdots k_{n-1}^-$. If $B_p = k_0^- \cdots k_{n-1}^-$, $(b_p)_0 = L$ so, since $(b_1)_0 = R$, $\exists 1 \leq k < p$ (maximum) such that $(b_k)_0 = R$ and $B_k = k_0^+ \cdots k_{m-1}^+$, then we have $B_k \cdots B_p B_{p+1} \cdots = k_0^+ \cdots k_{m-1}^+ (k_0^- \cdots k_{n-1}^-)^{p-k} B_{p+1} \cdots$. Using the hypothesis and the fact that A is minimal (recall that $(b_{p+1})_0 = L$ and $p-k+1 \leq p$), $A = k_0^+ \cdots k_{m-1}^+ (k_0^- \cdots k_{n-1}^-)^{p-k+1} \cdots$ so $B_{p+1} \cdots \geq k_0^- \cdots k_{n-1}^- \cdots \Rightarrow B_{p+1} = k_0^- \cdots k_{n-1}^-$.

If $(b_{p+1})_0 = R$, as A is minimal $B_{p+1} \cdots \geq k_0^+ \cdots k_{m-1}^+ \cdots$. If $B_p = k_0^- \cdots k_{n-1}^-$, since B is minimal $B_p B_{p+1} \cdots \leq B_0 B_1 \cdots = k_0^- \cdots k_{n-1}^- k_0^+ \cdots k_{m-1}^+ \cdots \Rightarrow B_{p+1} = k_0^+ \cdots k_{m-1}^+$. If $B_p = k_0^+ \cdots k_{m-1}^+$ then $(b_p)_0 = R$. Since $(b_0)_0 = R$, $\exists 0 \leq k < p$ (maximum) such that $(b_k)_0 = L$,

and $B_k \cdots B_p B_{p+1} \cdots = k_0^- \cdots k_{n-1}^- (k_0^+ \cdots k_{m-1}^+)^{p-k} \cdots$. If $k = 0$, $B \leq k_0^- \cdots k_{n-1}^- (k_0^+ \cdots k_{m-1}^+)^{\infty} \Rightarrow B_{p+1} = k_0^+ \cdots k_{m-1}^+$. If $k > 0$, $p - k < p$ and using the hypothesis and the fact that B is maximal, $B = k_0^- \cdots k_{n-1}^- (k_0^+ \cdots k_{m-1}^+)^{p-k+1} \cdots \Rightarrow B_{p+1} = k_0^+ \cdots k_{m-1}^+$.

To finish the proof we will now show that $((a_0)_0 (a_1)_0 \cdots, (b_0)_0 (b_1)_0 \cdots)$ is an admissible kneading invariant.

Supposing $((a_0)_0 (a_1)_0 \cdots, (b_0)_0 (b_1)_0 \cdots)$ not admissible, one of the following situations must happen.

1. $\exists p \geq 0$ such that $(b_p)_0 (b_{p+1})_0 \cdots > (b_1)_0 (b_2)_0 \cdots$.
2. $\exists p \geq 0$ such that $(a_p)_0 (a_{p+1})_0 \cdots > (b_1)_0 (b_2)_0 \cdots$.
3. $\exists p \geq 0$ such that $(a_p)_0 (a_{p+1})_0 \cdots < (a_1)_0 (a_2)_0 \cdots$.
4. $\exists p \geq 0$ such that $(b_p)_0 (b_{p+1})_0 \cdots < (a_1)_0 (a_2)_0 \cdots$.

Since all situations are very similar we will just make the proof for the first situation.

The case 1 is equivalent to the existence an $r > 0$ such that $(b_{p+i})_0 = (b_{i+1})_0$ for all $i < r$ and $(b_{p+r})_0 = R > (b_{r+1})_0 = L$. We can take p such that $(b_{p-1})_0 = L$, because if $(b_{p-1})_0 = R$ then $(b_{p-1})_0 (b_p)_0 \cdots > (b_p)_0 (b_{p+1})_0 \cdots$. But in that case, $B = k_0^- \cdots k_{n-1}^- B_1 \cdots B_r k_0^- \cdots k_{n-1}^- \cdots < k_0^- \cdots k_{n-1}^- B_p \cdots B_{p+r-1} k_0^+ \cdots k_{m-1}^+ \cdots$.

REMARK. Since $(K * S) * T = K * (S * T)$, the box associated to $K * S$ is inside the box associated to K and so on.

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