Annales Mathematicae Silesianae 12 (1998), 9–13

Prace Naukowe Uniwersytetu Śląskiego nr 1751, Katowice

ON CARLITZ THEOREM FOR BERNOULLI POLYNOMIALS

KRYSTYNA M. BARTZ

Abstract. The well-known Carlitz theorem for the Bernoulli numbers B_n (see [3]) is extended to the case of values of the Bernoulli polynomials $B_n(y)$ at rational points $\frac{a}{b}$, where $(b, n!) = 1$.

In the present note we will prove the following generalization of the Carlitz theorem (see Lemma 3 below) to the values of Bernoulli polynomials $B_n(y)$ at rational points.

THEOREM. *Let $m > 1$, a and b be positive integers, $(b,(2m)!) = 1$ and $(a, b) = 1$. If p is any prime number such that $(p-1)p^h$ divides $2m$, then the numerator of $B_{2m}\left(\frac{a}{b}\right) + \frac{1}{p} - 1$ is divisible by p^h. That is,*

$$pB_{2m}\left(\frac{a}{b}\right) \equiv p - 1 \pmod{p^{h+1}}.$$

REMARK. Putting here $a = 0$, $b = 1$, we get the Carlitz theorem, since $B_n(0) = B_n$.

In the proof we will use the following easily proved property of the sums

$$S_k(n) = 1^k + 2^k + \cdots + (n-1)^k.$$

Received on June 18, 1998.

1991 *Mathematics Subject Classification.* 11B68.

Key words and phrases: Bernoulli polynomials.

Supported by the State Committee for Scientific Research (KBN) of Poland under Grant 2 P03A 028 09.

LEMMA 1. *Let* m *be a positive integer and let* $n = \prod_{\substack{p \text{ prime} \\ p > m+1}} p.$ *Then* n *divides* $S_m(n)$.

PROOF. We first remark that

$$(k+1)^{m+1} - k^{m+1} = 1 + \binom{m+1}{1} k + \cdots + \binom{m+1}{m} k^m$$

and putting $k = 0, 1, \ldots, n-1$ and adding we get

$$n^{m+1} = m + \binom{m+1}{1} S_1(n) + \cdots + \binom{m+1}{m} S_m(n) .$$

Now, by the induction on m, first for $m = 1 : 2S_1(n) = n^2 - n$ and since $(2, n) = 1$ we have $n \mid S_1(n)$.

Assume the lemma is true for $m = 1, 2, \ldots, k-1$ and if we let $m = k$ and $n = \prod_{\substack{p \text{ prime} \\ p > k+1}} p$ (of course this n is also good for $m = 1, 2, \ldots, k-1$) we get $n \mid S_1(n), \ldots, n \mid S_{k-1}(n)$ and

$$(k+1) S_k(n) = n^{k+1} - n - \binom{k+1}{1} S_1(n) - \cdots - \binom{k+1}{k-1} S_{k-1}(n) .$$

Since $(k+1, n) = 1$, we conclude that $n \mid S_k(n)$ and the proof is complete.

The next lemma is a suitable version of the von Staudt-Clausen theorem for Bernoulli polynomials at rational points (compare [1] and [2]).

LEMMA 2. *Let* a, b *and* n *be positive integers and let* $(b, n!) = 1$ *and* $(a, b) = 1$. *Then* $b^n(B_n((\frac{a}{b}) - B_n)$ *is an integer divisible by* n.

PROOF. As is well known, the Bernoulli polynomials $B_n(y)$ may be defined by

$$\sum_{m=0}^{\infty} \frac{B_m(y) x^m}{m!} = \frac{x e^{yx}}{e^x - 1}$$

and Bernoulli numbers similarly (for $y = 0$)

$$\sum_{m=0}^{\infty} \frac{B_m x^m}{m!} = \frac{x}{e^x - 1} .$$

Then we have

$$(*) \qquad \sum_{m=1}^{\infty} \frac{(B_m(y) - B_m) x^m}{m!} = \frac{x(e^{yx} - 1)}{e^x - 1}$$

Putting $y = a$, we get in the simplest case in which $b = 1$,

$$\sum_{n=1}^{\infty} \frac{B_n(a) - B_n}{n!} x^n = x(1 + e^x + \cdots + e^{(a-1)x})$$
$$= x\left(1 + \sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} + \cdots + \sum_{n=0}^{\infty} \frac{(a-1)^n x^n}{n!}\right)$$

and we see that

$\frac{B_n(a)-B_n}{n} = 1 + 2^{n-1} + \cdots + (a-1)^{n-1}$ is an integer.

Let us consider next the case $b = \prod_{\substack{p \text{ prime} \\ p>n}} p$ and $a = 1$, putting $y = \frac{1}{b}$ in (*).

Then we get

$$\sum_{n=1}^{\infty} \frac{(B_n(\frac{1}{b}) - B_n)x^n}{n!} = \frac{x}{1 + e^{\frac{1}{b}x} + \cdots + e^{\frac{b-1}{b}x}}$$

and further putting $S_m(b) = 1^m + \cdots + (b-1)^m$ we have

$$\left(\frac{B_1(\frac{1}{b}) - B_1}{1!} + \frac{B_2(\frac{1}{b}) - B_2}{2!}x + \cdots + \frac{B_n(\frac{1}{b}) - B_n}{n!}x^{n-1} + \cdots\right)$$
$$\left(b + \frac{S_1(b)}{b \cdot 1!}x + \frac{S_2(b)}{b^2 2!}x^2 + \cdots\right) = 1\,.$$

Now, comparing the coefficients of like powers of x and doing induction on n, we get $b(B_1(\frac{1}{b}) - B_1) = 1$ and next for $n > 1$

$$\frac{(B_n(\frac{1}{b}) - B_n)b}{n!} + \frac{(B_{n-1}(\frac{1}{b}) - B_{n-1})}{(n-1)!}\frac{S_1(b)}{b \cdot 1!} + \frac{(B_{n-2}(\frac{1}{b}) - B_{n-2})}{(n-2)!}\frac{S_2(b)}{b^2 2!}$$
$$+ \cdots + \frac{B_2(\frac{1}{b}) - B_2}{2!}\frac{S_{n-2}(b)}{b^{n-2}(n-2)!} + \frac{B_1(\frac{1}{b}) - B_1}{1!}\frac{S_{n-1}(b)}{b^{n-1}(n-1)!} = 0$$

and multiplying by $b^{n-1}(n-1)!$, we obtain that

$$\frac{B_n(\frac{1}{b}) - B_n}{n} b^n = -\sum_{k=1}^{n-1} \frac{(B_k(\frac{1}{b}) - B_k)b^k S_{n-k}(b)}{b \cdot k} \binom{n-1}{n-k}$$

is an integer, since by Lemma 1: $b \mid S_{n-k}(b)$ and by induction hypothesis for $k < n$ we have that $k \mid (B_k(\frac{1}{b}) - B_k)b^k$. So, the result holds for $a = 1$. Now, by the addition formula, doing induction on a with b fixed we get for some integers k and l

$$b^n B_n\left(\frac{a+1}{b}\right)=\sum_{m=0}^{n}\binom{n}{m}B_m\left(\frac{a}{b}\right)b^m=1+\sum_{m=1}^{n}\binom{n}{m}(mk+b^mB_m)$$
$$=1+kn\sum_{m=0}^{n-1}\binom{n-1}{m}+\sum_{m=1}^{n}\binom{n}{m}b^mB_m=2^{n-1}nk+\sum_{m=0}^{n}\binom{n}{m}b^mB_m$$
$$=2^{n-1}kn+b^nB_n\left(\frac{1}{b}\right)=ln+b^nB_n\ .$$

This completes the proof.

The next lemma is the well-known Carlitz theorem for Bernoulli numbers.

LEMMA 3. (Carlitz theorem, see [3]) *Let $m>1$ be any positive integer. If p is any prime number and if $(p-1)p^h$ divides $2m$, then p^h divides the numerator of $B_{2m}+\frac{1}{p}-1$. That is,*

$$pB_{2m}\equiv p-1 \pmod{p^{h+1}}.$$

We conclude with a proof of our theorem that depends only on Lemmas 2 and 3.

PROOF. By Lemma 2 we have

$$b^{2m}\left(B_{2m}\left(\frac{a}{b}\right)\frac{1}{p}-1\right)=b^{2m}\left(B_{2m}+\frac{1}{p}-1\right)+2mk\ ,$$

where k is an integer. Now, by Carlitz theorem (Lemma 3), if $(p-1)p^h \mid 2m$, then p^h divides the numerator of $B_{2m}+\frac{1}{p}-1$ and since $p^h\mid 2m$, we get that p^h divides the numerator of $b^{2m}(B_{2m}\left(\frac{a}{b}\right)+\frac{1}{p}-1)$, and since $(p,b)=1$, we obtain the assertion of our theorem.

REFERENCES

[1] G. ALMKVIST, A. MEURMAN, *Values of Bernoulli polynomials and Hurwitz's zeta function at rational points*, C. R. Math. Rep. Acad. Canada, **XIII.3** (1991), 104–108.

[2] K. BARTZ, J. RUTKOWSKI, *On the von Staudt–Clausen theorem*, C. R. Math. Rep. Acad. Canada, **XV.1** (1993), 46–48.

[3] L. CARLITZ, *Some congruences for the Bernoulli numbers*, Amer. J. Math., **75** (1953), 163–172.

FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
ADAM MICKIEWICZ UNIVERSITY
MATEJKI 48/49
60-769 POZNAŃ
POLAND

e-mail:
kbartz@math.amu.edu.pl