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CYCLES OF RATIONAL MAPPINGS IN ALGEBRAICALLY CLOSED FIELDS OF POSITIVE CHARACTERISTICS

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1. Let K be a field. Let us define $\bar{K}$ as a formal set $K \cup \{\infty\}$ (which can be identified with $P^1(K)$). For convenience we put that the degree of a zero polynomial is zero (not $-\infty$). For relatively prime polynomials $f, g \in K[X]$ of degrees n, m and leading coefficients a_n, b_m, respectively, we define a rational function $\phi(X) = \dfrac{f(X)}{g(X)}$ as a mapping $\phi : \bar{K} \mapsto \bar{K}$ as follows:

$$\phi(\xi) = \begin{cases} f(\xi)/g(\xi) & \text{for} \quad \xi \in K, g(\xi) \neq 0 \\ \infty & \text{for} \quad \xi \in K, g(\xi) = 0 \\ a_n/b_m & \text{for} \quad \xi = \infty, n = m \\ \infty & \text{for} \quad \xi = \infty, n > m \\ 0 & \text{for} \quad \xi = \infty, n < m, \end{cases}$$

where we put $1/0$ as ∞.

More generally for $(f, g) \neq (0, 0)$ and $\phi = f/g$ we put $\phi = f_1/g_1$, where $f_1 = f/d, g_1 = g/d, d = \gcd(f, g)$.

A k-tuple $x_0, \dots, x_{k-1}$ of distinct elements of $\bar{K}$ is called a cycle of ϕ of length k if

$$\phi(x_i) = x_{i+1} \quad \text{for} \quad i = 0, 1, \dots, k-2 \quad \text{and} \quad f(x_{k-1}) = x_0.$$

The set of all positive integers which are not lengths of a cycle for ϕ will be denoted by $Exc(\phi)$.

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The case of a rational mapping not of the form $(aX+b)/(cX+d)$ over algebraically closed field of characteristic zero was solved by I. N. Baker [2], who showed that $Exc(\phi)$ is always finite and gave all possible examples of $Exc(\phi)$. However for positive characteristic the situation differs, in fact it was shown in [3] that for some polynomials ϕ the set $Exc(\phi)$ is infinite.

The aim of this paper is to prove that for a large class of rational ϕ over algebraically closed field of positive characteristic the set $Exc(\phi)$ is "lacunar", i.e. either $Exc(\phi)$ is finite or $Exc(\phi)=\{a_1<a_2<\dots\}$ with $a_{i+1}/a_i>\lambda>1$ for all i.

2. We start with some simple properties of rational mappings. For $\phi,\psi\in K(X)$ we define $\phi\circ\psi$ as a rational function which occurs by putting $\psi(X)$ for X in $\phi(X)$. So for $\phi(X)=f(X)/g(X), f(X)=a_nX^n+\dots+a_0, g(X)=b_mX^m+\dots+b_0,\ \psi(X)=r(X)/s(X), \gcd(f,g)=\gcd(r,s)=1$ we get

$$\phi\circ\psi(X)=\begin{cases}\dfrac{(a_nr^n+\dots+a_0s^n)s^{m-n}}{b_mr^m+\dots+b_0s^m} & \text{for}\quad m\geqslant n\\[2ex] \dfrac{a_nr^n+\dots+a_0s^n}{(b_mr^m+\dots+b_0s^m)s^{n-m}} & \text{for}\quad n>m.\end{cases}\tag{1}$$

Notice that the numerators and denominators in the last formula are co-prime. Notice also that we could define $\phi\circ\psi$ for non-proper (i.e. of shape 1/0) functions.

Lemma 1. *$(\phi\circ\psi)(\xi)=\phi(\psi(\xi))$ for all $\xi\in\bar K$ and rational ϕ,ψ.*

Proof. Standard computation. □

Lemma 2. *If for $a,b,c,d\in K, ad-bc\neq 0$, then a homography $\phi(X)=\dfrac{aX+b}{cX+d}$ is an invertible mapping $\phi:\bar K\mapsto\bar K$.*

Proof. Obvious. □

Definition. Two rational mappings ϕ,ψ are called associated ($\phi\sim\psi$), provided $\phi\circ h=h\circ\psi$ holds for some homography h.

Lemma 3. *Every non-constant rational ϕ over algebraically closed field K is associated with ψ of shape $f(X)/g(X)$, where $\deg f>\deg g$.*

Proof. Let $\phi(X)=f(X)/g(X)$, $\gcd(f,g)=1$, $f(X)=a_nX^n+\cdots$, $g(X)=b_mX^m+\cdots$. As ϕ is non-constant $f,g\neq 0$. If $n<m$ and $\xi\neq 0$ then

$$\frac{a_n(X+\xi)^n+\cdots}{b_m(X+\xi)^m+\cdots}-\xi=\frac{-\xi b_mX^m+\cdots}{b_mX^m+\cdots}.$$

So, we see that ϕ is associated with some $\psi = F/G$, $\deg F = \deg G$. Therefore we can restrict ourself to the case $n = m$. Let α be a root of $Xg(X) - f(X), h(X) = (\alpha X + 1)/X$. Then $\phi \sim h^{-1} \circ \phi \circ h$, and the last function has the needed property. In fact, it equals

$$\frac{g(\alpha + \frac{1}{X})}{f(\alpha + \frac{1}{X}) - \alpha g(\alpha + \frac{1}{X})} = \frac{X^m g(\alpha + \frac{1}{X})}{X^m(f(\alpha + \frac{1}{X}) - \alpha g(\alpha + \frac{1}{X}))},$$

and the degree of the numerator is m (owing to $g(\alpha) \neq 0$, which follows from $\gcd(f, g) = 1$, and $f(\alpha) - \alpha g(\alpha) = 0$), whereas the degree of the denominator is smaller. □

3.

THEOREM. *Let K be an algebraically closed field of positive characteristic, and ϕ a rational function over K associated with $\psi(X) = f(X)/g(X)$, where $\deg g < \deg f - \sqrt{\deg f}$. Then $Exc(\phi)$ is lacunar.*

PROOF. Let us define for a natural n the set

$$Z(n) = \{j : j|n, j < n\}.$$

Because of $\phi \sim \psi$ we have $Exc(\phi) = Exc(\psi)$ and so it suffices to consider $Exc(\psi)$. Put $F = \deg f, G = \deg g, d = F - G, F = (F - G)^{1+\Delta}$. Our assumptions imply $d > 1$ and $0 \leqslant \Delta < 1$. For $j = 1, 2, \ldots$ denote by ψ_j the j-th iterate of ψ, $\psi_j(X) = \dfrac{A_{(j)}(X)}{B_{(j)}(X)}$, and $\gcd(A_{(j)}, B_{(j)}) = 1$.

By simple induction we get

$$\deg A_{(j)} = F^j, \deg B_{(j)} = F^j - (F - G)^j. \tag{2}$$

Assume that there are no cycles of length $n, k, n > k$ for ψ. Let us consider (like in [1]) the function

(3)

$$T(X) = \frac{\psi_n(X) - X}{\psi_{n-k}(X) - X} = \frac{(A_{(n)} - XB_{(n)})B_{(n-k)}}{(A_{(n-k)} - XB_{(n-k)})B_{(n)}} = \frac{R(X)}{Q(X)} = \left(\frac{r(X)}{q(X)}\right)^{p^M},$$

where $\gcd(R, Q) = 1$, p is the characteristic of K, M is as big as possible, i.e. $(r/q)' \neq 0$.

Notice, that $\gcd(A_{(n)}-XB_{(n)}, B_{(n)})=\gcd(A_{(n-k)}-XB_{(n-k)}, B_{(n-k)})=1$. Put $m = \deg Q$, so in view of (2) we have

$$\deg R = d^n - d^{n-k} + m, \quad \deg r = p^{-M}(d^n - d^{n-k} + m), \quad \deg q = p^{-M}m.$$

LEMMA 4. *Under the above assumptions*

$$\text{i) } \#\{\xi \in K : T(\xi) = 0\} \leqslant F^{n-k} - (F-G)^{n-k} + \textstyle\sum_{j\in Z(n)} F^j, \tag{4}$$

$$\text{ii) } \#\{\xi \in K : T(\xi) = 1\} \leqslant 2F^{n-k} - (F-G)^{n-k} + \textstyle\sum_{j\in Z(k)}(F^{n-k+j} + F^{n-k} - (F-G)^{n-k}). \tag{5}$$

PROOF. i) If $\xi \in K$ and $T(\xi) = 0$, then we have $R(\xi) = 0$ and $((A_{(n)} - XB_{(n)})B_{(n-k)})(\xi) = 0$.

$$\#\{\xi : B_{(n-k)}(\xi) = 0\} \leqslant F^{n-k} - (F-G)^{n-k} \tag{6}$$

If $(A_{(n)} - XB_{(n)})(\xi) = 0$ then $B_{(n)}(\xi) \neq 0$ so $\dfrac{A_{(n)}(\xi) - \xi B_{(n)}(\xi)}{B_{(n)}(\xi)} = 0$ and $\psi_n(\xi) = \xi$.

As there are no cycles of length n then $\psi_j(\xi) = \xi$ for some $j \in Z(n)$. That means $\dfrac{A_{(j)}(\xi)}{B_{(j)}(\xi)} = \xi$ and $A_{(j)}(\xi) - \xi B_{(j)}(\xi) = 0$. So

$$\#\{\xi : (A_{(n)} - XB_{(n)})(\xi) = 0\} \leqslant \sum_{j\in Z(n)} F^j. \tag{7}$$

(6) and (7) give the statement.

ii) If $\xi \in K$ is such that $T(\xi) = 1$ then $R(\xi) = Q(\xi) \neq 0$ and

$$((A_{(n)} - XB_{(n)})B_{(n-k)})(\xi) = ((A_{(n-k)} - XB_{(n-k)})B_{(n)})(\xi). \tag{8}$$

So $B_{(n)}(\xi) = 0$ implies $B_{(n-k)}(\xi) = 0$. Hence

$$\#\{\xi : ((A_{(n-k)} - XB_{(n-k)})B_{(n)})(\xi) = 0\} \leqslant 2F^{n-k} - (F-G)^{n-k}. \tag{9}$$

For $\xi \in K : T(\xi) = 1$ and $((A_{(n-k)} - XB_{(n-k)})B_{(n)})(\xi) \neq 0$ we have

$$\frac{A_{(n)} - XB_{(n)}}{B_{(n)}}(\xi) = \frac{A_{(n-k)} - XB_{(n-k)}}{B_{(n-k)}}(\xi), \; \frac{A_{(n)}}{B_{(n)}}(\xi) = \frac{A_{(n-k)}}{B_{(n-k)}}(\xi),$$

and finally $\psi_k(\psi_{n-k}(\xi)) = \psi_{n-k}(\xi)$.

There are no cycles of length k for ψ, so there is $j \in Z(k)$ such that $\psi_j(\psi_{n-k}(\xi)) = \psi_{n-k}(\xi)$. So for some $j \in Z(k)$ we have $\psi_{n-k+j}(\xi) = \psi_{n-k}(\xi)$.

Observe that $\psi_{n-k}(\xi) \neq \infty$. Indeed, otherwise we would have $B_{(n-k)}(\xi) = 0$ and by (8) also $B_{(n)}(\xi) = 0$ which is not possible. Moreover we have

$$\frac{A_{(n-k+j)}}{B_{(n-k+j)}}(\xi) = \frac{A_{(n-k)}}{B_{(n-k)}}(\xi)$$

and finally

$$(A_{(n-k+j)}B_{(n-k)} - A_{(n-k)}B_{(n-k+j)})(\xi) = 0.$$

Therefore

$$\#\{\xi : T(\xi) = 1, ((A_{(n-k)} - XB_{(n-k)})B_{(n)})(\xi) \neq 0\} \leqslant \sum_{j\in Z(k)} (F^{n-k+j} + F^{n-k} - (F-G)^{n-k}).$$

This and (9) give the statement. □

In the text below $C, \tilde{C}, C_1, C_2, \ldots$ mean some absolute constants.

COROLLARY. $\#\{\xi : T(\xi) \in \{0,1\}\} \leqslant CF^{n-k/2}$.

PROOF. As $G < F - \sqrt{F}$ then $F \geqslant 2$. We have

$$F^{n-k} - (F-G)^{n-k} + \sum_{j\in Z(n)} F^j \leqslant F^{n-k} + C_1F^{n/2} \leqslant C_2F^{n-k/2}$$

and (as $k \leqslant 2F^{k/2}$)

$$\begin{aligned} 2F^{n-k} - (F-G)^{n-k} + \sum_{j\in Z(k)} (F^{n-k+j} + F^{n-k} - (F-G)^{n-k}) \\ \leqslant C_3(F^{n-k} + F^{n-k/2} + kF^{n-k}) \\ \leqslant C_4F^{n-k/2}. \end{aligned}$$ □

REMARK. If $\xi \in K$ is a zero of R/Q (where $\gcd(R,Q) = 1$) then $R(X) = (X-\xi)^w R_1(X), R_1(\xi) \neq 0$, w-multiplicity of ξ (as a root of R/Q). In that case ξ has multiplicity $\geqslant w-1$ (as a root of $(R/Q)'$).

REMARK. If $R/Q = (r/q)^{p^M}$, then $\frac{R}{Q}(\xi) = 1 \iff \frac{r}{q}(\xi) = 1$.

These remarks and the Corollary imply that the total number of $\xi \in K$ such that $\frac{r}{q}(\xi)$ equals 0 or 1 (counted with multiplicities) does not exceed $CF^{n-k/2} + \deg r + \deg q - 1$, (remembering that $(r/q)' = \frac{r'q-rq'}{q^2} \neq 0$).

On the other hand the total number of zeros and units of r/q equals $2\deg r$. So we obtained $2\deg r \leqslant CF^{n-k/2} + \deg r + \deg q - 1$.

Hence $\deg r - \deg q \leqslant CF^{n-k/2}$, $p^{-M}(d^n - d^{n-k}) \leqslant CF^{n-k/2}$, and finally

$$(F-G)^n \leqslant \tilde{C} p^M F^{n-k/2}, \tag{10}$$

as $d \leqslant F$.

Lemma 5. *For functions R, Q, ψ and numbers M, p, k, n, d defined just below the formula (3) we have*

i) *if p does not divide d then $p^M \leqslant C(\psi)k$;*

ii) *if $p|d$ then $p^M \leqslant d^{n-k}$.*

Proof. As $p^M | \deg R - \deg Q$ then $p^M | d^{n-k}(d^k - 1)$.

i) $p^M | d^k - 1$, so $p^M \leqslant C(\psi)k$ could be proved by considering the Newton binomial coefficients. It was also used (and proved) in [4].

ii) Obvious. □

To end the proof we will consider the following two cases separately.

Case 1. p does not divide d.

The formula (10) and Lemma 5(i) give

$$(F-G)^n \leqslant \tilde{C} C(\psi) k F^{n-k/2} \leqslant F^{n-\frac{k}{2}(1-\delta)}$$

for every $\delta > 0$ and $k \geqslant k(\delta, \psi)$. Hence for sufficiently large k we have $n \leqslant (1+\Delta)$

$(n - \frac{k}{2}(1-\delta))$ and $n/k \geqslant \frac{1+\Delta}{2\Delta}(1-\delta)$.

As $\frac{1+\Delta}{2\Delta} > 1$ then for sufficiently small δ we have $\frac{1+\Delta}{2\Delta}(1-\delta) > 1$.

Case 2. $p|d$.

The formula (10) and Lemma 5(ii) give

$$(F-G)^n \leqslant \tilde{C}(F-G)^{n-k} F^{n-k/2} \leqslant (F-G)^{n-k+(1+\Delta)(n-\frac{k}{2}(1-\delta))},$$

for every $\delta > 0$ and sufficiently large k. That means that for sufficiently small $\delta > 0$ and large k we have

$$n/k \geqslant \frac{1}{1+\Delta} + \frac{1-\delta}{2} > 1.$$

The two already considered cases imply that for sufficiently large k we have $n/k \geqslant \lambda(\psi) > 1$, where $n, k \in Exc(\psi), n > k$. This ends the proof. □

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