Annales Mathematicae Silesianae 12 (1998), 131–139

Prace Naukowe Uniwersytetu Śląskiego nr 1751, Katowice

WITT RINGS OF INFINITE ALGEBRAIC EXTENSIONS OF GLOBAL FIELDS

KRZYSZTOF KOZIOŁ AND MIECZYSŁAW KULA

Abstract. In this paper we discuss the problem to carry over the well-known Minkowski-Hasse local-global principle to the context of an infinite algebraic extension of the rationals or the rational function fields $\mathbb{F}_q(x)$ over finite fields. Applying this result we give a new proof of the elementary type conjecture for Witt rings of infinite algebraic extensions of global fields. This generalizes a result of I. Efrat [Ef] who proved, using Galois cohomology methods, a similar fact for algebraic extensions of the rationals.

1. Preliminaries

Let K denote a field of characteristic different from 2. By an n-dimensional quadratic form over K we mean any polynomial $f = a_1x_1^2 + \cdots + a_nx_n^2$ with $a_1, \ldots, a_n \in K^*$. Such form is denoted by $f = \langle a_1, \ldots, a_n\rangle$. The set $D_Kf = a_1K^{*2} + \cdots + a_nK^{*2} \cap K^*$ is called *the set of elements represented by the form* f over K. The set $D_K\langle 1, a\rangle$ is known to be a subgroup of K^* for every $a \in K^*$. If $0 \in a_1K^{*2} + \cdots + a_nK^{*2}$, then the form is said to be *isotropic*. We say that f is *universal* if $D_Kf = K^*$. The *Kaplansky radical* of K is defined by $R(K) := \bigcap_{a\in K^*} D_K\langle 1, a\rangle$. One can show that $c \in R(K)$ if and only if the binary form $\langle 1, -c\rangle$ is universal.

The quotient group $G(K) := K^*/K^{*2}$ will be called a *square class group*. The *Witt ring* of K is the quotient ring $W(K) := \mathbb{Z}[G(K)]/J$ where J is the ideal in $\mathbb{Z}[G(K)]$ generated by $[1]+[-1]$ and $[1]+[a]-[1+a]-[a(1+a)]$ for all $a \in K^*$ and $a \neq -1$. (Here [a] denotes the image of the coset $aK^{*2} \in G(K)$

Received on September 22, 1998.

1991 *Mathematics Subject Classification.* 11E81.

Key words and phrases: Infinite extensions of global fields, local–global principle, Witt ring.

in $\mathbb{Z}[G(K)]$). Every element of the Witt ring $W(K)$ is represented by a non-isotropic quadratic form over K.

Consider a pair (W, G_W) where W is a commutative ring with unity and G_W is a subgroup of the multiplicative group of units of W which has exponent 2 and contains -1. Let I_W denote the ideal of W generated by the set $\{a+b \in W \ : \ a, b \in G_W\}$. The pair is called *an abstract Witt ring* if the following axioms are satisfied:

$\mathcal{W}_1$. W is additively generated by G_W.

$\mathcal{W}_2$. $I_W \cap G_W = \emptyset$

$\mathcal{W}_3$. If $a+b+c \in I_W^2$ with $a, b, c \in G_W \cup \{0\}$, then $a+b+c = 0$ in W.

$\mathcal{W}_4$. If $a_1 + \cdots + a_n = b_1 + \cdots + b_n$ and $n \geqslant 3$, then there exist $a, b, c_3, \ldots, c_n \in G_W$ such that $a_2 + \cdots + a_n = a + c_3 + \cdots + c_n$, $b_2 + \cdots + b_n = b + c_3 + \cdots + c_n$ and $a_1 + a = b_1 + b$.

For any field K (char $K \neq 2$), the pair $(W(K), G(K))$ is an abstract Witt ring determined by the field K.

For an abstract Witt ring (W, G_W) the set

$$R(W) := \{a \in G_W \ : \ \bigwedge_{c \in G_W} 1 - a + c \in G_W\}$$

is referred to as the *Kaplansky radical* of the Witt ring. In the field case we have $R(W(K)) = R(K)/K^{*2}$. If $R(W) = \{1\}$, then the Witt ring is said to be *non-degenerated*. Otherwise W is said to be *degenerated*. If $R(W) = G_W$, then W is referred to as a *totally degenerated* Witt ring. A Witt ring is *of local type* if it is non-degenerated and $|I_W^2/I_W^3| = 2$ (here, W is allowed to be infinitely generated).

A *homomorphism* of abstract Witt rings (W_1, G_1) and (W_2, G_2) is a ring homomorphisms $\Phi : W_1 \longrightarrow W_2$ such that $\Phi(G_1) \subseteq G_2$. Abstract Witt rings and homomorphisms form a category closed with respect to forming finite direct products and group rings. The *elementary type conjecture* states that every finitely generated Witt ring of a field can be constructed from the Witt rings of $\mathbb{C}$, $\mathbb{F}_3$ and local fields by direct products and group ring formations.

It is easy to check, that the set $I := \{\langle 1, -a\rangle \ : \ a \in R(K)\}$ is an ideal of the Witt ring $W(K)$. We denote by $W_{\mathrm{nd}}(K)$ the factor-ring $W(K)/I$. The pair $(W_{\mathrm{nd}}(K), G(K)/R(K))$ is an abstract Witt ring. It is called the *non-degenerated part of the Witt ring*. According to [M, Chapter 5.5], every Witt ring is a product of its non-degenerated part and suitable totally degenerated Witt ring. Moreover, every totally degenerated Witt ring is a product of suitable numbers of copies of $W(\mathbb{F}_3)$ and $W(\mathbb{F}_5)$.

The rest of used notation is standard and can be found as well as the basic facts from the quadratic form theory in [L], [S] and [M].

2. Localizations of infinite algebraic extensions of global fields

Let K be an algebraic extension of a global field F and let σ be a fixed valuation of K. The valuation induces (by the restriction) valuations on finite (degree) subextensions of K/F. The completions of these subextensions in a fixed completion of K form a direct system of valuated fields. The direct limit of this system will be denoted by K_σ and called the *localization* of K with respect to σ.

LEMMA 1. *If L is a localization of an algebraic extension of a global field, then $[L^* : D_L\langle 1, a\rangle] \leqslant 2$ for all $a \in L^*$.*

PROOF. Let L be a localization of an algebraic extension of a global field F. If every binary form $\langle 1, a\rangle$ over L is universal, then $[L^* : D_L\langle 1, a\rangle] = 1$. Now suppose that there is $a \in L$ such that $\langle 1, a\rangle$ is not universal. Take $x, y \in L^* \setminus D_L\langle 1, a\rangle$. Then there exists a local field $F \subseteq M \subseteq L$ such that $a, x, y \in M$. Since $D_M\langle 1, a\rangle$ has index 2 in M^*, we have $xy \in D_M\langle 1, a\rangle \subseteq D_L\langle 1, a\rangle$. This shows that $[L^* : D_L\langle 1, a\rangle] \leqslant 2$ for all $a \in L^*$.

COROLLARY 2. *The Witt ring of a localization of an algebraic extension of a global field can be represented as a direct product of a Witt ring of local type and a totally degenerated Witt ring.*

PROOF. From Lemma 1 it may be concluded that the non-degenerated part of $W(L)$ is of local type. The corollary follows from the fact that every Witt ring is a product of its non-degenerated part and a totally degenerated Witt ring.

We get immediately the following conclusion.

COROLLARY 3. *Every finitely generated Witt ring of a localization of an algebraic extension of a global field is of elementary type.*

3. Local-global theorem for infinite algebraic extensions of global fields.

In this section we prove an analogue of the Minkowski–Hasse principle for infinite algebraic extensions of global fields.

THEOREM 4. *Let K be an algebraic extension of a global field F and let f be a quadratic form of dimension at least 3 over K. Then f is isotropic over K if and only if it is isotropic over every localization K_σ of K.*

PROOF. If f is isotropic over K, then it is isotropic over K_σ because K_σ is an extension of K.

Now we assume $f = \langle a_1, \ldots, a_n \rangle$ is anisotropic over K and we construct a valuation σ such that f is anisotropic over K_σ. There is a tower of fields:

$$F \subseteq F(a_1, \ldots, a_n) = M_1 \subseteq M_2 \subseteq \ldots \subseteq M_s \subseteq \ldots \subseteq K$$

such that all M_s are finite extensions of F and $K = \bigcup_{s \in \mathbb{N}} M_s$.

Let $s \geqslant 1$ be a fixed integer. By the local–global principle of Minkowski–Hasse (cf. [S, Chapter 6, Theorem 6.5]), the form f is anisotropic at least over one completion of M_s. On the other hand, the entries $a_1, \ldots a_n$ of f are units with respect to almost all valuations of M_s. Thus, according to [L, Chapter 6, Proposition 1.9] f remains anisotropic only over finite number of completions of M_s, whenever $\dim f \geqslant 3$.

We consider a graph where the set of nodes consists of all these completions of M_s (for all $s \geqslant 1$) over which f is anisotropic and arrows correspond to the field extension relation between compatible completions of M_s and M_{s+1}.

This graph is a forest satisfying assumptions of König's infinity lemma [KM, Ch.IX, Th. 1,p. 326]. By the lemma, there is an infinite chain of the completions in the graph defined above. The chain can be interpreted as follows:

There is a compatible chain $(M_{s,\sigma})_{s \in \mathbb{N}}$ of completions of the fields M_s such that f is anisotropic over each $M_{s,\sigma}$. (Here σ is a common name of the chosen valuations on the fields M_s) These valuations define a valuation on $K = \bigcup_{s \in \mathbb{N}} M_s$. Here $K_\sigma = \bigcup_{s \in \mathbb{N}} M_{s,\sigma}$ because any finite degree subextension of K/F is contained in some M_s.

The form f is anisotropic over K_σ because otherwise it would be isotropic over some $M_{s,\sigma}$ as the isotropy relation involves only finitely many elements of a field.

COROLLARY 5. *Let K be as in Theorem 4, f be a quadratic form over K of dimension at least 2 and $a \in K$. Then a is represented by f over K if and only if a is represented by f over every localization K_σ of K.*

PROOF. Observe that $a \in D_K f$ if and only if the form $f \perp \langle -a \rangle$ is isotropic, and then apply Theorem 4 to this form.

COROLLARY 6. *Let K be as in Theorem 4 and $a \in K$. Then $a \in R(K)$ if and only if $a \in R(K_\sigma)$ for every localization K_σ of K.*

PROOF. To prove this Corollary one should only observe that that every square class of K_σ can be represented by an element of K by the Local Square Theorem (cf. [L], Corollary 2.20) and density. Now apply Corollary 5.

REMARK 7. In Corollary 6 we can replace the Kaplansky radical $R(K)$ by the Yucas radicals R_n $(n \geqslant 1)$ defined in [Y].

COROLLARY 8. *Let K be as in Theorem 4 and $a \in K$. If $a \in K_\sigma^{*2}$ for every localization K_σ of K, then $a \in R(K)$.*

PROOF. It suffices to use the inclusion $K_\sigma^{*2} \subseteq R(K_\sigma)$.

Let H denote the subgroup of K^* consisting of all "local squares", i.e.,

$$H = \{a \in K^* : a \in K_\sigma^{*2} \text{ for every localization } K_\sigma \text{ of } K\}.$$

It is clear (by Corollary 8) that $H \subseteq R(K)$.

If K is a global field, then every element of K which is a local square is actually a square in K, i.e., $H = K^{*2}$. This property is not true for infinite extensions of global fields, in general, but we show, in the next theorem, that this is the necessary and sufficient condition for the generalized local–global principle to hold.

PROPOSITION 9. *Let K be as in Theorem 4. Then the following statements are equivalent:*

(1) $H = K^{*2}$.

(2) *For every quadratic form f over K, f is isotropic over K if and only if f is isotropic over every localization K_σ of K.*

(3) *For every quadratic form f over K and every $a \in K^*$ we have $a \in D_K f \iff a \in D_{K_\sigma} f$ for every localization K_σ of K.*

PROOF. (1) $\Rightarrow$ (2) If $\dim f \geqslant 3$ we apply Theorem 4. A form of dimension 1 cannot be isotropic, hence it remains to consider the case $\dim f = 2$. Notice that $f = \langle a, b\rangle = a\langle 1, ab\rangle$ is isotropic if and only if $\langle 1, ab\rangle$ is isotropic. Therefore f is isotropic over every K_σ if and only if $-ab \in K_\sigma^{*2}$ for every σ. This is equivalent to $-ab \in H = K^{*2}$ and as a consequence to the isotropy of f over K.

(2) $\Rightarrow$ (3) We apply (2) to the form $f \perp \langle -a\rangle$

(3) $\Rightarrow$ (1) It is easily seen that the set of elements represented by the form $\langle 1\rangle$ coincides with the group of squares of the field. Combining this with (3) yields (1).

The question: *When does the group H defined above equal K^{*2}?* - remains open. Below we give 2 examples of the possible situation.

PROPOSITION 10. *Let F be a global field and let $F \subseteq M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n \subseteq \ldots$ be a tower of finite extensions of F. If all but a finite number of degrees $[M_{i+1} : M_i]$ are odd and $K = \bigcup_{n\in\mathbb{N}} M_n$, then $H = K^{*2}$.*

PROOF. This is a consequence of Springer's Theorem for odd degree extensions (cf. [L, Chapter 7, Theorem 2.3]).

EXAMPLE 11 (W. Scharlau [S1]). Here we construct an infinite algebraic number field where 2 is a local square but not a global square.

Using Dirichlet's theorem we choose an infinite sequence of different rational primes $p_2, p_3, p_5, \ldots$ with the following property:

1° p_2 is a square in $\mathbb{Q}_2$ (e.g. $p_2 = 17$)

2° $2 \cdot p_\ell$ is a square in $\mathbb{Q}_\ell$ for all odd primes ℓ.

Now we consider the multiquadratic extension of the rational number field:

$$K = \mathbb{Q}(\sqrt{2 \cdot p_2}, \sqrt{p_3}, \sqrt{p_5}, \ldots).$$

Obviously, every localization K_σ contains $\mathbb{Q}_2(\sqrt{2 \cdot p_2}) = \mathbb{Q}_2(\sqrt{2})$ or $\mathbb{Q}_\ell(\sqrt{p_\ell})$, so 2 is a square in K_σ.

On the other side 2 is not in K^{*2} because by Kummer's theory (cf. [N, p.15]).

$$K^{*2} \cap \mathbb{Q} = \mathbb{Q}^{*2} \cdot < 2 \cdot p_2, p_3, p_5, \ldots > .$$

4. Witt rings over infinite algebraic extensions of global fields

Now we consider the natural homomorphism of Witt rings

$$\Phi : W(K) \longrightarrow \prod W(K_\sigma),$$

where K_σ ranges over all localizations of K.

PROPOSITION 12. *The kernel of Φ is the ideal of $W(K)$ generated by the set*

$$\{\langle 1, -a\rangle : a \in H\}.$$

PROOF. Let us consider a form f in $\ker \Phi$ without loss of generality we can assume that F is anisotropic. Because $\Phi(f) = 0$, we have $\dim f \leqslant 2$ by Theorem 4. If $\dim f = 1$, then $\Phi(f) \neq 0$. Hence f can be written $f = b \cdot \langle 1, -a\rangle$. For every valuation σ of K, the form $b \cdot \langle 1, -a\rangle$ is hyperbolic over K_σ if and only if $a \in K_\sigma^{*2}$, thus $f = b \cdot \langle 1, -a\rangle$ with $a \in H$. This shows that $\ker \Phi \subseteq (\langle 1, -a\rangle : a \in H)$. The converse inclusion is obvious.

Now formulate an analogue of the local-global principle for the non-degenerated parts of Witt rings. Recall that

$$W_{\mathrm{nd}}(K) = W(K)/\{\langle 1, -a\rangle : a \in R(K)\}.$$

PROPOSITION 13. *Let K be an algebraic extension of a global field F. Then the natural homomorphism*

$$\Psi : W_{\mathrm{nd}}(K) \longrightarrow \prod W_{\mathrm{nd}}(K_\sigma)$$

where K_σ ranges over all localizations of K, is a monomorphism.

PROOF. Let f be a form representing a coset in $W_{\mathrm{nd}}(K)$ that goes to 0 under the homomorphism Ψ. There is no loss of generality in assuming that f is anisotropic. Of course, $f \otimes K_\sigma$ is represented by a form in $\{\langle 1, -a\rangle : a \in R(K_\sigma)\}$ for every σ. Obviously, if $\dim f \geqslant 3$, then $f \otimes K_\sigma$ is isotropic for every σ and Theorem 4 leads to a contradiction.
If $\dim f = 2$, then $f = c\langle 1, -d\rangle$ for some $c, d \in K$ and $f \otimes K_\sigma = \langle 1, -a_\sigma\rangle$ for suitable $a_\sigma \in R(K_\sigma)$. Now we have $\langle 1, -d\rangle = c\langle 1, -a_\sigma\rangle = \langle 1, -a_\sigma\rangle$ because $\langle 1, -a_\sigma\rangle$ is universal. By the Witt Cancellation Theorem $a_\sigma K_\sigma^{*2} = dK_\sigma^{*2}$. Thus $d \in R(K_\sigma)$ for all localizations K_σ. Applying Corollary 6 completes the proof.

5. Fields with finite square class group

In this section we assume that K is an extension of a global field and K^*/K^{*2} is finite. It is easy to see that this extension has infinite degree. Moreover every valuation of K has rank 1, because the value group of the valuation is contained in the additive group of the rationals. Denote by $\mathcal{S}(K)$ the set of all (mutually independent) valuations of K. Moreover define $\mathcal{S}_0(K) := \{\sigma \in \mathcal{S}(K) : |K_\sigma^*/K_\sigma^{*2}| > 1\}$. Applying the independence theorem for valuations one can prove the following.

LEMMA 14. *Let K be an extension of a global field and let $|K^*/K^{*2}| < \infty$. Then:*

(1) *The natural mapping $\varphi : K^*/K^{*2} \longrightarrow \prod_{\sigma\in\mathcal{T}} K_\sigma^*/K_\sigma^{*2}$ is an epimorphism for every finite subset $\mathcal{T}$ of $\mathcal{S}(K)$.*

(2) *Every localization of K has finite square class group.*

(3) *The set $\mathcal{S}_0(K)$ is finite.*

PROOF. The statement (1) follows from [K, Theorem 2.2] immediately.

It follows from (1) that $|K_\sigma^*/K_\sigma^{*2}| \leqslant |K^*/K^{*2}| < \infty$, so we have (2).

To prove (3), suppose $|K^*/K^{*2}| = 2^n$ and $|\mathcal{S}_0(K)| = k$. Since $|K_\sigma^*/K_\sigma^{*2}| \geqslant 2$ for all $\sigma \in \mathcal{S}_0(K)$, so $2^n = |K^*/K^{*2}| \geqslant \prod_{\sigma\in\mathcal{S}_0(K)} |K_\sigma^*/K_\sigma^{*2}| \geqslant 2^k$.

Now we are in a position to prove the final result of the paper, which implies the elementary type conjecture for infinite extensions of global fields.

THEOREM 15. *Every finitely generated Witt ring of an algebraic extension of a global field is a direct product of Witt rings of finite or local fields.*

PROOF. Let K be an algebraic extension of a global field. Using Proposition 13 and the fact that $W_{\mathrm{nd}}(K_\sigma)$ is trivial for every $\sigma \notin S_0(K)$ we get

$$\Psi : W_{\mathrm{nd}}(K) \longrightarrow \prod_{\sigma \in S_0(K)} W_{\mathrm{nd}}(K_\sigma)$$

is a monomorphism. Applying Lemma 14 and the well-known fact that $W(K)$ is fintely generated if and only if K^*/K^{*2} is finite we conclude that φ maps K^*/K^{*2} onto $\prod_{\sigma \in S_0(K)} K_\sigma^*/K_\sigma^{*2}$. Recall, the Witt rings $W_{\mathrm{nd}}(K)$ and $W_{\mathrm{nd}}(K_\sigma)$ are generated by the sets of all 1-dimensional forms (i.e. elements of $G(K)/R(K)$ and $G(K_\sigma)/R(K_\sigma)$, resp.). Thus Ψ is an isomorphism. Combining this and Corollary 2 we see that $W_{\mathrm{nd}}(K)$ is of elementary type. To complete the proof it is enough to apply the fact that every Witt ring is a direct product of its non-degenerate part and a suitable totally degenerated Witt ring.

It is worth noticing that the above theorem states more than the elementary type conjecture. In fact, we have proved that every finitely generated Witt ring of algebraic extension of a global field can be built from the basic indecomposables without using the group ring formation.

Acknowledgment. The first author is grateful to dr Marek Szyjewski for a useful comment concerning this subject.

REFERENCES

[Ef] I. EFRAT, *Pro-p Galois groups of algebraic extensions of* $\mathbb{Q}$, J. Number Theory, **64** (1997), 84.

[K] M. KULA, *Fields with prescribed quadratic form schemes*, Math. Z., **167** (1979), 201–212.

[KM] K. KURATOWSKI, A. M. MOSTOWSKI, *Set Theory with an Introduction to Descriptive Set Theory*, PWN - Polish Scientific Publishers, Warsaw & North-Holland Publishing Company, Amsterdam–New York–Oxford (1976).

[L] T. Y. LAM, *The Algebraic Theory of Quadratic Forms*, Benjamin, Reading (Mass.) (1980).

[M] M. MARSHALL, *Abstract Witt Rings*, Queen's Papers in Pure and Applied Mathematics - No. 57, Queen's University, Kingston (1980).

[N] J. NEUKIRCH, *Class field theory*, Grundlehren der mathematischen Wissenschaften, 280, Springer-Verlag, Berlin–New York (1986).

[S] W. SCHARLAU, *Quadratic and Hermitian Forms*, Springer Verlag, Berlin–Heidelberg–New York (1985).

[S1] W. SCHARLAU, private communication.

[Y] J. L. YUCAS, *Quadratic forms and radicals of fields*, Acta Arithmetica, **39** (1981), 313–322.

INSTYTUT MATEMATYKI
UNIWERSYTET ŚLĄSKI
BANKOWA 14
40–007 KATOWICE
POLAND

e-mail:
koziol@ux2.math.us.edu.pl
kula@ux2.math.us.edu.pl