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DENSITY THEOREMS FOR RECIPROCITY EQUIVALENCES

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Abstract. A reciprocity equivalence between two number fields is a Hilbert symbol preserving pair of maps (t, T), in which t is a group isomorphism between the global square class groups of the two fields, and T is a bijection between the sets of primes. For two reciprocity equivalent number fields, it is proved that: Theorem A: The Dirichlet density of the wild set of any reciprocity equivalence is zero. Theorem B: There exists a reciprocity equivalence whose wild set is infinite. Theorem C: Given (t, T), the bijection T determines the global square class isomorphism t.

1. Introduction

This paper contains the results of the dissertation [Pa]. I thank Robert Perlis and P. E. Conner for their insights and guidance.

In [PSCL], Perlis, Szymiczek, Conner, and Litherland investigated Witt rings of algebraic number fields. They proved that two number fields K and L have isomorphic Witt rings if and only if the fields are *reciprocity equivalent*, which is defined as follows:

K and L are reciprocity equivalent when there is a bijection

$$T : \Omega_K \to \Omega_L$$

between the set Ω_K of primes of K and the set Ω_L of primes of L, and a group isomorphism

$$t : K^*/K^{*2} \to L^*/L^{*2}$$

of global square classes such that Hilbert symbols are preserved; that is

$$(a, b)_P = (ta, tb)_{TP}$$

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for every P in Ω_K and a, b in K^*/K^{*2}. We call the pair of maps (t,T) a *reciprocity equivalence.*

Let P denote a finite prime of K. When (t,T) preserves P-orders, *i.e.* when

$$ord_P(a) \equiv ord_{TP}(ta) \ (mod\ 2)$$

for each a in K^*/K^{*^2}, then we say that (t,T) is *tame* at P. Otherwise (t,T) is *wild* at P. The *wild set* of the reciprocity equivalence (t,T) is the collection of all finite primes P where (t,T) is wild. If the wild set is empty, we say that the reciprocity equivalence (t,T) is tame.

2. Summary of P-S-C-L

This section contains a summary of those results from the paper [P-S-C-L] that will be used in this paper. Let P be a prime, finite or infinite, of the number field K, and let K_P denote the completion of K at P. Let (t,T) be a reciprocity equivalence from K to L. The following is Lemma 4, parts a and b, of [P-S-C-L]. For the purposes of this paper, we call it Lemma 1.

LEMMA 1.

1. *There are local symbol-preserving isomorphisms*

$$t_P : K_P^*/K_P^{*2} \rightarrow L_{TP}^*/L_{TP}^{*2}$$

for $P \in \Omega_K$ *making the following diagram commute:*

$$\begin{array}{ccc} K^*/K^{*2} & \longrightarrow & K_P^*/K_P^{*2} \\ \downarrow t & & \downarrow t_P \\ L^*/L^{*2} & \longrightarrow & L_{TP}^*/L_{TP}^{*2} \end{array}$$

2. *The map* T *sends real primes to real primes, complex primes to complex primes, dyadic primes to dyadic primes, and finite nondyadic primes to finite nondyadic primes.*

Let S be a finite set of primes of K. Then S is said to be *sufficiently large* when S contains all real and all dyadic primes of K and when the ring of S-integers

$$O_S = \{x \in K \mid ord_P(x) \geqslant 0 \text{ for all primes } P \in \Omega_K \setminus S\}$$

has odd class number. If S already contains the real and dyadic primes, then S is sufficiently large if and only if S also contains a set of generators of the Sylow 2-subgroup of the ideal class group of K.

Let U_S be the group of units of O_S. That is,

$$U_S = \{x \in K \mid ord_P(x) = 0 \text{ for all primes } P \in \Omega_K \setminus S\}.$$

By definition, an S-equivalence from K to L consists of:

1. A bijection T from a sufficiently large set S of primes of K to a sufficiently large set TS of primes of L.
2. A group isomorphism

$$t_S : U_S/U_S^2 \rightarrow U_{TS}/U_{TS}^2.$$

3. For each prime P of S a symbol-preserving isomorphism

$$t_P : K_P^*/K_P^{*2} \rightarrow L_{TP}^*/L_{TP}^{*2}.$$

4. A commutative diagram

$$\begin{array}{ccc} U_S/U_S^2 & \xrightarrow{diag} & \prod_{P \in S} K_P^*/K_P^{*2} \\ {\scriptstyle t_S}\downarrow & & \downarrow \prod_{P \in S} t_P \\ U_{TS}/U_{TS}^2 & \xrightarrow{diag} & \prod_{P \in S} L_{TP}^*/L_{TP}^{*2}. \end{array}$$

Our second lemma is Lemma 5 from [P-S-C-L].

LEMMA 2. *Let S be a sufficiently large set of primes of K. Then the map*

$$U_S/U_S^2 \xrightarrow{diag} \prod_{P \in S} K_P^*/K_P^{*2}$$

is injective.

We close this section by quoting two results from [P-S-C-L]. The first is [P-S-C-L] Theorem 2, which we relabel Theorem 1:

THEOREM 1. *An S-equivalence from K to L can be extended to a reciprocity equivalence that is tame outside of S.*

The next result is taken from Corollary 3 of [P-S-C-L], restated in terms appropriate for this paper:

Corollary. *Let (t,T) be a reciprocity equivalence between two number fields K and L with at most a finite wild set W. Let S be a sufficiently large set of primes of K containing W. If TS is also sufficiently large, then (t,T) restricted to U_S/U_S^2 is an S-equivalence.*

3. Main Lemma

Let F be an algebraic number field and let M be a set of primes of F.

The terminology *almost all* means 'with the possible exception of a set of Dirichlet density 0'. Define

$$G(M) = \{\bar{x} \in F^*/F^{*2} \text{ such that } \bar{x} = 1 \text{ in } F_P^*/F_P^{*2} \text{ for almost all } P \text{ in } M\}.$$

Main Lemma. *If $G(M)$ is infinite, then the Dirichlet density of M is zero.*

Proof. $G(M)$ is a vector-space over the field F_2 of order 2. Being infinite, $G(M)$ has infinite dimension over F_2. Hence, for any natural number k there are F_2-linearly independent elements $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k$ in $G(M)$. Set $E_k = F(\sqrt{x_1}, \sqrt{x_2}, \ldots, \sqrt{x_k})$, where x_i is any representative of $\bar{x}_i$. Then E_k has degree $[E_k : F] = 2^k$ over F. Let D_k be the set of finite primes of F that split completely in E_k, and let Δ_k denote the set of all primes P of F which ramify in E_k.

We assert that M is almost contained in $D_k \cup \Delta_k$.

For k fixed and for each i in the range $1 \leqslant i \leqslant k$, let S_i be the set of all primes P in M for which x_i is not a square in F_P. By definition of $G(M)$, each set S_i has density 0. And Δ_k is also finite. Thus

$$S[k] = \Big(\bigcup_{i=1}^{k} S_i\Big) \cup (\Delta_k)$$

has density 0. Let P be a finite prime in $M \setminus S[k]$ and let Q be a prime of E_k that lies over P. Since x_i is a square in F_P for $1 \leqslant i \leqslant k$, the completion

$$(E_k)_Q = F_P(\sqrt{x_1}, \sqrt{x_2}, \ldots, \sqrt{x_k}) = F_P.$$

Hence P splits completely in E_k; so P is contained in D_k. Since each prime of M outside of $S[k]$ lies in D_k, it follows that M is contained in the union of D_k with the set $S[k]$. By Cebotarev's Density Theorem, the density of D_k is $[E_k : F]^{-1} = 2^{-k}$. Since the set $S[k]$ has density 0, the set M is a subset of a set of density 2^{-k} for every natural number k. It follows that M has Dirichlet density 0, proving the lemma.

4. Theorems A, B, C

This section contains the proofs of the three theorems mentioned in the abstract.

LEMMA 3. *Each element x in K^* is a local square at every wild prime of (t,T) with the exception of at most finitely many wild primes.*

PROOF. Suppose not. Then, since x is locally a unit at all but finitely many primes, there is an infinite set C of finite nondyadic wild primes of K such that x is locally a non-square unit at every prime in C. Applying the square class map t then shows that $t(\bar{x})$ is locally the square class of a local prime element at TP for an infinite set of primes TP of L. This is impossible, proving the lemma.

THEOREM A. *If (t,T) is a reciprocity equivalence from K to L, then the density of its wild set is zero.*

PROOF. Let M be the wild set of (t,T).

We assert that $G(M)$ is equal to the infinite square class group K^*/K^{*2}.

The inclusion $G(M) \subset K^*/K^{*2}$ is clear. Conversely, take $\bar{x}$ in K^*/K^{*2} and let x be an element of $\bar{x}$. By Lemma 3, x is a local square at almost every element of M. Thus x lies in $G(M)$, proving the assertion that $G(M) = K^*/K^{*2}$. Hence $G(M)$ is infinite and so by the Main Lemma, M has density zero, proving Theorem A.

Let A be an abelian multiplicative group. Recall that the *rank* of A is the minimal size of a set of generators of A. If no finite set of elements generates A, then the rank of A is ∞.

REMARK. By the Dirichlet S-unit theorem, if S is a finite set of primes of K containing all infinite primes, then U_S is finitely generated (in fact, U_S has rank $|S|-1$). It follows that U_S/U_S^2 is also finitely generated.

Let P be a finite nondyadic prime of a number field K, let u be a nonsquare unit of the ring of local integers of K_P, and let π be a local uniformizing parameter of P. Then we have the following values for Hilbert symbols:

$$(u,u)_P = 1, (\pi,u)_P = -1, (u\pi,u)_P = -1.$$

Moreover $(\pi,\pi)_P = 1$ if and only if -1 is a square in K_P^*/K_P^{*2}.

LEMMA 4. *Let S be a sufficiently large set of primes of a number field K. Then there is a prime P' outside of S and a a self-equivalence*

(t', T') from $U_{S'}/U_{S'}^2$ onto $U_{S'}/U_{S'}^2$ where $S' = S \cup \{P'\}$ and where (t', T') is defined as follows:

1. *t' is the identity map on $U_{S'}/U_{S'}^2$.*
2. *T' is the identity map on S'.*
3. *For every prime P in S the local map t'_P of (t', T') is the identity on K_P^*/K_P^{*2}.*
4. *The local map $t_{P'} : K_{P'}^*/K_{P'}^{*2} \to K_{P'}^*/K_{P'}^{*2}$ is defined by $t_{P'}(\bar{1}) = \bar{1}$, $t_{P'}(\bar{u}') = \bar{u}'\bar{\pi}'$, $t_{P'}(\bar{\pi}') = \bar{\pi}'$ and $t_{P'}(\bar{u}'\bar{\pi}') = \bar{u}'$ where u' denotes $u_{P'}$ and π' denotes $\pi_{P'}$.*

PROOF. By Dirichlet's S-unit theorem, there exist $a_1, a_2, \cdots, a_n$ in K which generate U_S/U_S^2. Put $a_0 = -1$ and let L_S denote the field

$$K\left(\sqrt{a_0}, \sqrt{a_1}, \ldots, \sqrt{a_n}\right).$$

Infinitely many primes of K split completely in L_S; choose P' to be one of these primes that is finite and nondyadic. Thus a_i is a square at P' for $0 \leqslant i \leqslant n$. Let $S' = S \cup \{P'\}$ and let (t', T') be as in the statement of this claim. Then the following diagram commutes:

$$\begin{array}{ccc} U_{S'}/U_{S'}^2 & \xrightarrow{diag} & \prod_{P \in S'} K_P^*/K_P^{*2} \\ {\scriptstyle t_{S_1}} \downarrow & & \downarrow \prod_{P \in S'} t_P \\ U_{TS'}/U_{TS'}^2 & \xrightarrow{diag} & \prod_{P \in S'} L_{TP}^*/L_{TP}^{*2}. \end{array}$$

It remains to check that Hilbert symbols are preserved. This is automatic for all $P \in S$ since the local map is the identity, so it remains to check that the local map at P' preserves Hilbert symbols. Since $a_0 = -1$ is a square in $K_{P'}^*/K_{P'}^{*2}$, we have the following Hilbert symbol equalities:
$(u', u')_{P'} = (u', u')_{P'}(u', \pi')_{P'}^2(\pi', \pi')_{P'} = (u'\pi', u'\pi')_{P'}$
and $(u', \pi')_{P'} = (u', \pi')_{P'}(\pi', \pi')_{P'} = (u'\pi', \pi')_{P'}$.
From the two equalities above and the definition of $t_{P'}$, we see that $t_{P'}$ preserves local Hilbert symbols. Finally, since S' is sufficiently large, (t', T') is an S'-equivalence, proving Lemma 4.

LEMMA 5. *Let (t, T) be a reciprocity equivalence from the number field K to the number field L with a finite wild set W comprised of n elements (where n can be zero). Suppose that S and TS are sufficiently large sets of primes of K and L, respectively, and suppose that S contains W. Then* *there exists a prime P' of K outside of S, a set of primes $S' = S \cup \{P'\}$ and an S'-equivalence (t', T') from $U_{S'}/U_{S'}^2$ onto $U_{TS'}/U_{TS'}^2$ satisfying the following properties:*

1. *(t', T') has exactly $n+1$ wild primes;*
2. *(t', T') restricted to U_S/U_S^2 is precisely (t, T) restricted to U_S/U_S^2.*

PROOF. By the Corollary to Theorem 1, (t, T) restricted to U_S/U_S^2 is an S-equivalence onto $U_{TS}/\ U_{TS}^2$. By Lemma 4 applied to the field L, there exists a prime P' of K outside of S such that, for $S' = S \cup \{P'\}$ (and hence $TS' = TS \cup \{TP'\}$), there exists an TS'- self-equivalence (t', T') from $U_{TS'}/U_{TS'}^2$ onto $U_{TS'}/U_{TS'}^2$ which satisfies properties 1, 2, 3, and 4 of Lemma 4 (with the field L in place of K.)

Let (t'', T'') denote the composition $(t' \circ t,\ T' \circ T)$ of the given reciprocity equivalence (t, T) from K to L with the TS'-self-equivalence we just constructed. Then (t'', T'') is an S'-equivalence from $U_{S'}/U_{S'}^2$ onto $U_{TS'}/U_{TS'}^2$ with exactly $n + 1$ wild primes. The reader can easily see that property 2 holds under this construction. This proves Lemma 5.

Lemma 5 contains the main ingredients needed for constructing a reciprocity equivalence with an infinite wild set. However, there are some necessary technical details which are handled in the following lemma.

LEMMA 6. *Let (t, T) be a reciprocity equivalence from K to L with a finite wild set $W(t, T)$. Let $p_1, p_2, p_3, \ldots$ denote an ordering of the rational primes numbers. For every natural number n, let A_n denote the set of all prime ideals in K lying over a rational prime p_j with $j \leqslant n$. Similarly, let B_n denote the set of all prime ideals in L lying over a rational prime p_j with $j \leqslant n$. We asert:*

a). *There exists a sufficiently large set S_1 containing A_1, the wild set of (t, T), and containing at least one wild prime. There also exists an S_1-equivalence (t_1, T_1) for which $T_1(S_1) \supset B_1$.*

b). *Given a natural number n and given a sufficiently large set S_n containing S_1 and given an S_n-equivalence (t_n, T_n) that restricts to (t_1, T_1), and given that the wild set of (t_n, T_n) contains at least n primes, then there is a set S_{n+1} containing S_n and an S_{n+1}-equivalence (t_{n+1}, T_{n+1}) which restricts to (t_n, T_n) and whose wild set contains at least $n+1$ primes, with $S_{n+1} \supset A_{n+1}$ and $T_{n+1}(S_{n+1}) \supset B_{n+1}$.*

PROOF. Let C be a finite set of primes which generates the Sylow 2-subgroup of the ideal class group of K. Define S_0, a set of primes of K, to be the union of all infinite primes, dyadic primes, and the set C. Similarly define $\tilde{S}_0$, a set of primes of L, to be the union of all infinite primes, dyadic primes and a set D of generators of the Sylow 2-subgroup of the ideal class group of L. Clearly any finite set of primes containing either S_0 or $\tilde{S}_0$ is sufficiently large. Let $S_1' = S_0 \cup T^{-1}(\tilde{S}_0 \cup B_1) \cup W(t,T) \cup A_1$. By Lemma 5 there exists a prime P_0 of K outside of S_1', a set $S_1 = S_1' \cup \{P_0\}$ and an S_1-equivalence (t_1, T_1) from $U_{S_1}/U_{S_1}^2$ onto $U_{TS_1}/U_{TS_1}^2$ for which P_0 is wild and which restricts to (t,T). Thus, the wild set $W(t_1,T_1)$ contains at least one wild prime. This proves part a).

For b), we first extend the given S_n-equivalence (t_n,T_n) to a reciprocity equivalence (t_n', T_n'), by Theorem 1. In fact, this extension (t_n', T_n') is tame outside S_n although that is not needed here. Let $S_{n+1}' = S_n \cup A_{n+1} \cup {T'}_n^{-1}(B_{n+1})$. By Lemma 5 there exists a prime P_{n+1} of K outside of S_{n+1}' and an extension of the given S_n-equivalence (t_n,T_n) to an S_{n+1}-equivalence (t_{n+1},T_{n+1}) where $S_{n+1} = S_{n+1}' \cup \{P_{n+1}\}$, for which P_{n+1} is wild. Thus the wild set $W(t_{n+1},T_{n+1})$ contains at least $n+1$ primes. This proves b) and Lemma 6.

This brings us to

THEOREM B. *If K is reciprocity equivalent to L, then there exists a reciprocity equivalence between them with an infinite wild set.*

PROOF. Recall that Ω_K denotes the collection of all primes (finite, dyadic, infinite) of the field K and Ω_L denotes the set of all primes of L. Let (t,T) be a reciprocity equivalence from K to L. If the wild set $W(t,T)$ is infinite, there is nothing to prove. So assume the wild set is finite. By Lemma 6, there is a sequence of sufficiently large sets

$$S_1 \subset S_2 \subset \cdots \subset S_n \subset \cdots$$

and a corresponding sequence of S_n-equivalences (t_n,T_n), in which the $(n+1)$st extends the nth, and for which the wild set $W(t_n,T_n)$ has at least n primes. Then the union

$$\cup_{n=1}^{\infty} S_n = \Omega_K$$

since S_n contains the set A_n defined in Lemma 6, and similarly

$$\cup_{n=1}^{\infty} T_n(S_n) = \Omega_L.$$

By compatibility, the bijections T_n's canonically induce a bijection T_* from Ω_K to Ω_L. Moreover,

$$\cup_{n=1}^{\infty} U_{S_n}/U_{S_n}^2 = K^*/K^{*2},$$

and therefore the compatible group isomorphisms t_n canonically induce a group isomorphism t_* from K^*/K^{*2} to L^*/L^{*2}. Then the pair (t_*, T_*) preserves Hilbert symbols, since each pair (t_n, T_n) does, and the wild set of (t_*, T_*) exceeds n for every natural number n. Thus (t_*, T_*) is the desired reciprocity equivalence with an infinite wild set, proving Theorem B.

Having proved Theorems A and B, we turn our attention to the following question: Given a reciprocity equivalence (t, T), to what extent does either of the two maps determine the other? In [P-S-C-L], Lemma 4, part f, it is proved that the square class isomorphism t determines T at the non-complex primes. For use below, we cite a very special case. We refer to a reciprocity equivalence from a field K to itself as a *self-equivalence*.

LEMMA 7. *Let (t, T) be a self-equivalence on K. If $t = id$, then $T = id$ except possibly at the complex primes.*

It should be observed that, given a reciprocity equivalence (t, T), one can change T by arbitrarily permuting the complex primes, yielding a new bijection T' for which (t, T') is another reciprocity equivalence. This settles the question above in one direction. We now consider the question: Does T determine t? The answer is given in Theorem C, below. The proof will take some preparation; the key step involves the sets $G(M)$ of the Main Lemma, in section 3.

LEMMA 8. *Let (t, T) be a self-equivalence on K, and let η be the homomorphism from K^*/K^{*2} to K^*/K^{*2} sending $\bar{x}$ to $t(\bar{x})/(\bar{x})$. Fix an element $\bar{y}$ of K^*/K^{*2}. Then there is a finite set, $S(\bar{y})$, of primes so that for any tame prime $P \notin S(\bar{y})$ with $TP = P$, then $\eta(\bar{y}) = \bar{1}$ locally in K_P^*/K_P^{*2}.*

PROOF. Let $\bar{y} \in K^*/K^{*2}$, and $y \in \bar{y}$. We define $S(\bar{y})$ to be the set of all infinite primes, dyadic primes, and primes P for which $ord_P(y) \neq 0$. Now suppose that P is a tame prime outside of $S(\bar{y})$ for which $TP = P$. Let u_P be a local non-square unit at P. Then $t_P(u_P) = u_{TP} = u_P$, by tameness. But locally at P the class $\bar{y}$ is either a local square or the class of u_P at P. So $\eta(\bar{y}) = t_P(\bar{y})/\bar{y} = \bar{1}$ locally at P, proving Lemma 8.

LEMMA 9. *Let (t, T) be a self-equivalence on K and let η be as before. Suppose that the image of η is a finite set. Then $TP = P$ for every prime P of K outside of a finite exceptional set.*

PROOF. Let $\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\}$ denote the image of η. The set of all dyadic primes, infinite primes, and all primes P for which $ord_{TP}(\bar{x}_i) \not\equiv 0 \ (mod\ 2)$

for some index i is a finite set. Take P outside this finite set. We claim that $TP = P$; we argue by contradiction. If $TP \neq P$, then by approximation, there exist global square classes $\bar{a}, \bar{b}$ such that $\bar{a}$ is a local square at TP and a local prime element at P, while $\bar{b}$ is a local non-square unit at P and a local square at TP. Write $\eta(\bar{a}) = \bar{x}_j$ and $\eta(\bar{b}) = \bar{x}_k$. Then we compute Hilbert symbols as follows:

$$-1 = (\bar{a}, \bar{b})_P = (t_P(\bar{a}), t_P(\bar{b}))_{TP} = (\bar{x}_j\bar{a}, \bar{x}_k\bar{b})_{TP} =$$

$$= (\bar{a}, \bar{b})_{TP}(\bar{x}_j, \bar{b})_{TP}(\bar{a}, \bar{x}_k)_{TP}(\bar{x}_j, \bar{x}_k)_{TP} = 1$$

since $\bar{a}, \bar{b}$ are local squares at TP and $\bar{x}_j$ and $\bar{x}_k$ are locally the square classes of local units at the non-dyadic prime TP. This contradiction proves that $TP = P$, proving the lemma.

LEMMA 10. *Let (t, T) be a self-equivalence on K and η be as before. If the image of η is finite, then $t = id$ and $T = id$ except possibly at the complex primes of K.*

PROOF. We will show that $t = id$; then $T(P) = P$ for non-complex P follows immediately from Lemma 7. To show that $t = id$ we will show that the image of η is $\bar{1}$. We begin by partitioning the set of primes of K into three disjoint subsets A, B and C. Let A be the set of all dyadic primes, infinite primes, and all tame primes P of (t, T) such that $TP \neq P$. The set A is finite by Lemma 9. Let B be the set of all nondyadic tame primes P of (t, T) such that $TP = P$; and let C be the set of all nondyadic wild primes of (t, T). The subsets B and C can be infinite. Let $\bar{x} \in K^*/K^{*2}$. By Lemma 8, $\eta(\bar{x}) = 1$ locally at P for every prime P in B outside a finite exceptional set. By Lemma 3, $\eta(\bar{x}) = 1$ locally at P for every prime P of C outside of a finite exceptional set. Thus $\eta(\bar{x})$ is a local square at P for every prime P of K outside of a finite set, and so by the Global Square Theorem, $\eta(\bar{x}) = \bar{1}$ in K^*/K^{*2}. Hence $t = id$; whence $T = id$ except possibly at the complex primes.

Recall that, for a set M of primes of K, then $G(M)$ is the set of all global square classes that are local squares at P for almost all P in M.

LEMMA 11. *Let (t, T) be a self-equivalence on K and let M be the set of primes P of K such that $TP = P$. Then $\eta(\bar{x}) \in G(M)$ for every $\bar{x} \in K^*/K^{*2}$.*

PROOF: Let $\bar{x}$ be a fixed element of K^*/K^{*2} and let A (respectively B) be the set of all tame (respectively wild) primes P of (t, T) contained in M such that $\eta(\bar{x}) \neq 1$ in K_P^*/K_P^{*2}. By Lemma 8, the density of A is zero.

Since the wild set has density 0 by Theorem A, the subset B has density zero. Therefore the density of $A \cup B$ is zero. Hence $\eta(\bar{x})$ is a local square at P for every $P \in M$ outside $A \cup B$, proving the lemma.

COROLLARY. *Let (t, T) be a self-equivalence on K and M be the set of primes P of K such that $TP = P$. If the density of M is bigger than zero, then $t = id$ and $T = id$ except possibly at the complex primes of K.*

PROOF. Suppose that $t \neq id$ or $T \neq id$ except possibly at the complex primes of K. By Lemma 10 the image of η is infinite. It follows from Lemma 11 that $G(M)$ is infinite, and hence, by the Main Lemma, the density of M is zero, contrary to our hypothesis. This establishes the corollary.

THEOREM C. *Let (t_1, T_1) and (t_2, T_2) be reciprocity equivalences from K to L.*

1. *If $t_1 = t_2$, then $T_1 = T_2$ except possibly at the complex primes of K.*
2. *Let M be a set of primes of K of positive density. If $T_1 P = T_2 P$ for every prime P in M, then $t_1 = t_2$ and $T_1 = T_2$ except possibly at the complex primes of K.*

PROOF. Note that $(t_2^{-1} t_1, T_2^{-1} T_1)$ is a self-equivalence on K. If $t_1 = t_2$, then $t_2^{-1} t_1 = id$, and so part 1 follows from Lemma 7.

If $T_1 P = T_2 P$ for every prime P contained in M, then $T_2^{-1} T_1 P = P$ for $P \in M$. By the Corollary to Lemma 11, $t_2^{-1} t_1 = id$ and $T_2^{-1} T_1 = id$ except possibly at the complex primes of K, and 2 follows, proving the theorem.

COROLLARY. *Let (t, T) be a reciprocity equivalence from K to L. Fix two distinct noncomplex primes P_0, Q_0 of K. Define a new map T_1 on primes by $T_1(P) = T(P)$ for P not in $\{P_0, Q_0\}$, $T_1(P_0) = T(Q_0)$ and $T_1(Q_0) = T(P_0)$. Then for any square class map t_1, the pair (t_1, T_1) is not a reciprocity equivalence.*

PROOF. Suppose (t_1, T_1) is a reciprocity equivalence. The complement of the set $\{P_0, Q_0\}$ in the set of all primes of K has density 1. Hence, by Theorem C, $T = T_1$ at the non-complex primes, contrary to the definition of T_1, proving the corollary.

References

[L] S. Lang, *Algebraic Number Theory*, Addison Wesley, Reading, Massachusetts (1968).

[Pa] T. Palfrey, *Density Theorems for Reciprocity Equivalences*, LSU dissertation, August (1989).

[PSCL] R. Perlis, K. Szymiczek, P. E. Conner, and R. Litherland, *Matching Witts with Global Fields*, In: W.B. Jacob, T.Y. Lam, and R. O. Robson (ed.), *Recent Advances in Real Algebraic Geometry and Quadratic Forms* (Proceedings of the RAG-SQUAD Year, Berkeley 1990-91), Contemporary Mathematics, **155** (1994), 365–387, Providence, Rhode Island, American Mathematical Society.

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