Annales Mathematicae Silesianae 11. Katowice 1997, 7 – 20

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CONVEX FUNCTIONS ON LATTICE ORDERED GROUPS

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Abstract. The classical result about continuity of midconvex functions, which are bounded from above in a neighbourhood of a single point, is extended on functions on lattice ordered 2–divisible Archimedian groups.

Introduction

In the paper there is a proof of the analogue of classical theorem of Bernstein and Doetsch about continuity of J–convex real functions which are bounded from above on a neighbourhood of a single point. Theorem is proved in more general situation: on lattice ordered 2–divisible Archimedian groups.

In the first chapter there is the description of the basic properties of lattice ordered groups, especially the properties of the norm N on a such a group A. This part can be mostly find in [1].

In the second chapter there is a definition of topology which is naturally associated to lattice ordered groups (C–topology). This topology is defined using the norm N. Although N has the properties similar to ordinary norm, usually it is not a norm, so there is a need to make some corrections in a definition of the open balls (open C–balls). The set C (in the paper it is called the set of admissible elements), is of the special interest. The aim of the paper is to investigate J–convex functions, so we restrict our consideration to 2–divisible groups. It is shown that C–topology has "good" properties in the case of the groups satisfying a certain version of Archimedes' axiom (C–Archimedian groups). We called such groups C–groups. The examples of those groups are additive groups of real, rational and dyadic numbers, Cartesian products of such groups, various groups of functions, etc. The second chapter is preliminary for third chapter and is partially contained in [2].

In the third chapter is introduced the notion of AC–J convex function in a similar way as it had been introduced the notion of C–J convex function

1991 *Mathematics Subject Classification.* AMS classification: Primary: 26B25 Secondary: 52A41.

in [3]. It is proved that every AC–J convex function, defined on a C–group, is necessary continuous if it is bounded from above in a neighbourhood of a single point of the domain (Theorem 3). Also, it is proved that every AC–J convex function on a C–group, which is bounded from above in a neighbourhood of a single point, is necessary locally bounded (from above and below) on the whole domain (Theorem 2). The proofs of Theorem 1, Theorem 2, and Theorem 3 are similar to the proofs of corresponding theorems in [3, VIII], but there are some differences, especially in Theorem 2, where we propose a pure algebraic proof. The concept of AC–J convex functions is more general then the concept of the classical J–convex function, so the result also holds for J–convex functions on a C–group.

Acknowledgement. The author wishes thanks to Professor Zygfryd Kominek for his valuable advice, and for many useful seggestions.

1. Lattice ordered groups

Let A be an ordered Abelian group with additive operation and with the relation of order $\leqslant$. It means that for all $a, b, c \in A$ hold:

(i) $a \leqslant a$,
(ii) $(a \leqslant b) \wedge (b \leqslant c) \Rightarrow (a \leqslant c)$,
(iii) $(a \leqslant b) \wedge (b \leqslant a) \Rightarrow (a = b)$,
(iv) $(a \leqslant b) \Rightarrow (a + c \leqslant b + c)$.

Let $A^+ = \{a \in A : 0 \leqslant a\}$ be the positive cone. Note that $(a \leqslant b) \Leftrightarrow (b - a \in A^+)$. We are going to write $a \geqslant b$ if and only if the relation $b \leqslant a$ is valid. Put $a < b$ if and only if $(a \leqslant b) \wedge (a \neq b)$.

Definition 1. An ordered group A is lattice ordered group if there exists a function max: $A \times A \to A$ such that for all $x, y, z \in A$ hold:

(i) $\max(x, y) \geqslant x$,
(ii) $\max(x, y) \geqslant y$,
(iii) $((z \geqslant x) \wedge (z \geqslant y)) \Rightarrow (z \geqslant \max(x, y))$.

It is easy to see that $\max(x, y) = \max(y, x)$.

Define the function min, by: $\min(x, y) = -\max(-x, -y)$. It is easy to see that:

(i) $\min(x, y) \leqslant x$,
(ii) $\min(x, y) \leqslant y$,
(iii) $((z \leqslant x) \wedge (z \leqslant y)) \Rightarrow (z \leqslant \min(x, y))$

for all $x, y, z \in A$.

Note that the max and min functions are as the functions sup and inf in [1, VI, 8.].

In Lemma 1–Lemma 7 the proofs will be omitted since they are trivial or can be found in [1, VI].

LEMMA 1. *Let A be a lattice ordered group, and let $x, y \in A$. Then:*

(i) $((x \leqslant y) \Leftrightarrow (\min(x,y) = x) \Leftrightarrow (\max(x,y) = y))$,
(ii) $((x \leqslant y) \Leftrightarrow (-y \leqslant -x))$,
(iii) $\max(x+y,0) \leqslant \max(x,0) + \max(y,0)$,
(vi) $\min(x+y,0) \geqslant \min(x,0) + \min(y,0)$.

LEMMA 2. *Let A be a lattice ordered group, and let $x, y, z \in A$. Then:*

(i) $\max(x+z, y+z) = \max(x,y) + z$,
(ii) $\min(x+z, y+z) = \min(x,y) + z$.

LEMMA 3. *Let A be a lattice ordered group, and let $x, y \in A$. Then:*

(i) $x + y = \max(x,y) + \min(x,y)$,
(ii) $x = \max(x,0) + \min(x,0)$.

PROOF. Directly from Lemma 2.

LEMMA 4. *Let $n \in \mathrm{N}$. Than* $\max(nx,0) = n\max(x,0)$, *and* $\min(nx,0) = n\min(x,0)$.

PROOF. See [1, VI, Corollary 4 of Proposition 11]. Note that $\max(-x,0) = -\min(x,0)$.

DEFINITION 2. Norm N on lattice ordered group A is the function $N : A \to A$ defined by the equality $N(x) = \max(x,-x)$.

Note that the norm N is as the function $|\ |$ from [1, VI, Definition 4].

LEMMA 5. *Let N be the norm on a lattice ordered group. Then:*

(i) $N(x) = \max(x,0) - \min(x,0)$,
(ii) $Nx = x$ *if and only if* $x \geqslant 0$.

PROOF. Directly from Lemma 2, Lemma 3(ii), Lemma 4 and Lemma 1. See also [1, VI, Proposition 9.d)].

LEMMA 6. *Let N be the norm on a lattice ordered group A. Then:*

(i) $N(x) \geqslant 0$, *for every* $x \in A$,
(ii) $(N(x) = 0) \Leftrightarrow (x = 0)$, *for every* $x \in A$,
(iii) $N(mx) = |m|N(x)$, *for every* $x \in A$, *and for every* $m \in \mathbf{Z}$,
(vi) $N(x+y) \leqslant N(x) + N(y)$, *for all* $x, y \in A$.

PROOF. Directly from Lemma 1, Lemma 4 and Lemma 5.

REMARK. From Lemma 6 we can see that the function N is similar to norm on the linear normed space.

LEMMA 7. *Let N be the norm on a lattice ordered group A. Then: $N(x) \leqslant \varepsilon \Leftrightarrow -\varepsilon \leqslant x \leqslant \varepsilon$, for all $\varepsilon \geqslant 0$, $x \in A$. Specially, we have, $-N(x) \leqslant x \leqslant N(x)$, for every $x \in A$.*

PROOF. Directly for the Definition 2.

2. Topology on lattice ordered groups

DEFINITION 3. [1, VI, Definition 5]. Let A be lattice ordered group, and let $x, y \in A$. We say that x, y are coprime if $\min(x, y) = 0$.

It is obvious that coprime elements are necessarily positive.

Directly from Lemma 3, we conclude that elements $x, y \in A$ are coprime if and only if $\max(x, y) = x + y$.

The aim of the paper is to investigate convex functions on ordered groups. Therefore the groups have to be 2–divisible. In other words, for every $x \in A$, there has to be one and only one $y \in A$, such that $2y = x$. We are going to denote the element y as $\frac{x}{2}$.

LEMMA 8. *Let A be 2–divisible lattice ordered group and let $x \in A$. Then it holds:*

$$(x \geqslant 0) \Rightarrow \left(\frac{x}{2} \geqslant 0\right).$$

PROOF. By Lemma 5.(ii), and Lemma 6.(iii), we have $2N\left(\frac{x}{2}\right) = Nx = x$. Therefore, $N\left(\frac{x}{2}\right) = \frac{x}{2}$, hence $\frac{x}{2} \geqslant 0$.

DEFINITION 4. The set of admissible elements in a lattice ordered 2–divisible group A is any nonempty subset C of the set of all positive elements A^+ having the following properties:

(i) $0 \notin C$,
(ii) $(x \in C \wedge y \geqslant x) \Rightarrow (y \in C)$,
(iii) $(x, y \in C) \Rightarrow (\min(x, y) \in C)$,
(vi) $(x \in C) \Rightarrow (\frac{x}{2} \in C)$.

It is obvious that $C \subseteq A^+ \backslash \{0\}$. If there exist at least two coprime elements in the group, then the given inclusion is strict (by (iii), (i) of Definition 4, by definition of coprime elements, and by the fact that coprime elements are positive).

Note that (ii) in Definition 4 can be replaced by:

(ii)' If $x \in C$, and if $a \geqslant 0$, then $x + a \in C$.

REMARK. Suppose that there exists a strictly positive element a in a group. Denote $A_a = \{x \in A : \ x \geqslant a\}$. Then A_a satisfies the statements (i), (ii), (iii) from the Definition 4. Denote $A_{a,n} = \frac{1}{2^n - 1} A_a$, for $n \in \mathbf{N}$. Then we have $A_{a,n+1} \supseteq A_{a,n}$, for every $n \in \mathbf{N}$. Put $C = \bigcup\limits_{n \in \mathbf{N}} A_{a,n}$. Then C is a set of admissible elements. This is minimal set of admissible elements which contains a.

From now on, we are going to consider that in every lattice ordered 2–divisible group a set of admissible elements C is chosen.

DEFINITION 5. Let A be a lattice ordered 2–divisible group and let C be a set of admissible elements of A. The open C–ball of a radius $r \in C$, with the centre $x_0 \in A$ is the set of all $x \in A$ such that $r - N(x - x_0) \in C$. We denote this set by $U_{x_0,r}$.

REMARK. The given definition is similar, but not identical to the standard definition of the open ball in a normed space (which is defined by the relation $r - |x - x_0| > 0$).

LEMMA 9. [2, Lemma 2] *Let A be as in Definition 5. Then open C–balls form a base of topology on A.*

PROOF. Since, $x_0 \in U_{x_0,r}$ for all $r \in C$, then open C–balls form an open cover of the space A. Let $z_0 \in U_{x_0,r} \cap U_{y_0,R}$ be arbitrary. By the definition of C–balls, we have $r - N(z_0 - x_0) = c_1$ and $R - N(z_0 - y_0) = c_2$, for some $c_1, c_2 \in C$. Then $U_{z_0,\varepsilon} \subseteq U_{x_0,r} \cap U_{y_0,R}$, for $\varepsilon = \min(c_1, c_2)$. Namely, for $x \in U_{z_0,\varepsilon}$, we have $\varepsilon - N(x - z_0) = c_3$, for some $c_3 \in C$. Applying Lemma

6.(iv), we get $r-N(x_0-x) \geqslant r-N(x_0-z_0)+\varepsilon-N(z_0-x)-\varepsilon = c_1+c_3-\varepsilon \in C$. Similarly holds for the second ball. It completes the proof.

REMARK. Lemma 9 is true if we do not assume that A is 2–divisible group (we omit condition (iv) of Definition 4).

DEFINITION 6. C–topology is the topology generated by the open C–balls.

It is easy to see that if $V_{x_0,r} = \{x \in A : N(x-x_0) < r\}$, $F_{x_0,r} = \{x \in A : N(x-x_0) \leqslant r\}$, then we have $U_{x_0,r} \subseteq V_{x_0,r} \subseteq F_{x_0,r}$. Those inclusions can be strict. Directly from the definition of C–topology we obtain: $2^m U_{x_0,r} + a = U_{2^m x_0+a, 2^m r}$, for all $m \in \mathbb{Z}$, $a \in A$. Therefore, if $U \subseteq A$ is an open subset, then $2^m U + a$ is an open subset, too. However, if U is open, and s dyadic, then sU is not necessarily open. Note, also that generally we have $sU_{x_0,r} \neq U_{sx_0,sr}$.

LEMMA 10. [2, Lemma 3(i)] *Let A be a lattice ordered 2–divisible group. Then C is an open set in C–topology.*

PROOF. From Lemma 7, we conclude that inequality $N(x-c) \leqslant \frac{c}{2}$ is equivalent to the $\frac{c}{2} \leqslant x \leqslant \frac{3c}{2}$. Hence, if $c \in C$, then $U_{c,\frac{c}{2}} \subseteq C$.

LEMMA 11. [2, Lemma 3(ii)] *Let A be as in lemma 10. Then $A = C - C$.*

PROOF. According to Lemma 3(ii) we have:

$$x = \max(x,0) - \max(-x,0) = (\max(x,0)+c) - (\max(-x,0)+c)$$

for all $x \in A$, $c \in C$.

One of the most important axiom of the set of the real numbers is Archimedes' axiom. From now on we will consider that A is lattice ordered 2–divisible C–Archimedian group, e.g. that in the group A the following axiom holds:

$$(\forall x \in C)(\forall y \in C)(\exists n \in \mathrm{N})(n \cdot y > x). \tag{1}$$

Writing the same axiom in a different manner, we have the following statement:

$$(\forall x \in C)(\forall y \in C)(\exists n \in \mathrm{N})\left(\frac{x}{2^n} < y\right). \tag{2}$$

One of the consequences of the given statement is that $U \cap C \neq \emptyset$, for every neighbourhood U of zero. Namely, if U is open C–disc around zero of the radius y and if $x \in C$ is arbitrary, then there exists $n \in \mathbf{N}$, such that $y - \frac{x}{2^n} \in C$. Hence $\frac{x}{2^n} \in U$.

It is easy to see that the statement (1) is equivalent to the following statement, which seems to be stronger, but it is just in appearance.

$$(\forall x \geqslant 0)(\forall y \in C)(\exists n \in \mathbf{N})(n \cdot y > x). \tag{3}$$

DEFINITION 7. The group A is C–group if A is lattice ordered, 2–divisible, C–Archimedian group.

EXAMPLE 1. Let $A = \mathbb{R} \times \mathbb{R}$ be group with addition defined coordinately and with relation of order defined by $(a,b) \leqslant (c,d) \Leftrightarrow (a \leqslant c \wedge b \leqslant d)$. Then A is divisible lattice ordered group in which holds:

$$\max((a,b),(c,d)) = (\max(a,c), \max(b,d)), \quad A^+ = \{(a,b) :\ a \geqslant 0 \wedge b \geqslant 0\},$$

$C = \{(a,b) :\ a > 0 \wedge b > 0\}$ is a set of admissible elements. It is easy to see that A is C–group. Also:

$$\begin{aligned}
U_{(x_0,y_0),r} &= \{(x,y) \in \mathbb{R}^2 :\ -r < x - x_0 < r \wedge -r < y - y_0 < r\};\\
F_{(x_0,y_0),r} &= \{(x,y) \in \mathbb{R}^2 :\ -r < x - x_0 \leqslant r \wedge -r \leqslant y - y_0 \leqslant r\};\\
V_{(x_0,y_0),r} &= F_{(x_0,y_0),r} \setminus \{(x_0 + r, y_0 + r), (x_0 + r, y_0 - r),\\
&\qquad (x_0 - r, y_0 + r), (x_0 - r, y_0 - r)\}.
\end{aligned}$$

It can be shown that the set $\{(x,y) \in \mathbb{R}^2 :\ (x \geqslant 0) \wedge (y > 0)\}$ is a set of admissible elements, too. However, with this set of admissible elements, A is not C–Archimedian group.

EXAMPLE 2. Dyadic numbers are numbers of from $\frac{m}{2^n}$, $m \in \mathbb{Z}$, $n \in \mathbf{N} \cup \{0\}$. Group $\mathbf{Q}_2$ of all dyadic numbers, with standard ordering, is C–group (C is the set of strictly positive dyadic numbers). Closure $\mathrm{Cl}\ \mathbf{Q}_2$ is additive group of real numbers. Every C–group is a module over dyadic numbers.

LEMMA 12. *Let A be a C–group. Then for all $x \in A$, $c \in C$ there exists $n_0 \in \mathbf{N}$, such that $\frac{x}{2^n} + c \in C$ holds, for every natural $n \geqslant n_0$.*

PROOF. Let $c \in C$, $x \in A$. Then, by Lemma 11, there exist $c_1, c_2 \in C$, such that $x = c_1 - c_2$. Hence, $\frac{x}{2^n} = \frac{c_1}{2^n} - \frac{c_2}{2^n}$, for every $n \in \mathbf{N}$. If we choose n_0 such that $c - \frac{c_2}{2^{n_0}} > 0$ (it is possible because the group A is C–Archimedian), then it will be: $\frac{x}{2^n} + c = \frac{c_1}{2^n} + \left(c - \frac{c_2}{2^n}\right) \in C$, for every $n \geqslant n_0$.

3. C–J Convex Functions

In this chapter A, B are any two C–groups (2–divisible, C–Archimedian lattice ordered groups), and $D \subseteq A$ is on open J–convex subset. That means it holds:

$$x, y \in D \Rightarrow \frac{x+y}{2} \in D.$$

By induction, somewhat stronger statement, can be proved:

$$(x, y \in D) \Rightarrow (\alpha x + (1-\alpha)y \in D),$$

for every dyadic $\alpha \in [0,1]$. Specially,

$$\frac{x}{2^n} + \left(1 - \frac{1}{2^n}\right) y \in D, \qquad \text{for every} \quad n \in \mathbb{N}.$$

EXAMPLE 3. Let D be an open C–ball (see Definition 5). Then D is open J–convex subset. Namely,

$$r - N\left(\frac{x+y}{2} - x_0\right) \geqslant \frac{r - N(x - x_0) + r - N(y - y_0)}{2}.$$

It is evident that if D is convex then $sD + a$ is convex for every dyadic s, and every $a \in A$. Recall that $x, y \in A$ are comparable if $x \leqslant y$ or $y \leqslant x$.

DEFINITION 8. We say that a function $f : D \to B$ is

(i) $AC - J$ convex, if $f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2}$, for all comparable $x, y \in D$.
(ii) $AC - W$ convex, if $f(x+h) - f(x) \leqslant f(y+h) - f(y)$, for all $x \leqslant y$, $h \geqslant 0$, such that $x, y, x+h, y+h \in D$.
(iii) J–convex (or midconvex) if (i) holds, for all $x, y \in D$.

Note that the notion of $AC - J$ convex function is similar to the notion of the $C - J$ convex function on linear space (see [3, p. 54]).

LEMMA 13. (i) *Every $AC - W$ convex function is $AC - J$ convex function, too.*

(ii) *Every J-convex function is $AC - J$ convex function, too.*

PROOF. (i) Substituting $h \mapsto \frac{y-x}{2}$, $y \mapsto \frac{x+y}{2}$ into Definition 8(ii), we get Definition 8(i). Similarly in the case $x \geqslant y$.

(ii) Obvious.

EXAMPLE 4. [3, Example 8.1] Let the situation be as in Example 1, and let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by formula:

$$f((x,y))=\begin{cases} xy, & xy>0\\ 0, & xy\leqslant 0\end{cases}$$

Then f is $AC-J$ convex function, but it is not J–convex function.

LEMMA 14. *Let $f:D\to B$ be $AC-J$ convex function. Then,*

$$f(\alpha x+(1-\alpha)y)\leqslant \alpha f(x)+(1-\alpha)f(y),$$

for every dyadic $\alpha\in[0,1]$, and for all comparable $x,y\in D$. Specially,

$$f\left(\frac{1}{2^n}x+\left(1-\frac{1}{2^n}\right)y\right)\leqslant \frac{1}{2^n}f(x)+\left(1-\frac{1}{2^n}\right)f(y),\qquad \textit{for every}\quad n\in\mathbb{N}.$$

PROOF. By induction.

We will say that a function $f:D\to B$ is bounded (bounded from above, bounded from below) in a neighbourhood of a single point $x_0\in D$, if there exist open subset V and $\alpha,\beta\in B$, such that $x_0\in V\subseteq D$ and such that $\beta\leqslant f(x)\leqslant\alpha$ ($f(x)\leqslant\alpha$, $f(x)\geqslant\beta$), for every $x\in V$. We will say that f is locally bounded (locally bounded from above, locally bounded from below) on D, if f is bounded (bounded from above, bounded from below) in a neighbourhood of every point $x_0\in D$.

THEOREM 1. *Let $f:D\to B$ be $AC-J$ convex function. If f is locally bounded on D, then f is continuous function.*

At first, we are going to prove the following lemma:

LEMMA 15. *Let $f:D\to B$ be $AC-J$ convex function bounded in a neighbourhood of a single point $x_0\in A$. Then the function $g:D-x_0\to B$, given by $g(x)=f(x+x_0)$, is $AC-J$ convex function, bounded in a neighbourhood of 0.*

PROOF. We have

$$g\left(\frac{x+y}{2}\right)=f\left(\frac{x+y}{2}+x_0\right)=f\left(\frac{(x+x_0)+(y+x_0)}{2}\right)$$
$$\leqslant\frac{f(x+x_0)+f(y+x_0)}{2}=\frac{g(x)+g(y)}{2}.$$

It is easy to see that the function g is bounded and that $D - x_0$ is an open convex subset of A.

PROOF OF THEOREM 1. By Lemma 15 it is enough to prove continuity of the function f in zero with assumption that $0 \in D$. Boûndness and continuity are local properties, so we can assume that D is open C–ball around zero and that there exist $\alpha, \beta \in B$, such that $\beta \leqslant f(x) \leqslant \alpha$, for every $x \in D$. Choose $u_0 \in \frac{D}{2^{n+1}} \cap C$, for sufficiently large $n \in \mathbb{N}$ (it is possible because, by (2) every open C–ball around zero cuts C). If $U = \left(\frac{1}{2^n}D + \left(1 - \frac{1}{2^n}\right) u_0\right) \cap D$, then U is a neighbourhood of zero. Namely, $\frac{1}{2^n}u + \left(1 - \frac{1}{2^n}\right) u_0 = 0$, for $u = -(2^n - 1)u_0 \in (2^n - 1)\frac{D}{2^{n+1}} \subset D$. Put $W = U \cap (2u_0 - U)$. Then $0 \in 2u_0 - U$, because $2u_0 - \left(\frac{1}{2^n}u + \left(1 - \frac{1}{2^n}\right) u_0\right) = 0$, for $u = (2^n + 1)u_0 \in \frac{2^n+1}{2^{n+1}}D \subset D$. Now, we can conclude that W is a neighboourhood of zero. Take $x \in W$. Then $x = \frac{1}{2^n}u + \left(1 - \frac{1}{2^n}\right) u_0$, $u \in D$. Hence, $x - u_0 = \frac{1}{2^n}(u - u_0)$. Suppose that x and u_0 are comparable. Then, if we apply the previous equality, we see that u and u_0 are comparable, too. By Lemma 14, we have:

$$f(x) \leqslant \frac{1}{2^n}f(u) + \left(1 - \frac{1}{2^n}\right) f(u_0), \quad \text{e.g.} \quad f(x) - f(u_0) \leqslant \frac{1}{2^n}(\alpha - \beta).$$

Since, x and u_0 are comparable if and only if $2u_0 - x$ and u_0 are comparable, it also holds:

$$f(2u_0 - x) - f(u_0) \leqslant \frac{1}{2^n}(\alpha - \beta).$$

Also since f is $AC - J$ convex, it must be $f(u_0) \leqslant \frac{f(x)+f(2u_0-x)}{2}$, hence

$$f(u_0) - f(x) \leqslant f(2u_0 - x) - f(u_0) \leqslant \frac{1}{2^n}(\alpha - \beta).$$

Now we have:

$$-\frac{1}{2^n}(\alpha - \beta) \leqslant f(x) - f(u_0) \leqslant \frac{1}{2^n}(\alpha - \beta). \tag{4}$$

Note that the set $V = W \cap (u_0 - C)$ is a neighbourhood of zero. If $x \in V$, then $x < u_0$, and we can apply the previous consideration. Since, $0 \in V$ it holds:

$$-\frac{1}{2^n}(\alpha - \beta) \leqslant f(u_0) - f(0) \leqslant \frac{1}{2^n}(\alpha - \beta). \tag{5}$$

Summarising (4) and (5) we get:

$$-\frac{1}{2^{n-1}}(\alpha - \beta) \leqslant f(x) - f(0) \leqslant \frac{1}{2^{n-1}}(\alpha - \beta).$$

It is, by Lemma 7, equivalent to $N(f(x) - f(0)) \leqslant \frac{\alpha-\beta}{2^n-1}$, for every $x \in V$. This completes the proof of the continuity of the function f in zero, as well as the continuity of f on D.

Note that the proof of Theorem 1 is a modification of the proof od Theorem 8.1 in [3, VIII].

COROLLARY. Let $f : D \to B$ be a bounded J–convex, or $\dot{AC} - W$ convex function. Then f is continuous on D.

Now we are going to prove a stronger statement. Namely, we are going to show that in the Theorem 1 it is sufficient to suppose that f is bounded from above at least one point from the domain. The proof is carried out through a few lemmas.

LEMMA 16. *Let $f : D \to B$ be a $AC - J$ convex function, bounded from above in a neighbourhood of a single point $x_0 \in D$. Then f is locally bounded from above on D.*

PROOF. Applying Lemma 15, we can assume that $0 \in D$, and that f is bounded from above in a neigbourhood of zero. Let V be a neighbourhood of zero in D, such that $f(x) \leqslant \alpha$, for some $\alpha \in B$ and for every $x \in V$. Suppose, at first, that $x \in (D \setminus V) \cap C$. We are going to show that f is bounded from above in a neighbourhood of x. Let $\nu_0 = \frac{x}{2^n} \in V$, for some $n \in \mathbb{N}$. Let $y = x\left(1 + \frac{1}{2^n}\right)$. We can assume that n is sufficiently large, such that $y \in D$. Notice that $y > x$. It is easy to check that

$$x = \frac{1}{2^n}\nu_0 + \left(1 - \frac{1}{2^n}\right) y.$$

Hence, $U = \left(\frac{1}{2^n}V + \left(1 - \frac{1}{2^n}\right) y\right) \cap D$ is a niegbourhood of x. Then $W = U \cap (y - C)$ is also a neigbourhood of x. Namely, $x \in y - C$. Therefore, we can conclude that $W \subseteq U \subseteq D$. Let $\omega \in W$. Then $\omega < y$. There exists $t \in V$, such that $\omega = \frac{1}{2^n}t + \left(1 - \frac{1}{2^n}\right) y$. Hence, $\omega - y = \frac{1}{2^n}(t - y)$. Consequently $t < y$.

By Lemma 14, we can conclude that:

$$f(\omega) \leqslant \frac{1}{2^n}\alpha + \left(1 - \frac{1}{2^n}\right) f(y), \qquad \text{for every} \quad \omega \in W.$$

Hence, f is bounded from above in a neighbourhood of x.

If $x \in -C$, the proof is analogous to the previous one if we take $W = U \cap (y + C)$, as well as $y < x$, $\omega \geqslant y$, $t \geqslant y$. Up to now we have proved:

If f is bounded from above in a neighbourhood of zero, then f is locally bounded from above on $(C \cup \{0\} \cup (-C)) \cap D$.

Now, we can conclude that:

If f is bounded from above in a neigbourhood of $x_0 \in D$, then f is locally bounded from above in a neighbourhood of $x_0 + e \in D$, where $e \in (C \cup (-C)) \cap D$. Namely if we define g by $g(x) = f(x + x_0)$, then, by Lemma 15 g is bounded from above in a neighbourhood of zero, so g is bounded from above in a neigbourhood of e. As in the proof of Lemma 15 we obtain that f must be bounded from above in a neighbourhood of $x_0 + e$.

Suppose, now, that $x \in D$ is arbitrary. We are going to prove that f is bounded from above in a neighbourhood of x.

The set D is J–convex and $0 \in D$, hence $\frac{x}{2^n} \in D$, for all $n \in \mathrm{N}$. Let be $c \in C$. Applying Lemma 12, we can find $c' \in C$, such that $\frac{x}{2^n} = c' - c$, for sufficiently large $n \in \mathrm{N}$. Replacing c by $\frac{c}{2^k}$, $k \in \mathrm{N}$ if it is necessary, we can assume that $c, c' \in D$, and that $x + c' \in D$ (it is possible because D is open). Make up the sequence:

$$0, c', c' - c, 2c' - c, 2c' - 2c, \dots, kc' - kc, (k+1)c' - kc, \dots, 2^n c' - 2^n c = x.$$

The set D is J–convex, and $0 \in D$, hence $\frac{k}{2^n}x + \frac{2^n - k}{2^n} \cdot 0 = k(c' - c) \in D$, for $1 \leqslant k \leqslant 2^n$. Since $\frac{2^n - k}{2^n}c' + \frac{k}{2^n}(x + c') = k(c' - c) + c'$, we can conclude that $(k+1)c' - kc \in D$, for $1 \leqslant k \leqslant 2^n$. It means that every term of the sequence is element of D. Notice that the difference of any two consecutive terms of the sequence is an element from $C \cup (-C)$. Therefore, f is bounded from above in a neighbourhood of every term of the sequence (namely f is bounded from above in a neighbourhood of zero, so f is bounded from above in a neighbourhood of c', hence f is bounded from above in a neigbourhood of $c' - c$ etc.). Therefore f is bounded from above in a neighbourhood of x. This completes the proof.

LEMMA 17. *Let $f : D \to B$ be $AC - J$ convex function bounded from above in a neighbourhood of some point. Then f is bounded from below in a neighbourhood of some point.*

PROOF. As before, we can assume that f is bounded from above in a neighbourhood of zero. It means that there exists a neighbourhood V of zero, and that there exists $\alpha \in B$, such that for every $x \in V$ holds $f(x) \leqslant \alpha$. Without loosing on generality we can assume that V is some open C–disc around zero and that V is a subset of the set D.

Let $x_0 \in V \setminus \{0\}$. Then f is bounded from above in a neighbourhood of x_0, too. It means that it holds $f(x) \leqslant \alpha$, in a neighbourhood $U \subseteq V$ of the element x_0. It is easy to see that $-U \subseteq V$ is a neighbourhood of element $-x_0$. It holds $f\left(\frac{x+(-x)}{2}\right) \leqslant \frac{f(x)+f(-x)}{2}$, for all $x \in U$, hence $f(-x) \geqslant 2f(0) - f(x) \geqslant 2f(0) - \alpha$.

LEMMA 18. *Let $f : D \to B$ be $AC-J$ convex function bounded from below in a neighbourhood of some point. Then f is locally bounded from below on D.*

PROOF. We can assume that D is an open C–disc around zero and that f is bounded from below in a neighbourhood of zero. Let $V \subseteq D$ be an open C–disc around zero, and let $\beta \in B$ be such that $f(x) \geqslant \beta$ for all $x \in V$ and let $x \in (D \setminus V) \cap C$ be such that $\frac{x}{2^n} \in V$ for some $n \in \mathbb{N}$ and let $y = -\frac{x}{2^n}$. Then $y \in V$. Let $\nu_0 = \frac{x}{2^{2n}}$. Then $\nu_0 = \left(1 - \frac{1}{2^n}\right) y + \frac{1}{2^n}x$. Having that $\nu_0 \in V \cap (y+C)$ and that the function ϕ defined by $t \mapsto \left(1 - \frac{1}{2^n}\right) y + \frac{1}{2^n}t$, $t \in D$, is continuous and having $\phi(x) = \nu_0$, there is a neighbourhood W of x such that $\phi(W) \subseteq V \cap (y + C)$. Let $w \in W$. Then $\phi(w) > y$. Hence, $0 < \phi(w) - y = \frac{1}{2^n}(w - y)$ and consequently $w > y$. It is easy to see that:

$$\phi(w) = \left(1 - \frac{1}{2^n}\right) y + \frac{1}{2^n} w. \tag{6}$$

Applying Lemma 14 to (6), we get $f(\phi(w)) \leqslant \left(1 - \frac{1}{2^n}\right) f(y) + \frac{1}{2^n} f(w)$. Therefore, $f(w) \geqslant 2^n\beta - (2^n - 1)f(y)$, for all $w \in W$. Hence, f is bouded from below in a neighbourhood of x.

If $x \in -C$, then similarly we get $y > 0$, $\phi(w) < y$, $w < y$, $\mu_0 \in y - C$.

Now we can proceed similarly as in the proof of the corresponding part of Lemma 16.

Note that proof of Lemma 18 is a modification of the proof of Lemma 8.4 in [3, VIII]. Combining Lemmas 16, 17 and 18, we get the following:

THEOREM 2. *Let $f : D \to B$ be $AC - J$ convex function bounded from above in a neighbourhood of a single point. Then f is locally bounded on D.*

Directly from Theorem 1 and Theorem 2 we have the following:

THEOREM 3. *Let $f : D \to B$ be a $AC - J$ convex function bounded from above in a neighbourhood of a single point. Then f is continuous function.*

COROLLARY. *Let $f : D \to B$ be a J-convex or $AC - W$ convex function bounded from above in a neighbourhood of a single point. Then f is continuous.*

PROOF. Directly from Theorem 2, and Lemma 13.

REMARK. In the case of real J–convex functions this result proved Bernstein and Doetch 1915. In fact they extended the Jensen's result (see [4, VII, 72]). In [4, VII, 71] there is a proof of the result in the case of J–convex functions on normed linear spaces. In the case of $C-J$ convex functions the result is proved in [3, VII, Theorem 8.1].

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